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ON FOLDING SOLUTIONS OF THE BELTRAMI EQUATION

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Abstract. A sufficient condition for the existence of a local folding solution to an alternating Beltrami equation is presented, and the uniformization of local folding solutions is studied.

1. Introduction

1.1. Consider the Beltrami equation

$$w_{\bar z} = \mu(z) w_z \tag{B}$$

in the unit disk $D = \{z \in \mathbf{C} : |z| < 1\}$, where $\mu\colon D \to \overline{\mathbf{C}}$ is measurable, μ is locally bounded in $D_1 = \{z \in D : \operatorname{Im} z > 0\}$, i.e. $\|\mu|K\|_\infty < 1$ for every compact set K in D_1, and $1/\mu$ is locally bounded in $D_2 = \{z \in D : \operatorname{Im} z < 0\}$.

Equation (B) can be rewritten in the symmetric form

$$A(z) w_z + B(z) w_{\bar z} = 0 \tag{B$'$}$$

where $A(z)$ and $B(z)$ are complex-valued measurable functions such that $\mu = -A/B$ in D_1 and $1/\mu = -B/A$ in D_2. Let $E = D \cap \mathbf{R}$.

1.2. Let μ be as in 1.1. A function $f\colon D \to \mathbf{C}$ is said to be a *solution* of (B), if f is continuous, $f \mid (D \setminus \mathbf{R})$ is in $\mathrm{W}^{1,2}_{\mathrm{loc}}$, and its weak derivatives satisfy (B$'$) a.e. in D. A solution f in D is a *folding solution* if it is injective in D_1, in D_2 and in E. If, in addition, $f(D_1) = f(D_2)$ f is a *proper folding solution.*

In Section 2 below, we provide a sufficient condition for the existence of a local folding solution, i.e. a local solution which is a folding, and in Section 3, we check to what extend local folding solutions can be uniformized to a global folding solution.

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2. Existence of local folding solutions

2.1. Theorem. *Let* D, D_1, D_2, E *and* μ *be as in* 1.1. *If there exist* $r \in (0,1)$, *a non-negative integer* m, *a real-valued real analytic function* $\theta(z)$, $|z| < r$ *with* $|\theta(z)| < \frac{1}{2}\pi$ *and a complex-valued real analytic function* $M(x,y)$ *in* $|z| < r$, $z = x + iy$, *with*

$$\operatorname{Re} M(0,0) > 0 \tag{2.2}$$

and such that for $|z| < r$,

$$\mu(z) = e^{2i\theta(z)}[1 - y^{2m+1} M(x,y)], \tag{2.3}$$

then (B) *has a folding solution in some neighborhood of* 0.

Proof. In [SY2, Theorem 1.1] we considered a Beltrami equation (B) with

$$\mu(z) = e^{2i\theta(z)}\big[1 - \varrho(y) M(z)\big] \tag{M}$$

where θ and ϱ are real-valued functions, and M is a complex-valued function which satisfy certain conditions. We then applied a process, which we called a deformation of the complex dilatation, and reduced (B) to a Beltrami equation in the (ξ,η)-plane with complex dilatation

$$\tilde{\mu}(\xi,\eta) = 1 - \varrho(\eta)\widetilde{M}(\xi,\eta)$$

where ϱ is the same as in (M), and $\widetilde{M}$ satisfies a similar condition as M, see [SY2, pp. 475–477]. The same method can be applied here. Thus we will assume that $\theta(z) \equiv 0$. We will also assume that $m = 0$ and that M is a polynomial. The general case is proved in a similar way.

In view of (2.2), M has the form

$$M(x,y) = 2a + 2ib + \sum_{n+k=1}^{N} q_{nk} z^n \bar{z}^k \tag{2.4}$$

where $N \geq 1$, a and b are real and $a > 0$.

As in [L, pp. 23–24] we set in (B)

$$w = f(z) = z + c_{01}\bar{z} + \sum_{n+k=2}^{\infty} c_{nk} z^n \bar{z}^k \tag{2.5}$$

with $c_{n0} = 0$ for all $n > 1$. With application of (2.3), one obtains all c_{nk} (uniquely). Again, exactly as in [L], one can show that the series in (2.5) converges uniformly in $U_0 = \{z : |z| < r_0\}$ for some $r_0 \in (0,r)$.

It remains to show that f is a folding in some neighborhood U of 0 which is contained in U_0. Let $f = u + iv$, then by (2.5)

$$u = 2x - b(x^2 + y^2) + \sum_{n+k=3}^{\infty} \alpha_{nk} x^n y^k \tag{2.6}$$

and

$$v = a(x^2 + y^2) + \sum_{n+k=3}^{\infty} \beta_{nk} x^n y^k \tag{2.7}$$

for some α_{nk} and β_{nk} which depend on the c_{nk}'s.

Next, define $\varphi\colon U_0 \to \mathbf{C}$ by letting

$$\varphi(x, y) = \bigl(u(x, y), y\bigr) \tag{2.8}$$

with u as in (2.6). Then φ has a non-zero Jacobian at 0, and hence it is a diffeomorphism in some neighborhood $U_1 \subset U_0$ of 0. Consequently, there exists a mapping $F\colon \varphi(U_1) \to \mathbf{C}$ such that $f = F \circ \varphi$. Since $\varphi \mid U_1$ is injective, it sufficies to show that F is a folding. Now, $(\varphi \mid U_1)^{-1}$ has the form

$$\varphi^{-1}(u, y) = \bigl(x(u, y), y\bigr), \tag{2.9}$$

hence,

$$F(u, y) = \bigl(u, \tilde{v}(u, y)\bigr), \tag{2.10}$$

where, by (2.7),

$$\tilde{v}(u, y) = v\bigl(x(u, y), y\bigr) = a\bigl[x^2(u, y) + y^2\bigr] + \sum_{n+k=3}^{\infty} \beta_{nk} x^n(u, y) y^k, \tag{2.11}$$

or

$$\tilde{v}(u, y) = A_0(u) + A_1(u)y + A_2(u)y^2 + \sum_{k=3}^{\infty} A_k(u) y^k \tag{2.12}$$

for some functions $A_k(u)$.

We now show that $A_1(u) \equiv 0$ and $A_2(u) \neq 0$ in some neighborhood of 0. This will imply that F is a folding in some neighborhood of 0. Indeed, in view of (2.3), the Jacobian of f vanishes at all points $(x, 0)$. Hence for $y = 0$,

$$A_1(u) = \frac{\partial \tilde{v}(u, y)}{\partial y}\Big|_{y=0} = J_F|_{y=0} = J_{f\circ\varphi^{-1}}|_{y=0} = 0.$$

By (2.11) and (2.12)

$$2A_2(0) = \frac{\partial^2 \tilde{v}(0,0)}{\partial y^2} = 2a > 0,$$

and by continuity, $A_2(u) > 0$ in $|u| < \delta$ for some $\delta > 0$.

Finally, since $A_2(u) > 0$ and $A_1(u) \equiv 0$ for $|u| < \delta$ we can get from (2.12)

$$F(u,y) = \big(u, \tilde{v}(u,y)\big) = \big(u, A_0(u) + w^2(u,y)\big),$$

where

$$w(u,y) = A_2^{1/2}(u)\left(y + \sum_{k=2}^{\infty} b_k(u) y^k\right)$$

for some functions $b_k(u)$. This shows that F is a local folding in some neighborhood of 0, and hence so is f.

3. Uniformization

3.1. Let D, D_1, D_2, E and μ be as in 1.1. The main question addressed in this section is whether the existence of local folding solutions at every point of E implies the existence of a global folding solution in D.

In the proof of the following theorem, we will use the fact that if $f\colon U \to \mathbf{C}$ is a folding solution such that $U \cap E$ is connected, then $U \cap E$ has a simply connected neighborhood V, $V \subset U$ such that $V \cap E$ is connected, and $f \mid V$ is a proper folding solution, see [SY1, Lemma 2.2].

3.2. Theorem. *Let D, D_1, D_2, E and μ be as in* 1.1, *and suppose that every point x in E has a neighborhood U_x, $U_x \subset D$, where* (B) *has a folding solution. Then E has a neighborhood V, $V \subset \bigcup_{x\in E} U_x$, such that*

(i) (B) *has a proper folding solution in V.*

(ii) (B) *has a folding solution g in $D_1 \cup V$ and a folding solution h in $D_2 \cup V$.*

Proof. We first prove (ii). Let $x \in E$. Then x has a simply connected neighborhood V_x, $V_x \subset U_x$ such that $V_x \cap E$ is connected, and such that (B) has a proper folding solution $f_x\colon V_x \to \mathbf{C}$. Let $V = \bigcup_{x\in E} V_x$. Since μ is locally bounded in D_1, (B) has a homeomorphic solution g_1 in D_1, cf. [B]. We may assume that $g_1(D_1) = D_1$. We will show that g_1 has a homeomorphic extension, denoted again by g_1, on $D_1 \cup E$. Indeed, given $x \in E$, f_x maps $V_x \setminus D_2$ homeomorphically onto $f_x(V_x)$, and thus, every subarc of $f_x(V_x \cap E)$ is a free boundary arc of $f_x(V_x \cap D_1)$. Let $\hat{f}_x = f \mid V_x \cap D_1$. Then, $g_1 \circ \hat{f}_x^{-1}$ is conformal, and hence has a homeomorphic extension on $f_x(V_x \setminus D_2)$. Therefore, g_1 has a homeomorphic extension on $D_1 \cup (E \cap V_x)$. Consequently, g_1 has a homeomorphic extension on $D_1 \cup E$. We may assume that $g_1(E) = E$.

We now define g in $D_1 \cup V$ as follows. If $z \in D_1 \cup E$, we set $g(z) = g_1(z)$. Given x in E, we define g in $V_x \cap D_2$ by letting $g(z) = g_1 \circ \hat{f}_x^{-1} \circ f_x(z)$. The mapping g is well defined in $V \cap D_2$, since by the uniqueness theorem for proper folding solutions [SY1, Theorem 3.2], once two points have the same image by a proper folding solution, they will have the same image under any other folding solution. Clearly, g is a folding solution in $D_1 \cup V$.

The mapping h for (ii) is constructed in the same way. As for (i), note that $g \mid V$, with g and V as above, is a proper folding solution of (B) in V, as needed.

In spite of Theorem 3.2 and contrary to the uniformization theorem, we have the following two theorems.

3.3. Theorem. *Let D, D_1, D_2, E be as in* 1.1. *Given a point $z_0 \in D \setminus E$, there exists μ as in* 1.1 *such that*

(i) (B) *has a local folding solution at every point of E.*

(ii) (B) *has a global solution in D.*

(iii) *Every global solution of* (B) *in D branches at z_0, and in particular,* (B) *has no (global) folding solution in D.*

Proof. Let $T(z) = i(z - \frac{1}{4})(1 - \frac{1}{4}z)$ and $\psi(z) = z^2$. Let E' be the connected component of $\psi^{-1}\big(T^{-1}(E)\big)$, which contains the point $\frac{1}{2}$. Let φ be a diffeomorphism of D onto itself such that $\varphi(z_0) = 0$ and $\varphi(E) = E'$, and let $F(x,y) = (x, |y|)$. Set $f = F \circ T \circ \psi \circ \varphi$ and $\mu = \mu_f$. Then μ satisfies the conditions of 1.1, f is a solution of (B) in D, and every point x of E has a neighborhood U_x where $f \mid U_x$ is a local proper folding solution of (B). Note that f branches at z_0 with local index $i(z_0, f) = 2$.

We now show that every other global solution branches at z_0. Suppose that g is a solution in D. We may assume that $\operatorname{Im} z_0 < 0$. The case $\operatorname{Im} z_0 > 0$ is similar. Let $G_1 = D_1$, let G_2 be the subdomain of D which lies between E and $\varphi^{-1}(L)$ where L is the intersection of D with the imaginary axes, and let $G_3 = D_2 \setminus \overline{G_2}$. Next for $i = 1,2,3$, let $f_i = f \mid G_i$ and $h_i = g \circ f_i^{-1}$. Then h_1 and h_3 are defined in D_1, and h_2 is defined in $D_1 \setminus F \circ T \circ \psi(L)$. Since g and all f_i are $\mathrm{W}^{1,2}_{\mathrm{loc}}$ solutions of (B) off E it follows that each h_i is analytic. Also,

$$g = h_i \circ f_i. \tag{3.4}$$

The mappings f_1 and f_2 extend homeomorphically on $G_1 \cup E$ and $G_2 \cup E$, respectively, and they coincide on E. Therefore, h_1 and h_2 have the same boundary values on E, and hence $h_1 = h_2$ in $D_1 \setminus L'$, where $L' = T \circ \psi(L)$. However, h_1 is analytic in D_1, therefore h_2 can be extended analytically to D_1, and $h_1 = h_2$ in D_1.

Finally, f_2 and f_3 coincide on $\varphi^{-1}(L)$, therefore h_2 and h_3 have the same boundary values on L' and therefore $h_2 = h_3$ in $D_1 \setminus L'$, and hence in D_1, when continued analytically across L'.

We obtain that $h_1 = h_2 = h_3 := h$ is analytic in D_1, and has a continuous extension on $D_1 \cup E$, and in view of (3.4), satisfy $g = h \circ f$. Therefore g branches at z_0 and consequently there is no global folding solution in D.

In view of the last theorem, one may ask: Suppose that (B) has a global solution in D which folds along E, and which is locally injective off E. Does it follow that (B) has a global folding solution in D? The following example answers this question negatively.

3.5. Theorem. *Given* D, D_1, D_2, E *as in* 1.1, *there exists* μ *satisfying the conditions of* 1.1 *such that*

(i) (B) *has a global solution in* D, *which folds along* E *and is locally injective off* E.

(ii) (B) *has no global folding solution in* D.

Proof. For $k = 0, 1, \dots, 8$, let $z_k = e^{ik\pi/8}$, and let I_k denote the subarc of ∂D_1 having end points z_{k-1} and z_k. Let Q' denote the square having vertices at the points $\frac{1}{4}i$, $\frac{3}{4}i$, $\frac{3}{4}i - \frac{1}{2}$ and $\frac{1}{4}i - \frac{1}{2}$.

Let f be a mapping of $\overline{D}$ onto $\overline{D_1}$ which has the following properties:

(a) $f(z) = \bar{z}$ for $\operatorname{Im} z \le 0$.

(b) $f(z_0) = 1$, $f(z_1) = i$, $f(z_2) = \frac{3}{4}i$, $f(z_3) = \frac{3}{4}i - \frac{1}{2}$, $f(z_4) = \frac{1}{4}i - \frac{1}{2}$, $f(z_5) = \frac{1}{4}i$, $f(z_6) = \frac{1}{8}i$, $f(z_7) = i$ and $f(z_8) = -1$.

(c) f is a local diffeomorphism in D_1, injective on each arc I_k, and maps I_1 and I_8 into $|z| = 1$ and each other arc I_k into a line segment.

(d) $f(D_1) = D_1 \setminus \bigl(\bigl[\frac{1}{8}i, \frac{1}{4}i\bigr] \cup \bigl[\frac{1}{2}i, i\bigr]\bigr)$, where $f \mid D_1$ covers every point of Q' twice, and every point in $D_1 \setminus \bigl(\bigl[\frac{1}{8}i, i\bigr] \cup Q'\bigr)$ once.

Then $\mu = \mu_f$ satisfies the conditions of 1.1, and f is a solution of (B) which folds along E and is a local homeomorphism in $D \setminus E$. Let Q denote the connected component of $f^{-1}(Q')$ whose boundary meets I_4, and let $D_0 = D \setminus \bigl(Q \cup \bigl[-i, -\frac{1}{8}i\bigr]\bigr)$. Then $f_0 = f \mid D_0$ is a proper folding solution in D_0 which maps D_0 onto $E \cup D_1 \setminus \bigl[\frac{1}{8}i, i\bigr]$.

Suppose now that (B) has a global folding solution in D. Then, by the uniqueness theorem for proper folding solutions [SY1, Theorem 3.2], $g \mid D_0 = h \circ f_0$, for some mapping h which is conformal in $\operatorname{int} f(D_0) = D_1 \setminus \bigl[\frac{1}{8}i, i\bigr] := D_1'$. However, g is homeomorphic in D_2, and $g = h \circ f$, hence h has a homeomorphic extension on D_1, denoted again by h.

Now $f^{-1}(\frac{1}{2}i)$ consists of three points $b_1 \in D_1$, $b_2 \in D_2$ and $b_3 \in \partial D_1$, or more precisely $b_3 \in I_7$, and thus a small neighborhood V of b_3 relative D_1 is mapped by f into Q'. It follows that $g^{-1}(w)$ has two pre-image points in D_1, whenever $w \in g(V) = h\bigl(f(V)\bigr)$, contradicting the assumption that g is a folding.

3.6. Remark. In [SY3] we showed that the existence of local folding solution at every point of E does not imply the existence of a (global) folding solution in D. There, the proof was based on the fact that $|a_2| \le 2$ for every schlicht function $f(z) = z + a_2 z^2 + \cdots$ in D.

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