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WEAKLY COMPACT COMPOSITION OPERATORS ON ANALYTIC VECTOR-VALUED FUNCTION SPACES

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Abstract. Let X be a Banach space. It is proved that the composition operator on X-valued Hardy spaces, weighted Bergman spaces and Bloch spaces is weakly compact or Rosenthal if and only if both id: $X \to X$ and the corresponding composition operator on scalar valued spaces are weakly compact or Rosenthal, respectively.

1. Introduction

Let $\varphi\colon D \to D$ be an analytic self map of the complex unit disc D. It can be easily proved that if the composition operator $C_\varphi\colon f \mapsto f\circ\varphi$ on vector-valued (i.e. with values in a Banach space X) Hardy, Bergman or Bloch spaces belongs to some operator ideal, then both its scalar version and the identity operator on X belong to the same ideal. For the ideal of weakly compact operators Liu, Saksman and Tylli [LST] proved the converse for vector-valued Hardy spaces $H_1(X)$, Bergman spaces $B_1(X)$ and $B_\infty(X) = H^\infty(X)$ as well as for Bloch spaces using analytic methods.

If a vector-valued space of analytic functions $E[X]$ can be represented as the space $L({}^*E, X)$ of all linear bounded operators from the predual of the scalar version of $E[X]$ into X, then we give a very simple functional analytic argument which replaces the more analytic ones in [LST]. In this way we obtain the results for Bloch spaces and extend the results of [LST] to weighted Bergman spaces of infinite order $B^v_\infty(X)$. In that part of the paper our main idea is to use the following result due to Saksman and Tylli in [ST], see also [R], [LS]. *Let E, F,*

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E_1, F_1 be Banach spaces and let $R \in \mathscr{L}(E,F)$ and $B \in \mathscr{L}(E_1,F_1)$ be two weakly compact operators. If B or R is compact, then the map $T \mapsto R \circ T \circ B$ from $\mathscr{L}(F_1,E)$ into $\mathscr{L}(E_1,F)$ is weakly compact.

Unfortunately the operator representation mentioned above does not hold in general, for instance, for Hardy spaces $H_1(X)$ or Bergman spaces $B_1(X)$. Thus the main part of the paper is devoted to that case. We are able to extend the methods and the results of [LST] to the classical weighted Bergman spaces $B_1^\alpha(X)$, $\alpha \geq -1$, a class which includes both $H_1(X)$ and $B_1(X)$. An essential improvement is done in a formula derived from the so-called Stanton formula (see Lemma 3).

Let us observe that for $1 < p < \infty$ the weighted Bergman space $B_p^v(X)$ and $H_p(X)$ are reflexive whenever X is reflexive. Thus C_φ on these spaces is automatically weakly compact if and only if X is reflexive.

Further, we prove a characterization of compact composition operators on the Bloch space. The sets of interpolation for the Bloch space [Ro] play a crucial role in the proof.

2. Preliminaries

We denote by $H(D,X)$ the space of holomorphic functions from the unit disc D into a Banach space X. As usual $H_p(X)$ stands for the Hardy space of X-valued functions in $H(D,X)$ such that

$$\|f\|^p_{H_p(X)} := \sup_{0\leq r<1} \frac{1}{2\pi}\int_0^{2\pi} \|f(re^{i\theta})\|_X^p \, d\theta < \infty \qquad \text{for } p<\infty,$$
$$\|f\|_{H^\infty(X)} := \sup_{z\in D} \|f(z)\|_X < \infty \qquad \text{for } p=\infty.$$

Let $v\colon D \to \mathbf{R}_+$ be an arbitrary *weight*, i.e., bounded continuous positive (which means *strictly positive* throughout the paper) function. We define the weighted Bergman space $B_p^v(X)$ as the space of those functions $f \in H(D,X)$ with

$$\|f\|^p_{B_p^v(X)} := \frac{1}{\pi}\int_D \|f(z)\|_X^p v(z)\, dA(z) < \infty \qquad \text{for } p<\infty,$$
$$\|f\|_{B_\infty^v(X)} := \sup_{z\in D} \|f(z)\|_X v(z) < \infty \qquad \text{for } p=\infty,$$

where dA denotes the Lebesque area measure on the plane. If $v(z) = (1-|z|^2)^\alpha$, $\alpha > -1$, then we write $B_p^\alpha(X)$ and if $\alpha = 0$ we just omit α. If $X = \mathbf{C}$, then we omit X in the notation. For the definition of B_p^α cf. [CM]. The Bergman spaces B_∞^v appear naturally in the study of growth conditions on analytic functions and in the scalar-case have been considered in many papers, see for example, [BBG], [BBT], [BS], [BDL], [BDLT], [SW1], [SW2].

We denote by $\mathscr{B}(X)$ the X-valued Bloch space of analytic functions $f\colon D \to X$ with the norm

$$\|f\|_{\mathscr{B}(X)} = \|f(0)\|_X + \sup_{z\in D}(1-|z|^2)\|f'(z)\|_X < \infty.$$

In [CH] the composition operators on the Bloch space are treated as weighted composition operators on B^v_∞ spaces.

A map $T \in \mathscr{L}(X)$ from the Banach space X into X is called compact, weakly compact, Rosenthal, if it maps the closed unit ball of X onto a relatively compact, a relatively weakly compact, a conditionally weakly compact set in X. A subset A in X is called conditionally weakly compact, if every sequence in A admits a weak Cauchy subsequence. Clearly every weakly compact operator is Rosenthal. Rosenthal's l_1 theorem implies that $T\colon X \to Y$ is Rosenthal if and only if T is not an isomorphism on any copy of l_1 in X.

When we write $f \sim g$ for two functions f and g we mean there are strictly positive constants a, b such that $af \le g \le bf$ for all the values of the variable.

For the sake of completeness we give a general argument why the considered conditions are necessary for C_φ to belong to the considered ideals.

Proposition 1. *If $\mathscr{J}$ is an operator ideal and $C_\varphi\colon E(X) \to E(X)$ belongs to $\mathscr{J}$ whenever $E(X)$ is one of the spaces of vector-valued analytic functions $B^v_p(X)$, $H_p(X)$, $\mathscr{B}(X)$, then both* id: $X \to X$ *and $C_\varphi\colon E \to E$, E the scalar version of the space, belongs to $\mathscr{J}$.*

Proof. Let $0 \ne x_0 \in X$, $l_0 \in X^*$ with $l_0(x_0) = 1$ and $z_0 \in D$. We define the operators

$$\begin{aligned}
&\gamma\colon E \to E(X), && \gamma(f)(z) = f(z)x_0;\\
&\eta\colon E(X) \to E, && \eta(f) = l_0 \circ f;\\
&p\colon X \to E(X), && p(x)(z) = x;\\
&r\colon E(X) \to X, && r(f) = f(z_0).
\end{aligned}$$

All these operators are continuous, $\mathrm{id}_X = r \circ C_\varphi \circ p$ and $\eta \circ C_\varphi \circ \gamma$ is exactly the scalar composition operator on E. □

3. Consequences of the Stanton formula

In [Sh1] Shapiro introduced the generalized Nevanlinna counting function $N_{\varphi,\alpha}$ for $\alpha > 0$. It is defined by

$$N_{\varphi,\alpha}(w) = \sum_{z\in\varphi^{-1}(w)} \left(\log\left(\frac{1}{|z|}\right)\right)^\alpha, \qquad w \in D \setminus \{\varphi(0)\}.$$

For our purpose it is convenient to introduce the modified Nevanlinna counting function

$$\widetilde{N}_{\varphi,\alpha}(w) = \sum_{z\in\varphi^{-1}(w)} (1-|z|^2)^{\alpha-1}\log\left(\frac{1}{|z|}\right), \qquad w\in D\setminus\{\varphi(0)\}.$$

The standard Nevanlinna counting function is $N_\varphi = N_{\varphi,1} = \widetilde{N}_{\varphi,1}$ and the partial Nevanlinna counting function of φ is defined for $0<r<1$ by

$$N_\varphi(r,w) = \sum_{z\in\varphi^{-1}(w),\,|z|\le r} \log\left(\frac{r}{|z|}\right), \qquad w\in D\setminus\{\varphi(0)\}.$$

The following formula for a continuous subharmonic function u is due to Stanton [St, Theorem 2]:

$$\frac{1}{2\pi}\int_0^{2\pi} u\big(\varphi(re^{i\theta})\big)\,d\theta = u(0) + \frac{1}{2\pi}\int_D N_\varphi(r,w)\,d[\Delta(u)](w),$$

where $r\in(0,1)$ and $\varphi\colon D\to D$ is analytic, $\varphi(0)=0$. When $f\in H(D,X)$, $d[\Delta\|f\|_X](w)$ denotes integration with respect to the distributional Laplacian of $\|f\|_X$, which is a positive measure on D since the map $z\mapsto\|f(z)\|_X$ is subharmonic. This means that for every test function (infinitely differentiable function on $\mathbf{C}$ with compact support) τ we have

$$\int \tau(w)\,d[\Delta\|f\|_X](w) = \frac{1}{2\pi}\int \|f(w)\|_X\Delta\tau(w)\,dA(w).$$

The Stanton formula was applied to composition operators first by Shapiro [Sh1], see also [SS]. We use it to characterize weakly compact operators with the help of the following lemmas.

Lemma 2 [LST, p. 300–301]. *If $f\colon D\to X$ is analytic, $\varphi(0)=0$ and $0<r<1$, then*

$$\text{(1)}\qquad \frac{1}{2\pi}\int_0^{2\pi}\|f(\varphi(re^{i\theta})\|_X\,d\theta = \|f(0)\|_X + \frac{1}{2\pi}\int_D N_\varphi(r,w)\,d[\Delta(\|f\|_X)](w),$$

$$\text{(2)}\qquad \|C_\varphi(f)\|_{H_1(X)} = \|f(0)\|_X + \frac{1}{2\pi}\int_D N_\varphi(w)\,d[\Delta(\|f\|_X)](w).$$

The next result was proved in [LST] only for $\alpha=0$:

Lemma 3. *If* $f\colon D \to X$ *is analytic,* $\varphi(0) = 0$ *and* $\alpha > -1$, *then*

$$\|C_\varphi(f)\|_{B_1^\alpha(X)} \sim \|f(0)\|_X + \frac{1}{2\pi}\int_D \widetilde{N}_{\varphi,\alpha+2}(w)\, d[\Delta(\|f\|_X)](w). \tag{3}$$

Proof. If $0 < r_0 \le r < 1$, then $\frac{1}{2}(1-r^2) \le \log(1/r) \le C(1-r^2)$ for some C. By partial integration, for $z \in D$ away from the origin, we have

$$\begin{aligned}\int_{|z|}^1 2r(1-r^2)^\alpha \log\left(\frac{r}{|z|}\right) dr &= \int_{|z|}^1 \frac{(1-r^2)^{\alpha+1}}{r(\alpha+1)}\, dr \\ &\sim \int_{|z|}^1 \left(\log\left(\frac{1}{r}\right)\right)^{\alpha+1} \frac{dr}{r} \sim \left(\log\frac{1}{|z|}\right)^{\alpha+2} \\ &\sim (1-|z|^2)^{\alpha+1} \log\left(\frac{1}{|z|}\right).\end{aligned}$$

Further, we have that

$$\lim_{|z|\to 0^+} \frac{\int_{|z|}^1 2r(1-r^2)^\alpha \log(r/|z|)\, dr}{(1-|z|^2)^{\alpha+1}\log(1/|z|)} = \frac{1}{\alpha+1}.$$

Indeed, by partial integration

$$I(|z|) := \int_{|z|}^1 2r(1-r^2)^\alpha \log\left(\frac{r}{|z|}\right) dr = \int_{|z|}^1 \frac{(1-r^2)^{\alpha+1}}{r(\alpha+1)}\, dr.$$

Further, let $J(|z|) := (1-|z|^2)^{\alpha+1}\log(1/|z|)$. Then, by l'Hôpital's rule,

$$\begin{aligned}\lim_{|z|\to 0^+} \frac{I(|z|)}{J(|z|)} &= \lim_{|z|\to 0^+} \frac{I'(|z|)}{J'(|z|)} \\ &= \lim_{|z|\to 0^+} \frac{1}{2(\alpha+1)^2|z|^2\log(1/|z|)(1-|z|^2)^{-1} + \alpha + 1} = \frac{1}{\alpha+1}.\end{aligned}$$

Hence $\int_{|z|}^1 2r(1-r^2)^\alpha \log(r/|z|)\, dr$ and $(1-|z|^2)^{\alpha+1}\log(1/|z|)$ are comparable with uniform constant for all $|z| > 0$. Thus

$$\begin{aligned}\int_0^1 2r(1-r^2)^\alpha N_\varphi(r,w)\, dr &= \sum_{z\in\varphi^{-1}(w)} \int_{|z|}^1 2r(1-r^2)^\alpha \log\left(\frac{r}{|z|}\right) dr \\ &\sim \sum_{z\in\varphi^{-1}(w)} (1-|z|^2)^{\alpha+1}\log\left(\frac{1}{|z|}\right).\end{aligned} \tag{4}$$

Now multiplying (1) by $2r(1-r^2)^\alpha$, integrating with respect to r from 0 to 1 and applying Fubini's theorem, we get

$$\int_D \big\|f\big(\varphi(w)\big)\big\|_X (1-|w|^2)^\alpha\, dA(w)$$
$$\sim \|f(0)\|_X + \frac{1}{2\pi}\int_D \bigg(\int_0^1 N_\varphi(r,w) 2r(1-r^2)^\alpha\, dr\bigg)\, d[\Delta(\|f\|_X)](w)$$

and we conclude from (4). □

For the special case that φ is the identity map we obtain the following formulas:

$$(5)\qquad \frac{1}{2\pi}\int_0^{2\pi} \|f(re^{i\theta})\|_X\, d\theta = \|f(0)\|_X + \frac{1}{2\pi}\int_{rD} \log\bigg(\frac{r}{|w|}\bigg)\, d[\Delta(\|f\|_X)](w),$$

$$(6)\qquad \|f\|_{H_1(X)} = \|f(0)\|_X + \frac{1}{2\pi}\int_D \log\bigg(\frac{1}{|w|}\bigg)\, d[\Delta(\|f\|_X)](w)$$

and

$$(7)\qquad \|f\|_{B_1^\alpha(X)} \sim \|f(0)\|_X + \frac{1}{2\pi}\int_D (1-|w|^2)^{\alpha+1} \log\bigg(\frac{1}{|w|}\bigg)\, d[\Delta(\|f\|_X)](w).$$

The estimates (6) and (7) permit to define $B_1^{-1}(X)$ as $H_1(X)$, and therefore we can consider these Hardy spaces as weighted Bergman spaces.

4. Composition operators on weighted Bergman spaces

First we consider the continuity of C_φ on $B_1^\alpha(X)$. We start with the following result.

Lemma 4. *Let $\alpha \geq -1$. If $z \in D$ and $f \in B_1^\alpha(X)$, then*

$$\|f(z)\|_X \leq C\|f\|_{B_1^\alpha(X)}(1-|z|^2)^{-(\alpha+2)},$$

where C is independent of f.

Proof. By [Sm, Lemma 2.5],

$$\big|l\big(f(z)\big)\big| \leq C\|l\circ f\|_{B_1^\alpha}(1-|z|^2)^{-(\alpha+2)}, \qquad l \in X^*,$$

and C does not depend on l and f. Since $\|l \circ f\|_{B_1^\alpha} \leq \|l\|\,\|f\|_{B_1^\alpha(X)}$ we are done. □

It follows immediately from Lemma 4 that evaluations are continuous on $B_1^\alpha(X)$, the compact open topology is weaker than the norm one and that $B_1^\alpha(X)$ is a Banach space.

Proposition 5. *Let* $\alpha \geq -1$. *The composition operator* $C_\varphi\colon B_1^\alpha(X) \to B_1^\alpha(X)$ *is continuous. In fact, for each* $\alpha \geq -1$ *there exists a constant* $C(\alpha)$ *such that*

$$\|C_\varphi\| \leq C(\alpha)\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\alpha+2}.$$

Proof. The proof is standard but for completeness we include it. The cases $\alpha = 0$ and $\alpha = -1$ are proved in [LST, Proposition 1]. For $a = \varphi(0)$ let $\varphi_a(z) := (a-z)/(1-\overline{a}z)$. Then $\psi := \varphi_a \circ \varphi\colon D \to D$ is analytic, $\psi(0) = 0$ and $\varphi = \varphi_a \circ \psi$.

Since $z \mapsto \|f \circ \varphi_a(z)\|_X$ is subharmonic, Littlewood subordination theorem [CM, p. 30] yields

$$\int_0^{2\pi} \|f \circ \varphi(re^{i\theta})\|_X \, d\theta \leq \int_0^{2\pi} \|f \circ \varphi_a(re^{i\theta})\|_X \, d\theta$$

for all $0 < r < 1$. Therefore,

$$\|C_\varphi f\|_{B_1^\alpha(X)} \leq \frac{1}{\pi}\int_D \|f \circ \varphi_a(z)\|_X (1-|z|^2)^\alpha \, dA(z).$$

By changing the variable in the last integral, we get

$$\begin{aligned}\|C_\varphi f\|_{B_1^\alpha(X)} &\leq \frac{1}{\pi}\int_D \|f(w)\|_X \bigl(1-|\varphi_a(w)|^2\bigr)^\alpha \frac{(1-|a|^2)^2}{|1-\overline{a}w|^4}\, dA(w)\\ &\leq C(\alpha)\left(\frac{1+|a|}{1-|a|}\right)^{\alpha+2}\|f\|_{B_1^\alpha(X)}. \ \square\end{aligned}$$

By [L, Corollary 2.7], it follows for $\alpha > -1$ that the space B_1^α is isomorphic to l_1. Therefore, by the well-known properties of l_1, we have:

Proposition 6. *Let* $\alpha > -1$. *The following statements are equivalent*:

(a) $C_\varphi\colon B_1^\alpha \to B_1^\alpha$ *is non-compact.*
(b) $C_\varphi\colon B_1^\alpha \to B_1^\alpha$ *is non-Rosenthal.*
(c) *There exist continuous linear operators* $S\colon l^1 \to B_1^\alpha$ *and* $T\colon B_1^\alpha \to l^1$ *such that* $T \circ C_\varphi \circ S = \mathrm{id}_{l_1}$.

The proposition above can also be obtained using interpolating sequences in B_1^α (cf. [HRS, Theorem 3.1]) without referring to the isomorphic classification of B_1^α due to Lusky [L].

The following result was proved by Liu, Saksman and Tylli in [LST] for the spaces $H_1(X)$ and $B_1(X)$. We only have to check that the same argument is valid for all spaces $B_1^\alpha(X)$. Let us note that the Banach–Steinhaus theorem cannot be used to obtain (9) for any infinite dimensional Banach space X.

The operators V_k defined in the next proposition are related to de la Vallée–Poussin summability kernels.

Proposition 7. *Let $\alpha \geq -1$, $k \in \mathbf{N}$ and X a Banach space. Define the operator V_k by setting*

$$V_k f(z) = \sum_{n=0}^{k} \hat{f}_n z^n + \sum_{n=k+1}^{2k-1} \frac{2k-n}{k} \hat{f}_n z^n$$

for analytic $f\colon D \to X$ with the Taylor expansion $f = \sum_{n=0}^{\infty} \hat{f}_n z^n$. Then there is $C > 0$ such that

$$\|V_k f\|_{B_1^\alpha(X)} \leq C \|f\|_{B_1^\alpha(X)} \tag{8}$$

for all $f \in B_1^\alpha(X)$. Moreover, given $\varepsilon > 0$ and $r \in (0,1)$ there is $k_0 = k_0(\varepsilon, r) > 0$ such that for $k \geq k_0$

$$\|f(z) - V_k f(z)\|_X \leq \varepsilon \|f\|_{B_1^\alpha(X)} \qquad \text{for all } |z| \leq r \text{ and } f \in B_1^\alpha(X). \tag{9}$$

Further, if X is reflexive, respectively does not contain a copy of l_1, then the operator $V_k\colon B_1^\alpha(X) \to B_1^\alpha(X)$ is weakly compact, respectively Rosenthal.

Proof. By [LST, Proposition 2] we know that (8) and (9) are valid for $H_1(X)$ with $C = 2$. Let $f \in B_1^\alpha(X)$ and $\alpha > -1$. It is easily seen that

$$\|f\|_{B_1^\alpha(X)} = 2 \int_0^1 \|f_r\|_{H_1(X)} r (1-r^2)^\alpha \, dr, \tag{10}$$

where $g_s(z) = g(sz)$ for $0 < s < 1$. Thus (8) is a direct consequence of the corresponding result for $H_1(X)$ and the relation $V_k f_r = (V_k f)_r$.

For completeness we give the argument from [LST] to obtain (9). Assume that $r \in (\frac{1}{2}, 1)$ and $\varepsilon > 0$ are given. Let $f \in B_1^\alpha(X)$ with $\|f\|_{B_1^\alpha(X)} \leq 1$. It follows from (10) that there exist a radius $r' \in (\sqrt{r}, 1)$ and a constant C with $\|f_{r'}\|_{H_1(X)} \leq C(\alpha+1)(1-\sqrt{r})^{-(\alpha+1)}$. Further we can choose k_0 such that for $k \geq k_0$ we have $\|g(z) - V_k g(z)\|_X \leq \varepsilon(\alpha+1)^{-1}(1-\sqrt{r})^{\alpha+1} C^{-1} \|g\|_{H_1(X)}$ for $|z| \leq \sqrt{r}$ and all $g \in H_1(X)$. Thus, for $|z| \leq r$ we have that $|z/r'| \leq \sqrt{r}$, so we get

$$\|f(z) - V_k f(z)\|_X = \left\| f_{r'}\left(\frac{z}{r'}\right) - V_k f_{r'}\left(\frac{z}{r'}\right) \right\|_X \leq \varepsilon.$$

The final statement follows exactly as in [LST]. □

Let $\mathscr{U}$ be a closed ideal of operators between Banach spaces. For $T \in \mathscr{L}(X)$ define $\|T\|_{\mathscr{U}} = \inf\{\|T - S\| : S \in \mathscr{U}\}$. Let $\mathscr{W}$ and $\mathscr{R}$ be the closed ideal of weakly compact respectively Rosenthal operators on X.

Theorem 8. *Let X be reflexive, respectively a Banach space not containing a copy of l_1. For each $\alpha \geq -1$ there exists a constant $C(\alpha)$ such that for C_φ acting on $B_1^\alpha(X)$ we have*

$$\|C_\varphi\|_{\mathscr{U}} \leq C(\alpha) \limsup_{|w|\to 1} \frac{N_{\varphi,\alpha+2}(w)}{(-\log|w|)^{\alpha+2}},$$

where $\mathscr{U}$ is $\mathscr{W}$ respectively $\mathscr{R}$.

Proof. Let $f \in B_1^\alpha(X)$ be arbitrary and fix an arbitrary $r \in (0,1)$. Without loss of generality, we may assume that $\varphi(0) = 0$. We have that $\|f(0) - V_k f(0)\|_X = 0$. By (2) and (3) we get

$$\begin{aligned}\|C_\varphi(f - V_k f)\|_{B_1^\alpha(X)} &\sim \frac{1}{2\pi}\int_{rD} \widetilde{N}_{\varphi,\alpha+2}(w)\, d[\Delta(\|f - V_k f\|_X)](w)\\ &\quad + \frac{1}{2\pi}\int_{D\setminus rD} \widetilde{N}_{\varphi,\alpha+2}(w)\, d[\Delta(\|f - V_k f\|_X)](w)\\ &:= I_{r,k} + J_{r,k}.\end{aligned}$$

To estimate the first term $I_{r,k}$ observe that by [Sh2, Corollary 10.4(b)], $N_\varphi(w) \leq \log(1/|w|)$ for each $w \in D$. Hence for all $w \in D$

$$\widetilde{N}_{\varphi,\alpha+2}(w) \leq N_\varphi(w) \leq \log\left(\frac{1}{|w|}\right).$$

Therefore we get,

$$\begin{aligned}I_{r,k} &\leq \frac{1}{2\pi}\int_{rD} \log\left(\frac{r}{|w|}\right) d[\Delta(\|f - V_k f\|_X)](w)\\ &\quad + \frac{1}{2\pi}\log\left(\frac{1}{r}\right)\int_{rD} d[\Delta(\|f - V_k f\|_X)](w).\end{aligned}$$

Hence by (6),

$$\begin{aligned}&\frac{1}{2\pi}\int_{rD} \log\left(\frac{r}{|w|}\right) d[\Delta(\|f - V_k f\|_X)](w)\\ &\qquad = \frac{1}{2\pi}\int_{D} \log\left(\frac{1}{|w'|}\right) d[\Delta(\|(f - V_k f)_r\|_X)](w') = \|(f - V_k f)_r\|_{H_1(X)}.\end{aligned}$$

Further,

$$\|(f - V_k f)_r\|_{H_1(X)} = \frac{1}{2\pi}\int_0^{2\pi} \|(f - V_k f)_r(e^{i\theta})\|_X\, d\theta \leq \sup_{|w|=r} \|f(w) - V_k f(w)\|_X.$$

Let now τ be a test function on the plane with $0 \le \tau \le 1$, the support of τ is contained in $\frac{1}{2}(r+1)D$ and $\tau \equiv 1$ on rD. Then

$$\begin{aligned}\int_{rD} d[\Delta(\|f - V_k f\|_X)](w) &\le \int \tau(w)\, d[\Delta(\|f - V_k f\|_X)](w) \\ &= \frac{1}{2\pi} \int_{(r+1)D/2} \|f(w) - V_k f(w)\|_X \Delta\tau(w)\, dA(w) \\ &\le M \int_{(r+1)D/2} \|f(w) - V_k f(w)\|_X\, dA(w),\end{aligned}$$

where $M := (1/2\pi)\max\{|\Delta\tau(w)| : w \in \mathbf{C}\}$ is finite. By Proposition 7, we get for every $r \in (0,1)$ that

$$\lim_{k\to\infty} \sup_{\|f\|_{B_1^\alpha(X)}\le 1} I_{r,k} = 0.$$

For the second term $J_{r,k}$ we first notice that $\widetilde{N}_{\varphi,\alpha+2}(w) \le 2^{\alpha+1} N_{\varphi,\alpha+2}(w)$ for all $w \in D$. Therefore

$$J_{r,k} \le \sup_{w\in D\setminus rD} \left(\frac{N_{\varphi,\alpha+2}(w)}{(-\log|w|)^{\alpha+2}}\right) \frac{2^{\alpha+1}}{2\pi} \int_{D\setminus rD} \left(\log\left(\frac{1}{|w|}\right)\right)^{\alpha+2} d[\Delta(\|f - V_k f\|_X)](w).$$

Since $\log(1/|w|)$ and $1-|w|^2$ are comparable for all $w \in D\setminus rD$, there is $M(\alpha, r)$ such that by (6), (7) and (8),

$$\begin{aligned}J_{r,k} &\le M(\alpha,r) \sup_{w\in D\setminus rD} \frac{N_{\varphi,\alpha+2}(w)}{(-\log|w|)^{\alpha+2}} \|f - V_k f\|_{B_1^\alpha(X)} \\ &\le C M(\alpha,r) \sup_{w\in D\setminus rD} \frac{N_{\varphi,\alpha+2}(w)}{(-\log|w|)^{\alpha+2}} \|f\|_{B_1^\alpha(X)}.\end{aligned}$$

Consequently,

$$\begin{aligned}\|C_\varphi\|_{\mathscr{U}} &\le C(\alpha)\Big\{\lim_{k\to\infty} \sup_{\|f\|_{B_1^\alpha(X)}\le 1} I_{k,r} + \lim_{r\to 1} \sup_{\|f\|_{B_1^\alpha(X)}\le 1} J_{r,k}\Big\} \\ &\le C(\alpha) \limsup_{|w|\to 1} \frac{N_{\varphi,\alpha+2}(w)}{(-\log|w|)^{\alpha+2}}. \ \square\end{aligned}$$

Corollary 9. *Let $\alpha \ge -1$. Then $C_\varphi\colon B_1^\alpha(X) \to B_1^\alpha(X)$ is weakly compact, respectively Rosenthal, if and only if X is reflexive, respectively does not contain a copy of l_1, and*

$$\limsup_{|w|\to 1} \frac{N_{\varphi,\alpha+2}(w)}{(-\log|w|)^{\alpha+2}} = 0. \tag{11}$$

Proof. One direction follows from Proposition 1. Indeed, by Proposition 6 and Sarason [S] (cf. also [J]) for $\alpha = -1$, every Rosenthal operator C_φ on B_1^α is compact. By [CM, Example 3.2.6, Theorem 3.12] compactness of C_φ on B_p^α is independent of $0 < p < \infty$ for $\alpha \geq -1$. Thus with $p = 2$ [Sh1, Theorems 6.8 and 2.3] give that C_φ on B_1^α is compact if and only if condition (11) is valid.

The converse statement follows directly from Theorem 8. □

5. Composition operators on general vector-valued spaces

In this section E denotes a Banach space of analytic functions on the unit disc D which contains the constant functions and such that its closed unit ball $U(E)$ is compact for the compact open topology co. These assumptions imply the following properties of the space E which will be frequently used later.

(a) For every $z \in D$ the evaluation map $\delta_z \colon E \to \mathbf{C}$, $\delta_z(f) = f(z)$, is continuous and non-zero.

(b) The map $\Delta \colon D \to E^*$, $\Delta(z) = \delta_z$, $z \in D$, is a vector valued analytic function. Indeed, since E is a separating subset of the dual E^{**} of E^*, we can apply a result of Grosse-Erdmann [GE, Theorem 5.2] which ensures it is enough to check $f \circ \Delta \in H(D)$ for every $f \in E$. This is trivially satisfied.

(c) By the Dixmier–Ng theorem [N], the space

$$^*E := \{u \in E^* : u \mid U(E) \text{ is } co\text{-continuous}\},$$

endowed with the norm induced by E^*, is a Banach space and the evaluation map $E \to (^*E)^*$, $f \mapsto [u \mapsto u(f)]$ is an isometric isomorphism. In particular $^*E$ is a predual of E.

(d) The linear span of the set $\{\delta_z : z \in D\}$ is contained and norm dense in $^*E$. This follows easily from the Hahn–Banach theorem: if $f \in E = (^*E)^*$ vanishes on all the evaluation maps it must be zero.

Let X be a Banach space. The vector valued space $E[X]$ associated with E is defined as

$$E[X] := \{f \in H(D, X) : x^* \circ f \in E \text{ for every } x^* \in X^*\}.$$

Given $f \in E[X]$, the map $T_f \colon X^* \to E$, $T_f(x^*) = x^* \circ f$, is well defined, linear and weak*-pointwise continuous. By the closed graph theorem T_f is continuous and the supremum $\|f\|_{E[X]} := \sup_{\|x^*\| \leq 1} \|x^* \circ f\|_E$ is finite. We endow $E[X]$ with this norm. Observe that the map $\Delta \colon D \to {}^*E$ defined in (b) above (also see (d)) belongs to $E[^*E]$ and $\|\Delta\|_{E[^*E]} = 1$.

A version of the following linearization result for $E = H^\infty$ can be found in [M] and for $E = B_\infty^v$ in [BBG].

Lemma 10. *The space $E[X]$ is isomorphic to the space of operators $L(^*E, X)$ in a canonical way. In particular, it is a Banach space.*

Proof. First we define $\chi\colon L({}^*E,X)\to E[X]$ by $\chi(T):=T\circ\Delta$. The map χ is well defined, linear, continuous and its norm is less than or equal to 1.

Fix $g\in E[X]$ and $u\in{}^*E$ and define $\psi(g)(u):X^*\to\mathbf{C}$ by $\big(\psi(g)(u)\big)(x^*):=u(x^*\circ g)$ for $x^*\in X^*$. Clearly

$$\big|\big(\psi(g)(u)\big)(x^*)\big|\le\|u\|_{{}^*E}\|x^*\circ g\|_E\le\|u\|_{{}^*E}\|x^*\|_{X^*}\|g\|_{E[X]},$$

for all $x^*\in X^*$, by the definition of the norm in $E[X]$. This yields $\psi(g)(u)\in X^{**}$ and $\psi(g)\in L({}^*E,X^{**})$ with $\|\psi(g)\|\le\|g\|_{E[X]}$. On the other hand $\psi(g)(\delta_z)=g(z)\in X$ for all $z\in D$. By the property (d) above we conclude $\psi(g)\in L({}^*E,X)$, and the map $\psi\colon E[X]\to L({}^*E,X)$ is well defined, linear continuous and its norm is less than or equal to 1.

To complete the proof it is enough to observe that $\psi\circ\chi$ and $\chi\circ\psi$ coincide with the identities on $L({}^*E,X)$ and $E[X]$ respectively. ▫

Let $\varphi\colon D\to D$ be holomorphic. The closed graph theorem and the argument in Proposition 1 imply that the composition operator $C_\varphi\colon E[X]\to E[X]$ is continuous if and only if $C_\varphi\colon E\to E$ is continuous. Moreover the result stated in Proposition 1 remains valid for the spaces of type $E[X]$. In order to obtain a converse we proceed as follows. Assume C_φ is continuous on E. The transpose map $C'_\varphi\colon E^*\to E^*$ maps ${}^*E$ into itself; indeed, by the property (d) above it is enough to check that $C'_\varphi(\delta_z)=\delta_{\varphi(z)}$ belongs to ${}^*E$ for all $z\in D$ which is trivial. Now the isomorphism proved in Lemma 10 transforms the operator C_φ on $E[X]$ into the wedge operator $W_\varphi\colon L({}^*E,X)\to L({}^*E,X)$, $W_\varphi(T)=\mathrm{id}_X\circ T\circ(C'_\varphi|{}^*E)$. More precisely, with the notations introduced in the proof of Lemma 10, $(\psi\circ C_\varphi\circ\chi)(S)=S\circ(C'_\varphi|{}^*E)$ for every $S\in L({}^*E,X)$ which implies $C_\varphi=\chi\circ W_\varphi\circ\psi$. We are ready to prove the main results in this section.

Proposition 11. *Let $C_\varphi\colon E\to E$ be compact and let X be a Banach space.*

(1) *If X is reflexive, then $C_\varphi\colon E[X]\to E[X]$ is weakly compact.*

(2) *If X does not contain a copy of l_1, then $C_\varphi\colon E[X]\to E[X]$ is a Rosenthal operator.*

Proof. Since $C'_\varphi|{}^*E$ is a compact operator on ${}^*E$, we can apply [ST, Theorem 2.9] for part (1) and [LS, Corollary 2.13] for part (2) to the wedge operator W_φ to reach the conclusion. ▫

Corollary 12 [LST, Theorem 4]. *Let $\varphi\colon D\to D$ be holomorphic and let X be a Banach space. The operator C_φ on the Bloch space $\mathscr{B}(X)$ is weakly compact (respectively Rosenthal) if and only if C_φ is Rosenthal on $\mathscr{B}$ and X is reflexive (respectively X does not contain a copy of l_1).*

Proof. First observe that the Bloch space $\mathscr{B}$ satisfies the assumptions we impose on the general space E considered in this section. In fact, if $f\in\mathscr{B}$, it follows by integration that

$$\max_{|z|\le r}|f(z)|\le\left\{1+\frac{1}{2}\log\left(\frac{1+r}{1-r}\right)\right\}\|f\|_{\mathscr{B}}\qquad(0\le r<1).$$

Therefore, every bounded set in $\mathscr{B}$ is relatively compact with respect to the compact-open topology and point evaluations are bounded linear functionals on $\mathscr{B}$. To see that the closed unit ball $U(\mathscr{B})$ of $\mathscr{B}$ is a compact subset of $(\mathscr{B}, co)$ it is enough to observe that $U(\mathscr{B})$ is a normal family by Montel's theorem. If $f_n \to f$ with respect to the co-topology and $\|f_n\|_{\mathscr{B}} \le 1$ for all n, then also $f_n' \to f'$ in the co-topology and consequently $\|f\|_{\mathscr{B}} \le 1$.

It is now easy to see that the vector valued Bloch space $\mathscr{B}(X)$ coincides with the space $\mathscr{B}[X]$ defined in this section and that

$$\|f\|_{\mathscr{B}[X]} \le \|f\|_{\mathscr{B}(X)} \le 2\|f\|_{\mathscr{B}[X]}$$

for every $f \in \mathscr{B}[X]$.

By Proposition 11, it remains to show that every Rosenthal composition operator on $\mathscr{B}$ is compact. This is proved below. □

A sequence $(z_n) \subset D$ is called *δ-separated* if $\inf_{n\ne k} |(z_n - z_k)/(1 - \overline{z}_k z_n)| > \delta > 0$.

Proposition 13. *There is a constant $\delta > 0$ such that if (w_n) in D is δ-separated, then there exist a continuous linear operator R: $l^\infty \to \mathscr{B}$ and functions $h_k := R(e_k) \in \mathscr{B}$ such that*

$$h_k'(w_n) = 0, \quad \text{if } n \ne k, \qquad (1 - |w_n|^2)h_n'(w_n) = 1.$$

Proof. By the proof of Proposition 1 in [MM] (see [Ro]), there are two continuous linear operators

$$S\colon \mathscr{B} \to l^\infty, \qquad S(f) = \big((1 - |w_n|^2)f'(w_n)\big)_n$$

and

$$T\colon l^\infty \to \mathscr{B}, \qquad T\big((\xi_n)\big)z = \sum_{n=1}^{\infty} \xi_n \frac{1}{3\overline{w}_n} \frac{(1 - |w_n|^2)^3}{(1 - \overline{w}_n z)^3}$$

such that $\|\mathrm{id} - ST\| < 1$. Thus ST has an inverse $(ST)^{-1}\colon l^\infty \to l^\infty$, and therefore S has a right inverse $R := T(ST)^{-1}\colon l^\infty \to \mathscr{B}$. Since $SR(e_k) = e_k$ for all k, we get that $(1 - |w_n|^2)h_k'(w_n) = \delta_{nk}$ for all n and k. □

Proposition 14. *The following statements are equivalent*:

(a) C_φ: $\mathscr{B} \to \mathscr{B}$ *is non-compact.*

(b) *There exist continuous linear operators R: $l^\infty \to \mathscr{B}$ and Q: $\mathscr{B} \to l^\infty$ such that $Q \circ C_\varphi \circ R = \mathrm{id}_{l_\infty}$.*

(c) C_φ: $\mathscr{B} \to \mathscr{B}$ *is not a Rosenthal operator.*

In [LST] the equivalence (a) $\Leftrightarrow$ (c) is obtained by other methods.

Proof. (a) $\Rightarrow$ (b): Since C_φ is non-compact, by [MM, Theorem 2], there is a sequence $(z_n) \in D$ and a constant $\varepsilon > 0$ so that $|\varphi(z_n)| \to 1$ and

$$\frac{(1-|z_n|^2)|\varphi'(z_n)|}{1-|\varphi(z_n)|^2} \geq \varepsilon \qquad \text{for all } n \geq 1.$$

Since $|\varphi(z_n)| \to 1$, passing to a subsequence, we can apply Proposition 13 and get a continuous linear operator $R\colon l^\infty \to \mathscr{B}$ and functions $h_k := R(e_k) \in \mathscr{B}$ such that

$$h_k'(\varphi(z_n)) = 0, \quad \text{if} \quad n \neq k, \qquad \big(1-|\varphi(z_n)|^2\big)h_n'\big(\varphi(z_n)\big) = 1.$$

Hence $R(\xi) = \sum_{k=1}^\infty \xi_k h_k$ for all $\xi = (\xi_k) \in c_0$. Now we define a map

$$Q\colon \mathscr{B} \to l^\infty, \qquad Q(f) = \left(\frac{1-|\varphi(z_n)|^2}{\varphi'(z_n)} f'(z_n)\right)_n.$$

Since

$$\|Q(f)\| \leq \frac{1}{\varepsilon}\sup_n |f'(z_n)|(1-|z_n|^2) \leq \frac{1}{\varepsilon}\|f\|_{\mathscr{B}} \qquad \text{for all } f \in \mathscr{B},$$

the map is well defined, linear and continuous. For every $\xi = (\xi_n) \in c_0$,

$$Q \circ C_\varphi \circ R(\xi) = \left(\big(1-|\varphi(z_n)|^2\big)\sum_{k=1}^\infty \xi_k h_k'\big(\varphi(z_n)\big)\right)_n.$$

Consequently, we get that $Q \circ C_\varphi \circ R(\xi) = \xi$ for all $\xi \in c_0$. Using a result of Rosenthal [Rs, Proposition 1.2] we get the conclusion.

The implications (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) are obvious. ▫

Corollary 15. *Let v be a weight on D. Let C_φ be continuous on B_∞^v. The operator C_φ is weakly compact (respectively Rosenthal) on $B_\infty^v(X)$ if and only if C_φ is Rosenthal on B_∞^v and X is reflexive (respectively X does not contain a copy of l_1).*

Proof. It is well known (e.g. [BS], [BBT]) that the space B_∞^v satisfies the conditions imposed on the general space E considered in this section. Moreover it is easy to see that the vector valued space $B_\infty^v(X)$ coincides isometrically with the space $B_\infty^v[X]$ defined here.

The *associated weight* is defined by

$$\tilde{v}(z) = \big(\sup\{|f(z)| : \|f\|_v \leq 1\}\big)^{-1}, \qquad z \in D.$$

It is better tied to the space B_∞^v than v itself [BBT], and $B_\infty^v = B_\infty^{\tilde{v}}$ holds isometrically. By [BDLT] the operator C_φ is continuous on B_∞^v if and only if

$$\sup_{z\in D}\frac{v(z)}{\tilde{v}\big(\varphi(z)\big)} < \infty.$$

Moreover, by [BDL, Theorem 1], the operator C_φ is Rosenthal on B_∞^v if and only if it is compact. Hence the conclusion follows from Proposition 11. ▫

If we take $v(z) = 1$ for every $z \in D$ in Corollary 15, we obtain as a particular case Theorem 6 and part of Theorem 7 in [LST].

To conclude we consider only radial weights v, that is, $v(z) = v(|z|)$. A radial weight v is called *essential*, if there exists a $C > 0$ such that $v(z) \le \tilde{v}(z) \le Cv(z)$. We can now apply [BDLT, Theorem 3.3] to get the following corollary.

Corollary 16. *Let v be an essential weight. Then $C_\varphi\colon B^v_\infty(X) \to B^v_\infty(X)$ is weakly compact (respectively Rosenthal) if and only if X is reflexive (respectively does not contain a copy of l_1) and*

$$\lim_{r\to 1} \sup_{\{z:|\varphi(z)|>r\}} \frac{v(z)}{v\big(\varphi(z)\big)} = 0 \qquad \text{or} \qquad \|\varphi\|_\infty < 1.$$

As a consequence of Lemma 4 and Fatou's lemma, the weighted Bergman spaces B^α_p, $1 \le p < \infty$, $\alpha \ge -1$, satisfy the conditions imposed on the scalar valued Banach space E. This permits to use Proposition 6 and Proposition 11 to get consequences on vector-valued composition operators on spaces of type $B^\alpha_p[X]$ as defined in this section. It is important to point out that the classical vector-valued space $B^\alpha_p(X)$ is continuously included in but different from $B^\alpha_p[X]$. This is the reason why we had to treat composition operators defined on them with another method.

References

[BBG] Bierstedt, K.D., J. Bonet, and A. Galbis: Weighted spaces of holomorphic functions on bounded domains. - Michigan Math. J. 40, 1996, 271–297.

[BBT] Bierstedt, K.D., J. Bonet, and J. Taskinen: Associated weights and spaces of holomorphic functions. - Studia Math. 127, 1998, 70–79.

[BS] Bierstedt, K.D., and W.H. Summers: Biduals of weighted Banach spaces of analytic functions. - J. Austral. Math. Soc. Ser. A 54, 1993, 70–79.

[BDL] Bonet, J., P. Domański, and M. Lindström: Essential norm and weak compactness of composition operators on weighted Banach spaces of analytic functions. - Bull. Canad. Math. 42, 1999, 139–148.

[BDLT] Bonet, J., P. Domański, M. Lindström, and J. Taskinen: Composition operators between weighted Banach spaces of analytic functions. - J. Austral. Math. Soc. 64, 1998, 101–118.

[CH] Contreras, M., and A. Hernández-Díaz: Weighted composition operators between weighted Banach spaces of analytic functions. - Preprint, Sevilla, 1999.

[CM] Cowen, C., and B. MacCluer: Composition Operators on Spaces of Analytic Functions. - CRC Press, Boca Raton, 1995.

[GE] Grosse-Erdmann, K.-G.: The Borel–Okada theorem revisited. - Habilitationsschrift, Fern Universität, Hagen, 1992.

[HRS] Hedenmalm, H., S. Richter, and K. Seip: Interpolation sequences and invariant subspaces of given index in the Bergman spaces. - J. Reine Angew. Math. 477, 1996, 13–30.

[J] Jarchow, H.: Compactness properties of composition operators. - Preprint, Zürich, 1998.

[LS] Lindström, M., and G. Schlüchtermann: Composition of operator ideals. - Math. Scand. 84, 1999, 284–296.

[LST] Liu, P., E. Saksman, and H.-O. Tylli: Small composition operators on analytic vector-valued function spaces. - Pacific J. Math. 184, 1998, 295–309.

[L] Lusky, W.: On generalized Bergman spaces. - Studia Math. 119, 1996, 77–95.

[MM] Madigan, K.M., and A. Matheson: Compact composition operators on Bloch spaces. - Trans. Amer. Math. Soc. 347, 1995, 2679–2687.

[M] Mujica, J.: Linearization of bounded holomorphic mappings on Banach spaces. - Trans. Amer. Math. Soc. 324, 1991, 867–887.

[N] Ng, K.: On a theorem of Diximier. - Math. Scand. 29, 1971, 279–280.

[R] Racher, G.: On the tensor product of weakly compact operators. - Math. Ann. 294, 1992, 267–275.

[Ro] Rochberg, R.: Interpolation by functions in the Bergman spaces. - Michigan Math. J. 29, 1982, 229–236.

[Rs] Rosenthal, H.P.: On relatively disjoint families of measures with some applications to Banach space theory. - Studia Math. 37, 1970, 155–190.

[ST] Saksman, E., and H.-O. Tylli: Weak compactness of multiplication operators on spaces of bounded linear operators. - Math. Scand. 70, 1992, 91–111.

[S] Sarason, D.: Weak compactness of holomorphic composition operators on H^1. - Functional Analysis and Operator Theory, Lecture Notes in Math. 1511, Springer-Verlag, 1992, 75–79.

[Sh1] Shapiro, J.O.: The essential norm of a composition operator. - Ann. Math. 125, 1987, 375–404.

[Sh2] Shapiro, J.O.: Composition Operators and Classical Function Theory. - Springer-Verlag, 1993.

[SS] Shapiro, J.O., and C. Sundberg: Compact composition operators on L^1. - Proc. Amer. Math. Soc. 145, 1990, 443–449.

[St] Stanton, C.S.: Counting functions and majorizations theorems for Jensen measures. - Pacific J. Math. 125, 1986, 459–449.

[Sm] Smith, W.: Composition operators between Bergman and Hardy spaces. - Trans. Amer. Math. Soc. 348, 1996, 2331–2348.

[SW1] Shields, A.L., and D.L. Williams: Bounded projections, duality, and multipliers in spaces of harmonic functions. - J. Reine Angew. Math. 299/300, 1978, 256–279.

[SW2] Shields, A.L., and D.L. Williams: Bounded projections and the growth of harmonic conjugates in the disc. - Michigan Math. J. 29, 1982, 3–25.

[W] Wojtaszczyk, P.: Banach Spaces for Analysts. - Cambridge, 1991.

[Z] Zhu, K.: Operator Theory in Functions Spaces. - Dekker, 1990.

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