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MATRICES FOR FENCHEL–NIELSEN COORDINATES

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Abstract. We give an explicit construction of matrix generators for finitely generated Fuchsian groups, in terms of appropriately defined Fenchel–Nielsen (F-N) coordinates. The F-N coordinates are defined in terms of an F-N system on the underlying orbifold; this is an ordered maximal set of simple disjoint closed geodesics, together with an ordering of the set of complementary pairs of pants. The F-N coordinate point consists of the hyperbolic sines of both the lengths of these geodesics, and the lengths of arc defining the twists about them. The mapping from these F-N coordinates to the appropriate representation space is smooth and algebraic. We also show that the matrix generators are canonically defined, up to conjugation, by the F-N coordinates. As a corollary, we obtain that the Teichmüller modular group acts as a group of algebraic diffeomorphisms on this Fenchel–Nielsen embedding of the Teichmüller space.

1. Introduction

There are several different ways to describe a closed Riemann surface of genus at least 2; these include its representation as an algebraic curve; its representation as a period matrix; its representation as a Fuchsian group; its representation as a hyperbolic manifold, in particular, using Fenchel–Nielsen (F-N) coordinates; its representation as a Schottky group; etc. One of the major problems in the overall theory is that of connecting these different visions. Our primary goal in this paper is to construct a bridge between F-N coordinates, for an arbitrary 2-orbifold with finitely generated fundamental group, and matrix generators for the corresponding Fuchsian group.

The usual view of F-N coordinates is that they consist of the lengths and twists about a maximal number of disjoint simple closed geodesics, here called *coordinate geodesics* (for a closed surface of genus g, this maximal number is $3g-3$). We start with these geodesics being undirected, and we use the hyperbolic sines of these lengths (the twists can also be described as lengths of geodesic arcs) as coordinates. It is obvious that the lengths of the geodesics are intrinsic on the surface. Fenchel and Nielsen, in their original unpublished manuscript [6] showed that this space of coordinates is naturally homeomorphic to the Teichmüller space. It follows that the twists cannot be intrinsic on the surface, but it is generally known that

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the twists can be canonically defined on the surface marked with a basis for the fundamental group; a proof of this fact appears in Section 9.

One of our results is that the exponential map is universal in the following sense. Following Wolpert [18], we choose the twist to be independent of the length of the geodesic we are twisting about. Then, for any maximal set of simple disjoint geodesics on a hyperbolic orbifold S_0, with finitely generated fundamental group, a quasiconformal deformation of S_0 has all its corresponding F-N coordinates (real) algebraic if and only if the corresponding (appropriately normalized) Fuchsian group is a discrete subgroup of some $\mathrm{PSL}(2,\mathbf{k})$, where $\mathbf{k}$ is a (real) number field.

A corollary of the above is that the Teichmüller modular group acts on any such space of F-N coordinates as a group of algebraic diffeomorphisms.

An *F-N system* consists of a hyperbolic base orbifold, S_0, with finitely generated fundamental group, together with a maximal set, $L_1,\dots,L_p$, of simple disjoint geodesics, none parallel to the boundary. These *coordinate geodesics* divide S_0 into pairs of pants, $P_1,\dots,P_q$. We also assume that the geodesics and pairs of pants are given in a particular order; see Section 2.1. It is well known (see [1]) that, for any given F-N system, the space of F-N coordinates is real-analytically equivalent to the appropriate (reduced) Teichmüller space.

Our first major goal is to write down formulae for matrix generators for the Fuchsian group described by a point in the given F-N coordinate space. The necessary information concerning the topology is encoded in the signature and in the pairing table, defined in Section 8. In the first step, in Section 6, we explicitly describe a set of hyperbolic isometries which generate the corresponding Fuchsian group; then, in Section 7, we give formulae for these isometries as matrices in $\mathrm{PSL}(2,\mathbf{R})$. For the first step, we follow the procedure of Fenchel and Nielsen [6]; we start with Fuchsian groups representing pairs of pants, these are orbifolds of genus 0 with three boundary components, including orbifold points (matrices representing generators for these groups are constructed in Section 5), and then use combination theorems to glue these pants groups together.

Our construction yields a well-defined set of hyperbolic isometries, which depend only on the signature of the base orbifold, the pairing table, which describes the topology of the F-N system, and the point in the corresponding coordinate space; these isometries are independent of the conformal or hyperbolic structure on the base orbifold.

The formulae for our matrix generators are sufficiently explicit for us to immediately observe that the entries in the matrices are (real) algebraic functions of the F-N coordinates. In fact, as functions of the parameters, the entries in the matrices are obtained by taking a finite number of degree two field extensions of the field of rational functions of the parameters. We also immediately observe that the arguments of these square roots are bounded away from zero, so that the mapping from F-N coordinates to the space of discrete faithful representations of the fundamental group is smooth. We make these observations here, and again in the recapitulation, but will not repeat them at each stage of the process.

We write our result in the form of an explicit algorithm, which is stated in Section 8. For any given F-N system, the algorithm yields a set of explicit formulae for the entries in the matrix generators, where these entries depend on the particular point in the F-N coordinate space.

We also need to reverse the above process. In Section 9, we start with a set of matrices, generating a discrete group G; we assume these have been defined by the above process. We show that these matrices uniquely define the signature of the underlying orbifold, the pairing table describing the F-N system, and the coordinate point in this system. It follows that our map from F-N coordinates to an appropriate space of discrete faithful representations of the fundamental group is injective, and that the twists are canonically defined on the Teichmüller space.

In Section 10, we give precise statements of our results, which include a new version of the original Fenchel–Nielsen theorem.

In Section 11, we explicitly work through the algorithm for one case of a closed surface of genus 3. In this case, the F-N system has one dividing geodesic and 5 non-dividing geodesics.

The algorithm, as stated, yields matrices that could be simpler, even for genus 2; in Section 12, we give a variation of this algorithm which yields simpler matrices in most cases, and then, in Section 13, we work out this algorithm for the case of three non-dividing geodesics on a closed surface of genus 2. The other case of a closed surface of genus 2, with one dividing geodesic and two non-dividing geodesics, appears in [14].

We also present a second variation of the algorithm in Section 14. In this second variation, the zero twist coordinate position for the handle closing generators is always given by the common orthogonal between two geodesics in the universal covering lying over the same coordinate geodesic. However, this second variation is in some sense less explicit, in that, for any given F-N system, the entries in the matrices are defined algorithmically, rather than being given by explicit formulae.

For orbifolds of dimension 3, our results here extend almost immediately to quasifuchsian groups of the first kind. There are also related results for other classes of Kleinian groups; these will be explored elsewhere.

Some of the ideas and computations used here, as well as various versions of the negative trace theorem, have appeared in print. References for these include Abikoff [1], Fenchel [5], Fenchel and Nielsen [6], Fine and Rosenberger [7], Fricke and Klein [8], Gilman and Maskit [10]; Jörgensen [11], Rosenberger [16], Seppälä and Sorvali [17], Wolpert [18]; see also [13] and [15].

This work was in part inspired by the work of Buser and Silhol [3], who worked out explicit F-N coordinates for certain algebraic curves. The author also wishes to thank Irwin Kra and Dennis Sullivan for informative conversations.

2. Topological preliminaries

We assume throughout that all orbifolds are complete, orientable, of dimension 2, and have non-abelian, finitely generated (orbifold) fundamental group. We denote the hyperbolic plane by $\mathbf{H}^2$; we will usually regard $\mathbf{H}^2$ as being the upper half-plane endowed with its usual hyperbolic metric, so that the group of all orientation-preserving isometries of $\mathbf{H}^2$ is canonically identified with $\mathrm{PSL}(2,\mathbf{R})$.

Let S be a hyperbolic orbifold; that is, there is a finitely generated Fuchsian group F so that $S = \mathbf{H}^2/F$. Topologically, S is a surface of genus g with some number of boundary elements; there are also some number of orbifold points. Geometrically, we regard the orbifold points as boundary elements, so that there are three types of boundary elements. The *punctures* or *parabolic boundary elements*, are in natural one-to-one correspondence with the conjugacy classes of maximal parabolic cyclic subgroups of F; the *orbifold points*, or *elliptic boundary elements*, are in natural one-to-one correspondence with the conjugacy classes of maximal elliptic cyclic subgroups of F; the *order* of a puncture is ∞; the *order* of an orbifold point is the order of a corresponding maximal elliptic cyclic subgroup; and the *holes*, or *hyperbolic boundary elements*, are in natural one-to-one correspondence with the conjugacy classes of hyperbolic boundary subgroups of F.

A *boundary subgroup* $H \subset F$ is a maximal hyperbolic cyclic subgroup, whose axis, the *boundary axis*, bounds a half-plane that is precisely invariant under H in F. The elements of a boundary subgroup are called *boundary elements*; the boundary axis projects to the corresponding *boundary geodesic* on S, which is parallel to the boundary. The *size* of the corresponding boundary element is half the length of this boundary geodesic; that is, if a is a generator of the boundary subgroup H, then its size σ is given by $2\cosh(\sigma) = |\operatorname{tr}(a)|$. If a generates a boundary subgroup of G, then the corresponding axis A separates $\mathbf{H}^2$ into two half-planes. The *boundary half-plane* is precisely invariant under $\langle a\rangle$ [1] in G. The other half-plane, which is not precisely invariant (unless G is elementary), is called the *action half-plane.*

The orbifold S is completely described, up to quasiconformal deformation, by its genus g; the number of boundary elements n that are either punctures or elliptic orbifold points; the orders $\alpha_1, \ldots, \alpha_n$ of these points; and the number m of holes.

As usual, we encode this information in the *signature*

$$(g, n, m; \alpha_1, \ldots, \alpha_n).$$

When we do not need to know the actual values of the α_i, we write the signature as simply $(g, n+m)$. Since we require S to be hyperbolic, there are some well-known restrictions on these numbers.

[1] The group generated by $a, \ldots$ is denoted by $\langle a, \ldots\rangle$.

It is well known that there are at most $p = 3g - 3 + n + m$ simple disjoint geodesics $L_1, \dots, L_p$ on an orbifold of signature $(g, n+m)$, where none of the L_i is parallel to the boundary. There are also m boundary geodesics, which we label as $L_{p+1}, \dots, L_{p+m}$.

For the remainder of this section, we will consider a geodesic to be defined modulo orientation; that is, we do not distinguish between a geodesic and its inverse.

2.1. F-N systems. An F-N *system* on S is an ordered set of $p+m$ simple disjoint geodesics—this is the maximal possible number of such geodesics, together with an ordering of the other n boundary elements, where the ordering satisfies the conditions below. We write the F-N system either as $L_1, \dots, L_{p+m}, b_1, \dots, b_n$ or as $L_1, \dots, L_p, b_1, \dots, b_{n+m}$, or as $L_1, \dots, L_{p+n+m}$. It will always be clear from the context which system of notation we are using.

Except in Section 4, we will assume throughout that S is not a pair of pants; that is, $p > 0$.

None of the first p geodesics of an F-N system are parallel to the boundary; they are the *coordinate geodesics*. The coordinate geodesics divide S into $q = 2g - 2 + n + m$ pairs of pants, $P_1, \dots, P_q$, each of which is a hyperbolic orbifold of signature $(0, n_0 + m_0)$, $n_0 + m_0 = 3$. Each coordinate geodesic is either a boundary element of two distinct pairs of pants, or corresponds to two boundary elements of the same pair of pants.

There is in general no canonical way to order and direct the coordinate geodesics, and to order the pairs of pants they divide the surface into. From here on, we assume that the coordinate geodesics and boundary elements, and also the pairs of pants, have been ordered in accordance with the following set of rules.

2.1.1. *Rules for order.*

(i) If there is a dividing coordinate geodesic, then L_1 is dividing; in any case, if $q \geq 2$, then L_1 lies between P_1 and P_2.

(ii) If $q \geq 3$, then L_2 lies between P_1 and P_3.

(iii) The first $q-1$ coordinate geodesics, and the q pairs of pants, $P_1, \dots, P_q$, are ordered so that, for every $j = 3, \dots, q-1$, there is an $i = i(j)$, with $1 \leq i(j) < j$, so that L_j lies on the common boundary of P_i and P_j. The coordinate geodesics $L_1, \dots, L_{q-1}$ are called the *attaching* geodesics; the coordinate geodesics $L_q, \dots, L_p$ are called the *handle* geodesics.

(iv) The hyperbolic boundary elements $b_1, \dots, b_m$ precede the parabolic boundary elements, which, in turn precede the elliptic boundary elements. Also, the elliptic boundary elements are in decreasing order.

From here on, we reserve the indices, m, n, p and q, for the meanings given above.

2.2. F-N coordinates. Let G_0 be a given finitely generated Fuchsian group, and let $S_0 = \mathbf{H}^2/G_0$. A (quasiconformal) deformation of G_0 is a discrete faithful representation ψ of G_0 into $\mathrm{PSL}(2,\mathbf{R})$, where there is a quasiconformal homeomorphism $f\colon \mathbf{H}^2 \to \mathbf{H}^2$ inducing ψ. Two such deformations, ψ and ψ' are *equivalent* if there is an element $a \in \mathrm{PSL}(2,\mathbf{R})$ so that $\psi(g) = a\psi'(g)a^{-1}$ for all $g \in G_0$.

Let $\psi\colon G_0 \to G$ be a quasiconformal deformation. The *F-N coordinates* of (the equivalence class of) ψ are given by the following vector:

$$\Phi = (s_1, \dots, s_{p+m}, t_1, \dots, t_p) \in (\mathbf{R}^+)^{p+m} \times \mathbf{R}^p.$$

The geodesics $L_1, \dots, L_p, L_{p+1}, \dots, L_{p+m}$ are well defined on $S = \mathbf{H}^2/G$. The length of L_i on S, $i = 1, \dots, p+m$, is $2\sigma_i$, where $s_i = \sinh \sigma_i$. Also, for $i = 1, \dots, p$, the twist about L_i is $2\tau_i$, where $t_i = \sinh \tau_i$; this will be explained in Section 6.

2.3. Pairs of pants. Each pair of pants P has three boundary elements; in most cases, the ordering of the coordinate geodesics and boundary elements of S_0 induces an ordering of the boundary elements of P. There are two exceptional cases in which there are two boundary elements of P corresponding to just one coordinate geodesic of S_0.

In the first exceptional case, S_0 is a torus with one boundary component, so two of the boundary elements of P are hyperbolic, necessarily of the same size, and the other boundary element can be of any type. Since the torus with one boundary component is elliptic (i.e., admits a conformal involution with 3 or 4 fixed points), one cannot tell the difference between the two boundary elements of P corresponding to the one coordinate geodesic on S_0. We make an arbitrary choice of which of these two boundary elements precedes the other; since the elliptic involution acts ineffectively on the Teichmüller space, it makes no difference which choice we make.

In the second exceptional case, all three boundary elements of P are necessarily hyperbolic. Here, one of the boundary elements of P corresponds to an attaching geodesic, which is also a dividing geodesic of S_0, while the other two boundary elements both correspond to the same non-dividing handle geodesic. In this case, b_1, the first boundary element, necessarily corresponds to the dividing geodesic. The other two boundary elements, b_2 and b_3, are necessarily hyperbolic of the same size. As above, if S_0 is elliptic or hyperelliptic, then the choice of which boundary element of P to call b_2 and which to call b_3 is arbitrary, and it does not matter which choice we make. If S_0 is not elliptic or hyperelliptic, then one can make a canonical choice; this will be done in Section 6. Until then, we leave it that this choice is made somehow.

We now return to the general case. Between any two boundary elements of P, b_i and b_j, there is a unique simple orthogonal geodesic arc $N_{ij} \subset P$. That is, if b_i is parabolic, then N_{ij} has infinite length, with an infinite endpoint at the parabolic puncture; if b_i is elliptic, then N_{ij} has one endpoint at this elliptic orbifold point; if b_i is hyperbolic, then N_{ij} is orthogonal to the corresponding boundary geodesic.

In the case that b_i is hyperbolic, with boundary geodesic L_i, then the two common orthogonals to the two other boundary elements of P meet L_i at two distinct points of L_i; these two points divide L_i into two arcs of equal length.

We will use the following notation throughout. The boundary elements of the pair of pants, P_i, are labeled as $b_{i,1}, b_{i,2}, b_{i,3}$, in the order given above.

2.4. Directing the coordinate and boundary geodesics. We need to specify a direction for each coordinate geodesic. In general, we direct L_1 so that P_1 lies on the right as we traverse L_1 in the positive direction. In the exceptional cases that S_0 is elliptic or hyperelliptic, this choice of a first direction is necessarily arbitrary; however, as mentioned above, it is irrelevant which choice is made.

We say that two geodesics on the boundary of some pair of pants P are *consistently oriented* with respect to P, if P lies on the right as we traverse either geodesic in the positive direction, or if P lies on the left as we traverse either geodesic in the positive direction.

Assume that $L_1, \dots, L_j$, $j \geq 1$, have been directed. If L_{j+1} lies on the boundary of two distinct pairs of pants, or is a boundary geodesic, then there is a lowest index i, so that L_{j+1} lies on the boundary of P_i. Since $j+1 > 1$, L_{j+1} corresponds to either $b_{i,2}$ or $b_{i,3}$, for $b_{i,1}$ must correspond to some attaching geodesic, $L_{j'}$, $j' \leq j$. We direct L_{j+1} so that L_{j+1} and $L_{j'}$ are consistently oriented as boundary elements of P_i.

If L_{j+1} is a handle geodesic, with the same pair of pants, P_i, on both sides of L_{j+1}, then the direction of L_{j+1} is more complicated. As above, we will see in Section 6 that this choice can be made canonically; for the moment, we assume that this choice has been made somehow.

3. $\mathrm{SL}(2, \mathbf{R})$ and $\mathrm{PSL}(2, \mathbf{R})$

There is a canonical identification of $\mathrm{PSL}(2, \mathbf{R})$ with the group of orientation-preserving isometries of $\mathbf{H}^2$; each such transformation has two representatives in $\mathrm{SL}(2, \mathbf{R})$. There is likewise a canonical identification of $\mathrm{PGL}(2, \mathbf{R})$ with the group of all plane hyperbolic isometries; each such isometry has two representatives in $S^{\pm}L(2, \mathbf{R})$, the group of real 2×2 matrices with determinant ± 1.

We will use the following convention throughout. If $\tilde{a}$ is a matrix in $S^{\pm}L(2, \mathbf{R})$, then the corresponding hyperbolic isometry is denoted by a.

We will usually use this notation in reverse; that is, given the isometry a, we will choose a representative matrix $\tilde{a} \in S^{\pm}L(2, \mathbf{R})$. Also, all hyperbolic isometries (i.e., all transformations) that are not explicitly identified as reflections, are assumed to be orientation-preserving.

A Fuchsian group F is *algebraic* if there is a (real) number field $\mathbf{k}$ so that $F \subset \mathrm{PSL}(2,\mathbf{k})$. Correspondingly, the orbifold $S = \mathbf{H}^2/F$ is *algebraic* if F is algebraic.

We remark that a Fuchsian group F, with generators $a_1, \dots, a_i \dots$, is algebraic if and only if the numbers, $a_i(0), a_i(1), a_i(\infty)$, are all algebraic.

For our purposes, from here on, a Fuchsian group is a non-Abelian finitely generated discrete subgroup of $\mathrm{PSL}(2,\mathbf{R})$; it is elementary if it contains an Abelian subgroup of finite index, and non-elementary otherwise. Unless explicitly stated otherwise, all Fuchsian groups will be assumed to be non-elementary[2].

4. Reflections and geometric generators

It will often be convenient to have an order among the different kinds of hyperbolic isometries. We say that hyperbolic transformations are *higher* than the parabolic ones, which in turn are higher than the elliptic ones; further, elliptic transformations of higher (finite) order are *higher* than elliptic transformations of lower order.

An elliptic transformation a of order α is *primitive* if $|\operatorname{tr}(a)| = 2\cos(\pi/\alpha)$; that is, a is a geometrically primitive rotation.

4.1. Transformations with disjoint axes. Let a_1, a_2 and $a_3 = (a_1a_2)^{-1}$ be elements of $\mathrm{PSL}(2,\mathbf{R})$. If a_i is hyperbolic, then its *axis* A_i is, as usual, the hyperbolic line connecting its fixed points. If a_i is parabolic or elliptic, then its *axis* A_i is its fixed point, which lies either on the circle at infinity or is an interior point of $\mathbf{H}^2$.

We will use the following conventions throughout: If a_α^β is a given element of $\mathrm{PSL}(2,\mathbf{R})$, then its axis is denoted by A_α^β.

In general, if $X \subset \mathbf{H}^2$, then we denote the Euclidean closure of X by $\overline{X}$. Also, in general, two lines, L and L', are *disjoint* if $\bar{L} \cap \bar{L}' = \emptyset$, in particular, the axes of a_1 and a_2 are *disjoint* if $\bar{A}_1 \cap \bar{A}_2 = \emptyset$.

If a is elliptic or parabolic, then we say that the line M is *orthogonal* to A if M passes through A, or ends at A. We now have that, independent of the type of a_i and a_j, if a_i and a_j have disjoint axes, then these axes have a unique common orthogonal.

4.2. Reflections in lines. For every hyperbolic line M, there is a well-defined reflection r, whose fixed point set is equal to M.

Let M_1 and M_2 be distinct lines; denote reflection in M_i by r_i, and let $a = r_1r_2$. Then a is hyperbolic, respectively, parabolic, respectively, elliptic, if M_1 and M_2 are disjoint, respectively, meet at the circle at infinity, respectively, cross inside $\mathbf{H}^2$. If a is hyperbolic, then $|\operatorname{tr}(a)| = 2\cosh\lambda$, where λ is the distance

[2] Among Fuchsian groups, the $(2,2,\infty)$-triangle group is uniquely elementary but not Abelian; it shares many important properties with the non-elementary Fuchsian groups.

between M_1 and M_2; if a is elliptic, then $|\operatorname{tr}(a)| = 2|\cos\theta|$, where θ is the angle between M_1 and M_2.

4.3. Matrices for reflections. The matrices in $S^{\pm}L(2,\mathbf{R})$ of determinant -1 and trace 0 correspond to reflections in lines. One can regard the choice of a matrix for a reflection as being equivalent to a choice of a direction on the fixed line (see Fenchel [5]). More precisely, we choose the matrix

$$\tilde{r} = \frac{1}{x-y}\begin{pmatrix} x+y & -2xy \\ 2 & -x-y \end{pmatrix}$$

to correspond to the reflection in the upper half-plane with fixed line ending at x and y, where x is the *positive endpoint* of this line. Then by continuity, the reflection with matrix

$$\tilde{r} = \begin{pmatrix} 1 & -2y \\ 0 & -1 \end{pmatrix}$$

has its positive fixed point at ∞ and its negative fixed point at y.

If M_1 and M_2 are disjoint directed hyperbolic lines, then we say that the positive endpoints of M_1 and M_2 are *adjacent* to mean that both negative endpoints lie on the same arc of the circle at infinity between these positive endpoints.

Easy observations now show the following.

Proposition 4.1. *Let $\tilde{r}_1, \tilde{r}_2 \in S^{\pm}L(2,\mathbf{R})$ represent reflections in the disjoint directed lines M_1, M_2, respectively. Then $\operatorname{tr}(\tilde{r}_1\tilde{r}_2) > 0$ if and only if the positive endpoints of M_1 and M_2 are adjacent.*

Proposition 4.2. *Let $\tilde{r}_1, \tilde{r}_2 \in S^{\pm}L(2,\mathbf{R})$ represent reflections in the directed lines M_1, M_2, respectively, where M_1 and M_2 have exactly one endpoint on the circle at infinity in common. Then $\operatorname{tr}(\tilde{r}_1\tilde{r}_2) = +2$ if and only if the common endpoint is either the positive endpoint, or the negative endpoint, of both lines.*

Proposition 4.3. *Let $\tilde{r}_1, \tilde{r}_2 \in S^{\pm}L(2,\mathbf{R})$ represent reflections in the directed lines M_1, M_2, respectively, where M_1 and M_2 intersect at an interior point of $\mathbf{H}^2$. Then $\operatorname{tr}(\tilde{r}_1\tilde{r}_2) = 2\cos\theta$, where θ is the angle between the positive endpoints of M_1 and M_2.*

4.4. Hyperbolic triangles. In Euclidean geometry, a triangle is completely determined by three lines, no two of which are parallel; in hyperbolic geometry, the situation is somewhat more complicated. For our purposes, a *triangle* D is the intersection of the three closed half-planes, R_1, R_2, R_3, bounded by the three distinct lines, M_1, M_2, M_3, respectively, provided that, for $i = 1,2,3$, $M_i \cap D$ contains a non-trivial open arc of M_i. This arc, $M_i \cap D$, is called a *side* of D.

Every pair of these lines, M_i and M_j, defines a *vertex*, $v_{i,j}$, which is the common orthogonal of M_i and M_j; this vertex is hyperbolic, respectively, parabolic, respectively, elliptic, if M_i and M_j are disjoint, respectively, meet at the circle at infinity, respectively, meet at an interior point of $\mathbf{H}^2$.

We remark that the triangle D is not necessarily uniquely determined by the three lines, M_1, M_2, M_3.

We say that D is a *Poincaré triangle* if the interior angle at every elliptic vertex is of the form, π/α, $\alpha \in Z$, $\alpha \geq 2$.

A triangle D is *degenerate* if two of the bounding lines are each orthogonal to the third. The orientation preserving half of the group generated by reflections in the three sides of a degenerate triangle is elementary.

Let r_i denote reflection in M_i, $i = 1, 2, 3$; set $a_1 = r_2 r_3$, $a_2 = r_3 r_1$, and $a_3 = r_1 r_2$.

Poincaré's polygon theorem asserts that if D is a Poincaré triangle, then the group $\hat{J} = \langle r_1, r_2, r_3 \rangle$ is discrete; D is a fundamental polygon for $\hat{J}$; and $\hat{J}$ has the following presentation:

$$\hat{J} = \langle r_1, r_2, r_3 : r_1^2 = r_2^2 = r_3^2 = (r_1 r_2)^{\alpha_3} = (r_2 r_3)^{\alpha_1} = (r_3 r_1)^{\alpha_2} = 1 \rangle,$$

where the statement $(r_i r_j)^{\alpha_k} = a_k^{\alpha_k} = 1$ has its usual meaning if the vertex $v_{i,j}$ is elliptic of order α_k; it means that a_k is parabolic if $v_{i,j}$ is parabolic; and it has no meaning if $v_{i,j}$ is hyperbolic. That is, a_k is hyperbolic, respectively, parabolic, respectively, elliptic of order α_k, if and only if the vertex $v_{i,j}$ is hyperbolic, respectively, parabolic, respectively, elliptic of order α_k.

We note that A_k is the common orthogonal to M_i and M_j. Further, if a_k is hyperbolic, then $|\operatorname{tr}(a_k)| = 2\cosh\lambda_k$, where λ_k is the distance between M_i and M_j; if a_k is parabolic, then $|\operatorname{tr}(a_k)| = 2$, and if a_k is elliptic of order α_k, then $|\operatorname{tr}(a_k)| = 2\cos\pi/\alpha_k$.

Let J be the orientation-preserving half of $\hat{J}$. If D is a Poincaré triangle, then $\mathbf{H}^2/J$ is a pair of pants, where the boundary element b_i is the projection of A_i.

If D is a Poincaré triangle, then we say that $a_1 = r_2 r_3$ and $a_2 = r_3 r_1$ are *geometric generators of a pants group.* On course, in this case, a_2 and a_3, or a_3 and a_1, are also geometric generators of the same pants group.

It is well known that every pants group, including the triangle groups, has a set of geometric generators; in fact, these generators are unique up to conjugation in the pants group, up to orientation, and up to a choice of which of the three generators to call a_1, and which to call a_2.

4.5. Appropriate orientation. Let a_1 and a_2 be hyperbolic isometries with disjoint axes.

If a_1 and a_2 are both hyperbolic, then A_1 and A_2 are naturally directed. If the region between these two axes lies on the left as one traverses one of these axes in the positive direction, and it lies on the right as one traverses the other axis in the positive direction, then a_1 and a_2 are not *appropriately oriented.*

If either a_1 or a_2 is elliptic of order 2, then a_1 and a_2 are *appropriately oriented.*

Every parabolic element imparts a natural direction to the circle at infinity, as does every elliptic element of order at least 3. If a_1 and a_2 are both either parabolic or elliptic of order at least 3, then they are *appropriately oriented* if they impart the same direction to the circle at infinity.

Suppose a_1 is hyperbolic and a_2 is either parabolic or elliptic of order at least 3. Let H be the half-plane bounded by A_1, where $\overline{H} \supset A_2$, and let S be the arc of the circle at infinity on $\overline{H}$. Then a_1 and a_2 are *appropriately oriented* if they impart the same direction to S.

4.6. The negative trace theorem.

Theorem 4.1. *Let* $\tilde{a}_1$ *and* $\tilde{a}_2$ *be matrices in* $\mathrm{SL}(2,\mathbf{R})$, *where* a_2 *is not higher than* a_1.

A. *If* A_1 *and* A_2 *are not disjoint, then* a_1 *and* a_2 *are geometric generators of a pants group if and only if* a_1 *is hyperbolic and* a_2 *is elliptic of order* 2.

B. *If* A_1 *and* A_2 *are disjoint, then* a_1 *and* a_2 *are geometric generators of a pants group if and only if the following hold*:

(i) $T = \mathrm{tr}(\tilde{a}_1)\,\mathrm{tr}(\tilde{a}_2)\,\mathrm{tr}(\tilde{a}_1\tilde{a}_2) \le 0$; *and*

(ii) *if any of* a_1, a_2 *or* a_1a_2 *is elliptic, then it is primitive.*

Proof. All cases of two transformations with non-disjoint axes are well known. The group $G = \langle a_1, a_2\rangle$ is discrete only in the case above, and in various cases of two hyperbolic generators with crossing axes. In these latter cases, either $\mathbf{H}^2/G$ has signature $(1,1)$, or has signature $(0,3)$, but a_1 and a_2 are not geometric generators.

Now assume that the axes of a_1 and a_2 are disjoint. Let L_3 be the common orthogonal to A_1 and A_2. One easily finds lines, M_1 and M_2, so that, denoting reflection in M_i by r_i, $a_1 = r_2r_3$ and $a_2 = r_3r_1$.

Let M_1, M_2, M_3, be any three distinct directed lines. Let $\tilde{r}_i \in S^{\pm}L(2,\mathbf{R})$ be the matrix representing reflection in M_i, with the given orientation. Let $\tilde{a}_1 = \tilde{r}_2\tilde{r}_3$, $\tilde{a}_2 = \tilde{r}_3\tilde{r}_1$ and $\tilde{a}_3 = \tilde{r}_1\tilde{r}_2$.

Observe that T is unchanged if we replace any $\tilde{r}_i$ by $-\tilde{r}_i$.

There are five cases to consider; we do not need to consider the sixth case, where all three lines meet at a point, for we assume that $\bar{A}_1 \cap \bar{A}_2 = \emptyset$. In each case, we draw three lines, and direct them somehow; the sign of T does not depend on which direction we choose. Then we use Propositions 4.1–4.3, to compute the sign of T.

Case 1. If the three lines are pairwise disjoint, and one of the lines separates the other two inside $\mathbf{H}^2$, then a_1 and a_2 are not geometric generators, and $T > 0$.

Case 2. If the lines have no points of intersection inside $\mathbf{H}^2$, and the three lines bound a common region, then a_1 and a_2 are geometric generators, and $T < 0$.

Case 3. If exactly two of the lines meet inside $\mathbf{H}^2$, then there is exactly one of the five regions cut out by these three lines that can be a triangle. The angle at the one elliptic vertex is acute if and only if $T < 0$; that angle is a right angle if and only if $T = 0$.

Case 4. If say M_1 meets both M_2 and M_3, but $M_2 \cap M_3 = \emptyset$, then these three lines separate $\mathbf{H}^2$ into six regions, of which at most one can be a triangle with all angles $\leq \pi$. There is such a triangle if and only if $T \leq 0$.

Case 5. If M_1, M_2 and M_3 form a compact triangle, then $T \leq 0$ if and only if none of the angles are obtuse. □

5. Fully normalized pants groups

We assume that we are given three numbers λ_1, λ_2 and λ_3, where either $\lambda_i \geq 0$, or $\lambda_i = i\pi/\alpha$, $\alpha \in Z$, $\alpha \geq 2$. We need to write down matrices, $\tilde{a}_1$ and $\tilde{a}_2$, corresponding to geometric generators for a pants group, where $|\operatorname{tr}(\tilde{a}_1)| = 2\cosh\lambda_1$, $|\operatorname{tr}(\tilde{a}_2)| = 2\cosh\lambda_2$ and $|\operatorname{tr}(\tilde{a}_3)| = |\operatorname{tr}(\tilde{a}_1\tilde{a}_2)^{-1}| = 2\cosh\lambda_3$. We can assume without loss of generality that the λ_i are given so that the a_i are in non-increasing order; we can also assume that $\operatorname{tr}(\tilde{a}_1) \geq 0$ and $\operatorname{tr}(\tilde{a}_2) \geq 0$.

Since the normalizations are different, we will take up separately the different cases according to the types of a_1, a_2 and $a_3 = (a_1a_2)^{-1}$.

5.1. Standard normalizations. If a_1 is hyperbolic, then A_1 is the imaginary axis, pointing towards ∞. If a_2 and a_3 are both elliptic of order 2, then A_2 is the point i. Otherwise, A_1 and A_2 are disjoint, in which case A_2 lies in the right half-plane and M_3, the common orthogonal between A_1 and A_2, lies on the unit circle.

If a_1 is parabolic, then A_1 is the point at infinity, and M_3 lies on the imaginary axis. If a_2 is also parabolic, then A_2 is necessarily at 0; if a_2 is elliptic, then A_2 is at the point i.

If a_1 is elliptic, then we change our point of view; regard $\mathbf{H}^2$ as being the unit disc; place A_1 at the origin, and place A_2 on the positive real axis.

5.2. Three hyperbolics. In this case, $\lambda_i > 0$, $i = 1, 2, 3$. We need to find matrices $\tilde{a}_1$ and $\tilde{a}_2$, with $\operatorname{tr}(\tilde{a}_1) = 2\cosh(\lambda_1)$; $\operatorname{tr}(\tilde{a}_2) = 2\cosh(\lambda_2)$; and $\operatorname{tr}(\tilde{a}_1\tilde{a}_2) = -2\cosh(\lambda_3)$.

We need a_1 and a_2 to be appropriately oriented; hence we write our matrices so that the repelling fixed point of a_2 is greater than 1, while the attracting fixed point is less than 1.

We define μ by:

$$\coth\mu = \frac{\cosh\lambda_1\cosh\lambda_2 + \cosh\lambda_3}{\sinh\lambda_1\sinh\lambda_2}, \qquad \mu > 0. \tag{1}$$

We write:

$$\tilde{a}_1 = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{-\lambda_1} \end{pmatrix}; \qquad \tilde{a}_2 = \frac{1}{\sinh \mu} \begin{pmatrix} \sinh(\mu - \lambda_2) & \sinh \lambda_2 \\ -\sinh \lambda_2 & \sinh(\mu + \lambda_2) \end{pmatrix}.$$

Then a_1 is as desired, and a_2 has its attracting fixed point at $e^{-\mu}$, its repelling fixed point at e^{μ}, and $\mathrm{tr}(\tilde{a}_2) = 2\cosh \lambda_2$. Equation (1) yields that $\mathrm{tr}(\tilde{a}_1 \tilde{a}_2) = -2\cosh(\lambda_3)$.

We will also need a different matrix representation for $a_3 = a_2^{-1} a_1^{-1}$. We recall that M_3, the common orthogonal between A_1 and A_2 meets A_1 at i. Then, since a pair of pants is hyperelliptic, M_2, the common orthogonal of A_1 and A_3, meets A_1 halfway between i and $a_1(i)$. Hence M_2, lies on the circle $|z| = e^{\lambda_1}$.

We define ν by the following.

$$\text{(2)} \qquad \coth \nu = \frac{\cosh \lambda_1 \cosh \lambda_3 + \cosh \lambda_2}{\sinh \lambda_1 \sinh \lambda_3}, \qquad \nu > 0.$$

Using the above remark, together with the definition of ν, it is easy to see that we can write

$$\tilde{a}_3 = -\tilde{a}_2^{-1} \tilde{a}_1^{-1} = \frac{1}{\sinh \nu} \begin{pmatrix} \sinh(\nu - \lambda_3) & e^{\lambda_1} \sinh \lambda_3 \\ -e^{-\lambda_1} \sinh \lambda_3 & \sinh(\nu + \lambda_3) \end{pmatrix}.$$

Remark 5.1. One easily sees that μ is related to δ, the distance between A_1 and A_2, by $\coth \mu = \cosh \delta$, or, equivalently, $\sinh \mu \sinh \delta = 1$. The RHS of equation (1) is the well-known formula for the hyperbolic cosine of the length of one side of a hexagon with all right angles, given the lengths of three non-adjacent sides. Similar remarks hold for equation (2).

5.3. Closing a handle. The case that S_0 is a torus with one hole needs to be treated separately. In this case, S_0 has one coordinate geodesic, necessarily non-dividing, and one boundary geodesic. We change our usual order, and label the boundary geodesic of the one pair of pants P as b_1, and label the other two boundary elements, corresponding to the coordinate geodesic, as b_2 and b_3.

We proceed exactly as above, and construct $\tilde{a}_1$, $\tilde{a}_2$ and $\tilde{a}_3$, with $\lambda_2 = \lambda_3$. We need to find a matrix for the *handle-closer* d, which maps the action half-plane of a_2 onto the boundary half-plane of a_3, while twisting by 2τ in the positive direction along A_2.

We can write $d = r r_2 e_\tau$, where e_τ is the hyperbolic motion (or the identity) with the same fixed points as a_2 and with trace equal to $2\cosh \tau$, where a_2 and e_τ have the same attracting fixed point if $\tau > 0$, and have opposite attracting fixed points if $\tau < 0$; r_2 is the reflection in A_2; and r is the reflection in the line halfway between A_2 and A_3.

Using $\tilde{a}_2$ as a model, we already know how to find $\tilde{e}_\tau$:

$$\tilde{e}_\tau = \frac{1}{\sinh\mu}\begin{pmatrix} \sinh(\mu-\tau) & \sinh\tau \\ -\sinh\tau & \sinh(\mu+\tau) \end{pmatrix}.$$

The line halfway between A_2 and A_3 is the circle centered at the origin of radius $\exp(\frac{1}{2}\lambda_1)$. Hence we can write

$$\tilde{r} = \begin{pmatrix} 0 & \exp(\frac{1}{2}\lambda_1) \\ \exp(-\frac{1}{2}\lambda_1) & 0 \end{pmatrix}.$$

To find a matrix for r_2, observe that if

$$\tilde{a} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2, \mathbf{R}),$$

where a is hyperbolic and $\beta\gamma \neq 0$, then we can write the matrix for the reflection r_A in A as

$$\tilde{r}_A = \frac{1}{\sqrt{(\alpha+\delta)^2-4}}\begin{pmatrix} \alpha-\delta & 2\beta \\ 2\gamma & \delta-\alpha \end{pmatrix}. \tag{3}$$

In our case, we obtain

$$\tilde{r}_2 = \frac{1}{\sinh\mu}\begin{pmatrix} -\cosh\mu & 1 \\ -1 & \cosh\mu \end{pmatrix}.$$

Hence we can write

$$\tilde{d} = \tilde{r}\tilde{r}_2\tilde{e}_\tau. \tag{4}$$

5.4. Two hyperbolics, one parabolic or elliptic. As above, there is one special case, where S_0 has signature $(1,1)$; we take up that case below. Here we only assume that a_1 and a_2 are hyperbolic, and that a_3 is parabolic or primitive elliptic. As above, our normalization yields that $a_1(z) = e^{2\lambda_1}z$, and a_2 has its fixed points at $e^{\pm\mu}$, $\mu > 0$. Since a_1 and a_2 need to be appropriately oriented, we place the repelling fixed point of a_2 at $e^{+\mu}$. Then, if a_3 is parabolic, it has its fixed point at e^{λ_1}. If a_3 is primitive elliptic of order α, then it has its fixed point in the first quadrant on the circle $|z| = e^{\lambda_1}$. We write

$$\tilde{a}_1 = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{-\lambda_1} \end{pmatrix}; \qquad \tilde{a}_2 = \frac{1}{\sinh\mu}\begin{pmatrix} \sinh(\mu-\lambda_2) & \sinh\lambda_2 \\ -\sinh\lambda_2 & \sinh(\mu+\lambda_2) \end{pmatrix},$$

where μ is defined by

$$\coth\mu = \frac{\cosh\lambda_1\cosh\lambda_2 + \cosh\lambda_3}{\sinh\lambda_1\sinh\lambda_2}, \qquad \mu > 0. \tag{5}$$

Remark 5.2. The formula here for μ is the same as that in equation (1); the geometric meaning is the same in both cases.

5.5. The torus with one puncture or orbifold point. In this case we change our standard normalization, and require a_1 to be parabolic or primitive elliptic. Then $\lambda_2 = \lambda_3 > 0$. We normalize so that A_1 lies on the positive imaginary axis (A_1 is the point at infinity if a_1 is parabolic), and so that the unit circle is the common orthogonal between A_2 and A_3, where the fixed points of a_3 are positive, with the repelling fixed point larger than the attracting one, and the fixed points of a_2 are negative. Then r_0, the reflection in the imaginary axis, conjugates a_2 into a_3^{-1}. We can write the fixed points of a_2 as $-e^{\pm\mu}$ and the fixed points of a_3 as $e^{\pm\mu}$. We write

$$\tilde{a}_2 = \frac{1}{\sinh\mu}\begin{pmatrix} \sinh(\mu+\lambda_2) & \sinh\lambda_2 \\ -\sinh\lambda_2 & \sinh(\mu-\lambda_2) \end{pmatrix},$$
$$\tilde{a}_3 = \frac{1}{\sinh\mu}\begin{pmatrix} \sinh(\mu-\lambda_2) & \sinh\lambda_2 \\ -\sinh\lambda_2 & \sinh(\mu+\lambda_2) \end{pmatrix}.$$

Since we require $\operatorname{tr}(\tilde{a}_2\tilde{a}_3) = -2\cosh\lambda_1$, easy computations show that

$$\sinh^2(\mu) = \frac{2\sinh^2\lambda_2}{\cosh\lambda_1 + 1}. \tag{6}$$

Remark 5.3. The formula for μ given in equation (6) is different from that given in equations (1) and (5) because the underlying geometry is different. We still have $\coth\mu = \cosh\delta$, but here δ is the distance from A_2 to the imaginary axis, which is the common orthogonal of A_3 with the common orthogonal of A_1 and A_2.

As in Section 5.3 we also need a matrix representing the handle-closer d, which conjugates a_2 onto a_3^{-1} while introducing a twist of 2τ along A_2. We write $d = r_0 f_\tau r_2$, where r_2 is the reflection in A_2, f_τ is the twist by 2τ along the imaginary axis, and r_0 is the reflection in the imaginary axis. The corresponding matrices are given by

$$\tilde{r}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{r}_2 = \frac{1}{\sinh\mu}\begin{pmatrix} \cosh\mu & 1 \\ -1 & -\cosh\mu \end{pmatrix}, \quad \tilde{f}_\tau = \begin{pmatrix} e^\tau & 0 \\ 0 & e^{-\tau} \end{pmatrix}.$$

5.6. Exactly one hyperbolic and at least one parabolic. Here $\lambda_1 > 0$, $\lambda_2 = 0$, and either $\lambda_3 = 0$ or $\lambda_3 = i\pi/\alpha$. We revert to our standard normalization, so that $a_1(z) = e^{2\lambda_1}z$, and a_2 has its fixed point at $+1$. Then a_3 has its fixed point in the right half-plane on the circle $|z| = e^{\lambda_1}$.

We write the matrices

$$\tilde{a}_1 = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{-\lambda_1} \end{pmatrix}, \qquad \tilde{a}_2 = \begin{pmatrix} 1+\beta & -\beta \\ \beta & 1-\beta \end{pmatrix}.$$

Using Theorem 4.1, easy computations show that

$$\beta = -\frac{\cosh\lambda_1 + \cosh\lambda_3}{\sinh\lambda_1}.$$

5.7. The elementary pants groups. In the special case that $\lambda_1 > 0$ and $\lambda_2 = \lambda_3 = 0$, we write

$$\tilde{a}_1 = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{-\lambda_1} \end{pmatrix}, \qquad \tilde{a}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The group generated by these is of course elementary.

5.8. Exactly one hyperbolic and no parabolics. Here $\lambda_1 > 0$, $\lambda_2 = i\pi/\alpha_2$, and $\lambda_3 = i\pi/\alpha_3$. We have $a_1(z) = e^{2\lambda_1}z$, and a_2 has its fixed point on the unit circle in the (open) right half-plane. Then as above, a_3 has its fixed point in the right half-plane on the circle $|z| = e^{\lambda_1}$. Denote the fixed point of a_2 by $e^{\mu} = e^{i\theta}$, $0 < \theta < \frac{1}{2}\pi$.

As in [12, p. 6], we can write

$$\tilde{a}_1 = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{-\lambda_1} \end{pmatrix}, \qquad \tilde{a}_2 = \frac{1}{\sinh\mu}\begin{pmatrix} \sinh(\mu - \lambda_2) & \sinh\lambda_2 \\ -\sinh\lambda_2 & \sinh(\mu + \lambda_2) \end{pmatrix}.$$

As above, we obtain

$$\coth\mu = \frac{\cosh\lambda_1\cosh\lambda_2 + \cosh\lambda_3}{\sinh\lambda_1\sinh\lambda_2}. \tag{7}$$

Remark 5.4. Here, $\coth\mu = -i\sinh\delta$, where δ is the distance from A_1 to A_2. This gives a known formula for one side of a quadrilateral with two adjacent right angles, in terms of the other two angles, and the distance between the two right angles (see [5]).

5.9. The classical triangle groups. For the sake of completeness, since this form seems not to be known in the literature—although a related form can be found in [9], we write down matrices for geometric generators for the general Fuchsian $(\alpha_1, \alpha_2, \alpha_3)$-triangle group.

The case of three parabolics is well known and needs no further discussion.

If $\lambda_1 = \lambda_2 = 0$, and $\lambda_3 = i\pi/\alpha_3$, then we normalize so that $a_1(z) = z + 1$, and a_2 has its fixed point at 0; we write

$$\tilde{a}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{a}_2 = \begin{pmatrix} 1 & 0 \\ -2 - 2\cosh\lambda_3 & 1 \end{pmatrix}.$$

If $\lambda_1 = 0$, $\lambda_2 = i\pi/\alpha_2$ and $\lambda_3 = i\pi/\alpha_3$, then we normalize so that a_1 has its fixed point at ∞, with $a_1(0) > 0$, and so that a_2 has its fixed point at i. Then, since a_1 and a_2 are appropriately oriented, $a_2(\infty) < 0$. We write

$$\tilde{a}_1 = \begin{pmatrix} 1 & (2\cosh\lambda_2 + 2\cosh\lambda_3)/(-i\sinh\lambda_2) \\ 0 & 1 \end{pmatrix}, \ \tilde{a}_2 = \begin{pmatrix} \cosh\lambda_2 & -i\sinh\lambda_2 \\ i\sinh\lambda_2 & \cosh\lambda_2 \end{pmatrix}.$$

Finally, if $\lambda_j = i\pi/\alpha_j$, $j = 1, 2, 3$, then we change our view of $\mathbf{H}^2$, which we now regard as being the unit disc, and we normalize so that $a_1(z) = e^{2\pi i/\alpha_1}z = e^{2\lambda_1}z$, and a_2 has its fixed point at $e^{-\mu}$, $\mu > 0$. We write

$$\tilde{a}_1 = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{-\lambda_1} \end{pmatrix}; \qquad \tilde{a}_2 = \frac{1}{\sinh\mu}\begin{pmatrix} \sinh(\mu + \lambda_2) & -\sinh\lambda_2 \\ \sinh\lambda_2 & \sinh(\mu - \lambda_2) \end{pmatrix},$$

where μ is given by

$$\coth\mu = -\frac{\cosh\lambda_1\cosh\lambda_2 + \cosh\lambda_3}{\sinh\lambda_1\sinh\lambda_2}. \tag{8}$$

Remark 5.5. Here, as in equation (1), $\coth\mu = \cosh\delta$, where δ is the distance from A_1 to A_2. This equation for δ is one of the two hyperbolic laws of cosines (see [2]).

Note that we can solve equation (8) for $\mu > 0$ if and only if the right-hand side is > 1, which occurs if and only if

$$\frac{\pi}{\alpha_1} + \frac{\pi}{\alpha_2} + \frac{\pi}{\alpha_3} < \pi.$$

6. From F-N systems to homotopy bases

We now assume we are given an F-N system on the hyperbolic orbifold S_0, where the coordinate geodesics $L_1, \dots, L_p$, the pairs of pants, $P_1, \dots, P_q$, and the boundary elements, $b_1, \dots, b_{n+m}$, are ordered as in 2.1.1, and the coordinate geodesics are directed as in 2.4. We also assume we are given a point

$$\Phi = (s_1, \dots, s_{p+m}, t_1, \dots, t_p) \in (\mathbf{R}^+)^{p+m} \times \mathbf{R}^p,$$

in the corresponding space of F-N coordinates.

In this section, we give a canonical procedure for writing down a set of generators for the corresponding Fuchsian group, where these are described as hyperbolic isometries; we find explicit matrices for these generators in the next section.

We write the generators in the following order. The first $2p-q+1$ generators are hyperbolic; their axes project, in order, to the p coordinate geodesics, followed by the $p - q + 1$ handle closers. The axes of the remaining $n + m$ generators project, in order, to the boundary elements of S_0. We note that the total number of generators, $d = 2q + 1 = 4g - 3 + 2n + 2m$, is in general far from minimal.

6.1. Normalization. Our normalization is somewhat unusual in that we require more than can normally be achieved by conjugation by an orientation-preserving isometry. We achieve this by including the possibility of replacing a_1 by a_1^{-1} and/or replacing a_2 by a_2^{-1}.

We will use the following *standard normalization* throughout. Let $a_1, \dots, a_d$ be a set of generators for a Fuchsian group. For our purposes, we can assume that a_1 and a_2 are both hyperbolic with disjoint axes. We require that A_1 lie on the imaginary axis, pointing towards ∞; A_2 lies in the right half-plane, with the attracting fixed point smaller than the repelling one; and the common orthogonal between A_1 and A_2 lies on the unit circle.

6.2. Cutting and extending holes. Let $S = \mathbf{H}^2/G$ be an orbifold with a hole. Let H be an open half-plane in $\mathbf{H}^2$ lying over the hole, and let $\widehat{K}$ be the complement of the union of the translates of H. Then $S' = \widehat{K}/G$ is the orbifold obtained from S by *cutting off the infinite end of the hole.*

There is an obvious process that reverses the above; we say that S is obtained from S' by *completing the hole.*

6.3. Basic building blocks. The cases where $q = 1$ have already been dealt with; we assume $q > 1$. Each P_i has three distinct boundary elements, which are labeled as $b_{i,1}, b_{i,2}, b_{i,3}$. Except for the cases where we have not yet distinguished between $b_{i,2}$ and $b_{i,3}$, the order of these boundary elements is determined by the order of the coordinate geodesics and boundary elements of the F-N system. Further, the size of each hyperbolic boundary element is specified by Φ.

Let P_i be one of the pairs of pants of our F-N system, and let P_i' be P_i with its incomplete holes completed—these are the holes corresponding to the coordinate geodesics. For each $i = 1, \dots, q$, there is a unique fully normalized pants group H_i representing P_i'. That is, $\mathbf{H}^2/H_i = P_i'$; H_i has three distinguished generators $a_{i,1}, a_{i,2}, a_{i,3}$, where $a_{i,1}a_{i,2}a_{i,3} = 1$, and $A_{i,k}$ projects onto $b_{i,k}$, $k = 1, 2, 3$. Even in the cases where we have not yet distinguished between $b_{i,2}$ and $b_{i,3}$, the fully normalized pants group H_i, with its three distinguished generators, is uniquely determined.

6.4. Base points. We will need a canonical base point on each $A_{i,k}$. For $i = 1, \dots, q$, the point i is the canonical base point on $A_{i,1}$; for $i = 1, \dots, q$, and for $k = 2, 3$, the base point on $A_{i,k}$ is the point of intersection of $A_{i,k}$ with the common orthogonal between $A_{i,1}$ and $A_{i,k}$.

6.5. The primary chain. For $j = 1, \dots, q-1$, we define the suborbifold Q_j as the interior of the closure of the union of the P_i, $i \leq j$. In general, Q_j is incomplete, let Q_j' be the orbifold obtained by completing the incomplete holes of Q_j.

Each Q_j' has a naturally defined F-N system, where the coordinate geodesics are $L_1, \dots, L_{j-1}$, and the pairs of pants are $P_1, \dots, P_j$. The order of the boundary elements of Q_j' will be described below.

Let $H_1 = J_1$ be the fully normalized pants group representing P_1; then the imaginary axis, which is the axis of $a_{1,1}$, projects to L_1; the positive direction of $A_{1,1}$ projects to the positive direction of L_1.

Let H_2 be the fully normalized pants group representing P_2. Let c_2 be the hyperbolic isometry which maps the right half-plane onto the left half-plane, while introducing a twist of $2\tau_1$ in the positive direction on L_1; that is, $c_2(0) = \infty$, $c_2(\infty) = 0$, and $c_2(i) = e^{2\tau_1}i$.

Let $\widehat{H}_2 = c_2 H_2 c_2^{-1}$. Then the action half-plane of $a_{1,1} = \hat{a}_{1,1}$ is equal to the boundary half-plane of $\hat{a}_{2,1} = c_2 a_{2,1} c_2^{-1}$. It follows that one can use the AFP combination theorem (First combination theorem in [12]) to amalgamate $\widehat{H}_2$ to H_1. Set $J_2 = \langle H_1, \widehat{H}_2 \rangle$. The subgroups $\widehat{H}_1 = H_1$ and $\widehat{H}_2$ are the *distinguished* subgroups of J_2. The following now follow from the AFP combination theorem.

(i) J_2 is Fuchsian.

(ii) J_2 is generated by $a_{1,1}, a_{1,2}, a_{1,3}, \hat{a}_{2,2}, \hat{a}_{2,3}$; in addition to the defining relations of H_1 and H_2, these satisfy the one additional relation: $a_{1,1} = \hat{a}_{2,2}\hat{a}_{2,3}$.

(iii) $\mathbf{H}^2/J_2$ has signature $(0,4)$; the corresponding boundary subgroups are generated by the above four generators.

It is clear that one can canonically identify $\mathbf{H}^2/J_2$ with Q_2'. This imposes a new order on the boundary elements of $\mathbf{H}^2/J_2$, as follows. If b and b' are boundary elements of Q_2, where b precedes b' as coordinate geodesics or as boundary elements of S_0, then b precedes b' as a boundary element of Q_2. If b and b' both correspond to the same coordinate geodesic on S_0, and b corresponds to a boundary geodesic on P_1, while b' corresponds to a boundary geodesic on P_2, then, as boundary elements of Q_2, b precedes b'. Finally, if b and b' both correspond to boundary elements of either P_1 or P_2, then b precedes b' if b corresponds to $A_{i,2}$ and b' corresponds to $A_{i,3}$.

In the case that P_i has two boundary elements corresponding to the same handle geodesic, L, we now direct L so that the positive direction of $A_{i,2}$ projects onto the positive direction of L. We note that we have now given an order to the boundary elements of Q_2, and directed them.

We introduce a new ordered set of generators for J_2 as $a_1^2, \ldots, a_5^2$, where $a_1^2 = a_{1,1} = \hat{a}_{2,1}^{-1}$, and $a_2^2, \ldots, a_5^2$, are the generators $a_{1,2}, a_{1,3}, \hat{a}_{2,2}$, $\hat{a}_{2,3}$, where these have been rearranged so as to be in proper order; i.e., A_j^2 projects onto b_{j-1}^2. We remark that those A_j^2 that are hyperbolic are all directed so that the attracting fixed point of a_j^2 is smaller than the repelling fixed point.

Each A_j^2 has a canonical base point on it. In the case that $a_j^2 = a_{1,i}$, the base point is the canonical base point for $a_{1,i}$; in the case that $a_j^2 = c_2 a_{2,i} c_2^{-1}$, then the canonical base point is the c_2 image of the canonical base point for $a_{2,i}$.

We now iterate the above process. Assume that we have found J_k, with distinguished subgroups, $\widehat{H}_1, \ldots, \widehat{H}_k$, representing Q_k', where $k < q$. Assume that we have ordered the boundary elements of Q_k, and that we have found the distinguished generators, $a_1^k, \dots, a_{2k+1}^k$, for J_k, where each distinguished generator lies in some distinguished subgroup, so that, for $i = 1, \dots, k$, A_i^k projects onto the coordinate geodesic L_i, and, for $i = k+1, \dots, 2k+1$, A_i^k projects onto the boundary element b_{k-i}^k of Q_k. We assume that we have assigned a canonical base point on each of the geodesics A_i^k, $i = k+1, \dots, 2k+1$, and we also assume that all the hyperbolic boundary generators of J_k are directed so that the attracting fixed point is smaller than the repelling fixed point. Then there is some j so that A_{k+j}^k projects onto L_{k+1}. We need to renormalize H_{k+1} so that the renormalized $A_{k+1,1}$ agrees with A_{k+j}^k, but with the opposite orientation, and with an appropriate twist.

We define the *conjugator* c_{k+1} to be the unique orientation-preserving hyperbolic isometry mapping the left half-plane onto the action half-plane of A_{k+j}^k, while mapping the base point i onto the point whose distance from the base point on A_{k+j}^k, measured in the positive direction along A_{k+j}^k, is exactly $2\tau_k$. As above, the AFP combination theorem assures us that $J_{k+1} = \langle J_k, c_{k+1} H_{k+1} c_{k+1}^{-1} \rangle$ satisfies the following.

(i) J_{k+1} is Fuchsian.

(ii) $J_{k+1} = \langle a_1^k \dots, a_{2k+1}^k, c_{k+1} a_{k+1,1} c_{k+1}^{-1}, c_{k+1} a_{k+1,2} c_{k+1}^{-1}, c_{k+1} a_{k+1,3} c_{k+1}^{-1} \rangle$; these satisfy the defining relation(s) of J_k, the defining relation(s) of H_{k+1}, together with the additional defining relation: $a_{k+j}^k = (c_{k+1} a_{k+1,1} c_{k+1}^{-1})^{-1}$.

(iii) $\mathbf{H}^2 / J_{k+1}$ has signature $(0, 2k+1)$.

The *distinguished* subgroups of J_{k+1} are the distinguished subgroups of J_k, together with $\widehat{H}_{k+1} = c_{k+1} H_{k+1} c_{k+1}^{-1}$.

As above, we rewrite the generators of J_{k+1} as $a_1^{k+1}, \dots, a_{2k+3}^{k+1}$, where the first k generators correspond in order to the k coordinate geodesics of Q_{k+1}, and the remaining generators correspond to the boundary elements of Q_{k+1} in the following order. Each boundary generator a_i of J_{k+1} corresponds to either a coordinate geodesics L_j on S_0, or it corresponds to a boundary element b_j of S_0; pulling back the order of the coordinate geodesics and boundary elements from S_0 imposes a partial order on the boundary generators of J_{k+1}. The only ambiguities occur when the generators a_i and $a_{i'}$ both correspond to the same coordinate geodesic. If a_i, respectively, $a_{i'}$, lies in the distinguished subgroup $\widehat{H}_j$, respectively, $\widehat{H}_{j'}$, where $j < j'$, then a_i precedes $a_{i'}$; if a_i and $a_{i'}$ both lie in the same distinguished subgroup, $\widehat{H}_j$, then $\hat{a}_{j,2}$ precedes $\hat{a}_{j,3}$.

The hyperbolic boundary generators of J_k all have distinguished base points; we assign the c_k image of the distinguished base point on $a_{k+1,2}$ and $a_{k+1,3}$ as the distinguished base point on the new boundary generators of J_{k+1}. We also observe that the hyperbolic boundary generators of J_{k+1} are all directed so that their attracting fixed points are smaller than their repelling fixed points.

When $k = q-1$, we reach the group J_q, representing Q'_q. Note that, as part of the above process, we have ordered and directed those boundary geodesics of Q_q that correspond to handle geodesics on S_0, where the same pair of pants appears on both sides of the handle geodesic.

6.6. Closing the handles. We rename the group J_q, and now call it K_0. We also rename its ordered set of generators and call them, in order, $a_1^0, \ldots, a_{2q+1}^0$. The first $q-1$ of these generators correspond to the attaching coordinate geodesics of S_0; the next $2(p-q+1)$ generators correspond to the boundary elements of Q_q that are handle geodesics on S_0; the remaining generators correspond to boundary elements of both Q_q and S_0.

We define the first handle-closer d_1 as the orientation-preserving hyperbolic isometry mapping the action half-plane of a_q^0 onto the boundary half-plane of a_{q+1}^0, while mapping the point at distance $-2\tau_q$ from the base point on A_q^0 to the base point on A_{q+1}^0. Note that d_1 conjugates a_q^0 onto $(a_{q+1}^0)^{-1}$.

Set $K_1 = \langle K_0, d_1 \rangle$. The following follow from the second combination theorem (HNN extension):

(i) K_1 is Fuchsian.

(ii) K_1 is generated by $a_1^0 \ldots, a_{2q+1}^0, d_1$; these satisfy the defining relations of K_1, together with the additional relation: $a_{q+1}^0 = d_1 (a_q^0)^{-1} d_1^{-1}$.

(iii) $\mathbf{H}^2/K_1$ is an orbifold of signature $(1, q)$.

The *distinguished* subgroups of K_1 are the distinguished subgroups of K_0. The generators of K_1 are the generators of K_0, in the same order, but with a_{q+1}^0 deleted, and d_1 added to the list. In the list of generators for K_1, d_1 appears after all the generators corresponding to coordinate geodesics on S_0, and before the first generator corresponding to a boundary element of S_0.

There is no difficulty (other than notation) in iterating the above process. In the next iteration, we eliminate the generator a_{q+3}^0, and the new generator d_2 appears immediately after d_1.

After g iterations, we reach the discrete group $G = K_g$, where $\mathbf{H}^2/G = S_0$. Further, G has q distinguished subgroups, representing in order the q pairs of pants, $P_1, \ldots, P_q$; and G has $2q+1$ distinguished generators, $a_1, \ldots, a_{2q+1}$. For $i = 1, \ldots, p$, the axis A_i projects onto the coordinate geodesic L_i; the generators $a_{p+1}, \ldots, a_{2p-q+1}$ are handle closers; and the axes of the remaining generators project, in order, onto the boundary elements of S_0.

6.7. Summary. We started with an F-N system on S_0, together with the point Φ in the corresponding coordinate space, and constructed from these a set of generators for the Fuchsian group representing the deformation of S_0 determined by Φ. It is easy to observe that if Φ, respectively, Φ', are points in this F-N coordinate space, and $a_1, \ldots, a_d$, respectively, $a'_1, \ldots, a'_d$, are the corresponding sets of generators, then there is a quasiconformal deformation of the hyperbolic plane conjugating each a_i onto the corresponding a'_i.

We note that, among other things, we have shown the following.

Proposition 6.1. *Let $L_1, \dots, L_p$ be an F-N system on the hyperbolic orbifold S_0. Then there is a canonical procedure for choosing a basis for the (orbifold) fundamental group of S_0, so that, for $i = 1, \dots, p$, L_i is the shortest geodesic in the free homotopy class of the i-th generator.*

Remark 6.1. The above definition of the conjugators has the unfortunate consequence that the untwisted handle-closers do not in general preserve the common orthogonal between the axes of the two generators they conjugate. There is a relatively easy way to solve this problem, but this entails the loss of the explicit formulae for the entries in the matrices; see Section 14.

7. Explicit matrices

7.1. Reduction to primitive conjugators. Let H_i and H_{j+1} be the fully normalized pants groups representing the pairs of pants, P_i and P_{j+1}, respectively. We assume that $i < j+1$, and that there is a k, $1 \le k \le 3$, so that $a_{j+1,1}$ and $a_{i,k}$ represent the same attaching geodesic, L_j on S_0, but with reverse orientations. Then $|\operatorname{tr}(\tilde a_{j+1,1})| = |\operatorname{tr}(\tilde a_{i,k})|$. The *elementary conjugator* $e_{i,k}$ maps the left half-plane (the boundary half-plane of $a' = a_{j+1,1}$) onto the action half-plane of $a = a_{i,k}$, while introducing a twist of $2\tau_j$; that is, $e_{i,k}$ maps the base point on $A_{j+1,1}$ to the point on $A_{i,k}$ at distance $2\tau_j$ from the base point on $A_{i,k}$; since $i < j+1$, this is the positive direction on the projection of these axes. The *untwisted* elementary conjugator, $e^0 = e^0_{i,k}$, which is independent of the index j, also maps the left half-plane onto the action half-plane of a, but maps the base point on $A_{j+1,1}$ (this is the point i) to the base point on $A_{i,k}$.

Once we have found matrices for the elementary conjugators, then we can inductively find matrices for all the conjugators. If P_i and P_{j+1} are adjacent pairs of pants in the F-N system on S_0, with $i < j+1$, and, as above, $b_{j+1,1}$ attached to $b_{i,k}$, then, once we have found the matrix for the conjugator c_i, and we have found the matrix for the elementary conjugator, $e_{i,k}$, the matrix for the conjugator c_{j+1} is given by

$$\tilde c_{j+1} = \tilde c_i \tilde e_{i,k}. \tag{9}$$

For each $j = q, \dots, p$, the handle closer d_j conjugates one distinguished boundary generator of K_0 onto the inverse of another distinguished boundary generator. These two boundary generators either lie in the same distinguished subgroup, or they lie in different distinguished subgroups.

If the two boundary generators lie in the same distinguished subgroup, $\widehat H_i$, then there is a conjugator c_i, and there is a fully normalized pants group H_i, so that $\widehat H_i = c_i H_i c_i^{-1}$. In this case, we can choose the matrix for the handle closer as

$$\tilde d_j = \tilde c_i \tilde d \tilde c_i^{-1}, \tag{10}$$

where $\tilde{d}$ is given in equation (4).

If these two boundary generators lie in distinct distinguished subgroups, say $\widehat{H}_i$ and $\widehat{H}_{i'}$, where $i < i'$, where the first boundary generator corresponds to $\hat{a}_{i,k}$, and the second corresponds to $\hat{a}_{i',k'}$ then we write d_j as a product of four transformations: first we twist along the axis $\hat{A}_{i,k}$; then we map the boundary half-plane of $a_{i,k}$ onto the right half-plane; then we interchange left and right half-planes; and last, we map the right half-plane onto the boundary half-plane of $\hat{a}_{i',k'}$.

The first transformation preserves both sides of $\hat{A}_{i,k}$ and maps the point on $\hat{A}_{i,k}$, whose distance from the base point is $-2\tau_j$, to the base point on $\hat{A}_{i,k}$. We can write the matrix for this transformation as

$$(\tilde{c}_i\tilde{e}^0_{i,k})\tilde{f}_{\tau_j}(\tilde{c}_i\tilde{e}^0_{i,k})^{-1}, \tag{11}$$

where f_τ is the universal twist map, $f_\tau(z) = e^{-2\tau}z$.

The second transformation maps the boundary half-plane of $\hat{a}_{i,k}$ onto the right half-plane, and maps the base point on $\hat{A}_{i,k}$ to the base point on the imaginary axis. We can write the matrix for this transformation as

$$(\tilde{c}_i\tilde{e}^0_{i,k})^{-1}. \tag{12}$$

The *interchange* transformation g interchanges left and right half-planes and preserves the base point on the imaginary axis; that is, $g(z) = -1/z$.

The final transformation maps the left half-plane onto the boundary half-plane of $\hat{a}_{i',k'}$; the matrix for this transformation can be written as

$$\tilde{c}_{i'}\tilde{e}^0_{i',k'}. \tag{13}$$

Combining equations (11)–(13), we obtain

$$\tilde{d}_j = \tilde{c}_{i'}\tilde{e}^0_{i',k'}\tilde{g}\tilde{f}_{\tau_j}(\tilde{c}_i\tilde{e}^0_{i,k})^{-1}. \tag{14}$$

Hence, in this case as well, we can write the matrix for the handle closer, once we know the sizes of the attaching geodesics, and we have the matrices for the twisted and untwisted elementary conjugators.

7.2. The universal twist map and interchange. We represent the *universal twist map*, f_τ, which twists by -2τ in the positive direction along the imaginary axis, and the interchange transformation g, as follows:

$$\tilde{f}_\tau = \begin{pmatrix} e^{-\tau} & 0 \\ 0 & e^{\tau} \end{pmatrix}; \qquad \tilde{g} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

7.3. The elementary conjugator for $a_{i,1}$. We first take up the case that the attaching coordinate geodesic L_j corresponds to both $a' = a_{j+1,1}$ and $a = a_{i,1}$. This case occurs only for $i = j = 1$. We note that the left half-plane is the boundary half-plane for both a' and a.

The interchange transformation g interchanges the right and left half-planes in $\mathbf{H}^2$, and fixes the point i, which is the base point on both A and A'. Hence, we can choose $\tilde{g}$ as the matrix for the untwisted elementary conjugator $\tilde{e}^0_{2,1}$.

Then the matrix for $e_{2,1}$, is given by

$$\tilde{e}_{2,1} = \tilde{e}^0_{2,1}\tilde{f}_{\tau_1} = \begin{pmatrix} 0 & -e^{\tau_1} \\ e^{-\tau_1} & 0 \end{pmatrix}.$$

7.4. The elementary conjugator for $a_{i,2}$. We next take up the case that $k = 2$. Since the base point of $A = A_{i,2}$ is the point of intersection of A with the common orthogonal of A and $A_{i,1}$, we can choose $e^0_{i,2}$ to be the composition of the reflection r_0 in $A_{i,1}$, followed by the reflection r_{12} in the line halfway between $A_{i,1}$ and $A_{i,2}$.

We have already found the matrix for r_0 as

$$\tilde{r}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since we can locate the fixed points of a at $e^{\pm\mu_i}$, easy computations show that we can choose

$$\tilde{r}_{12} = \frac{1}{\sqrt{2\sinh\mu_i}} \begin{pmatrix} \exp(\frac{1}{2}\mu_i) & -\exp(-\frac{1}{2}\mu_i) \\ \exp(-\frac{1}{2}\mu_i) & -\exp(\frac{1}{2}\mu_i) \end{pmatrix},$$

where μ_i is defined by equation (1) or (5) or (7), depending on the type of $a_{i,3}$.

We can also write the above as

$$\tilde{r}_{12} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\coth\mu_i + 1} & -\sqrt{\coth\mu_i - 1} \\ \sqrt{\coth\mu_i - 1} & -\sqrt{\coth_i\mu + 1} \end{pmatrix}. \tag{15}$$

Remark 7.1. Up to this point, the entries in all our matrices were rational functions of hyperbolic sines and cosines of lengths of closed geodesics, or of geodesic arcs. We see from equation (15) that, for Fuchsian groups representing sufficiently complicated surfaces or orbifolds, one must introduce square roots of these hyperbolic sines and cosines.

We can now write

$$\tilde{e}^0_{i,2} = \tilde{r}_{12}\tilde{r}_0 = \frac{1}{\sqrt{2\sinh\mu_i}} \begin{pmatrix} \exp(\frac{1}{2}\mu_i) & \exp(-\frac{1}{2}\mu_i) \\ \exp(-\frac{1}{2}\mu_i) & \exp(\frac{1}{2}\mu_i) \end{pmatrix}, \tag{16}$$

and

$$\tilde{e}_{i,2} = \tilde{e}^0_{i,2}\tilde{f}_{\tau_j} = \frac{1}{\sqrt{2\sinh\mu_i}}\begin{pmatrix} \exp(\frac{1}{2}\mu_i - \tau_j) & \exp(-\frac{1}{2}\mu_i + \tau_j) \\ \exp(-\frac{1}{2}\mu_i - \tau_j) & \exp(\frac{1}{2}\mu_i + \tau_j) \end{pmatrix}. \tag{17}$$

7.5. The primitive conjugator for a_3. For $k = 3$, the base point on $A_{i,3}$ is at the point where $A_{1,3}$ meets the common orthogonal with $A_{i,1}$. However, the base point on $A_{i,1}$ is the point where it meets the common orthogonal with $A_{i,2}$. We write $e^0_{i,3} = r_{13}r_0f_{-\lambda_i}$, where r_0 is as above, r_{13} is the reflection in the line halfway between $A_{i,1}$, and $A_{i,3}$, and λ_i is the size of $a_{i,1}$; that is, $|\operatorname{tr}(a_{i,1})| = 2\cosh\lambda_i$.

We have already observed that the common orthogonal between $A_{i,1}$ and $A_{i,3}$ lies on the circle of radius e^{λ_i}. Also, the fixed points of $a = a_{i,3}$ are at $e^{\lambda_i \pm \nu_i}$, where ν_i is defined by equation (2). Hence, we can choose

$$\tilde{r}_{13} = \frac{1}{\sqrt{2\sinh\nu_i}}\begin{pmatrix} \exp\big(\frac{1}{2}(\nu_i + \lambda_i)\big) & \exp\big(\frac{1}{2}(-\nu_i + \lambda_i)\big) \\ \exp\big(\frac{1}{2}(-\nu_i - \lambda_i)\big) & \exp\big(\frac{1}{2}(\nu_i - \lambda_i)\big) \end{pmatrix}.$$

We now write

$$\tilde{e}^0_{i,3} = \tilde{r}_{13}\tilde{r}_0\tilde{f}_{-\lambda_i} = \frac{1}{\sqrt{2\sinh\nu_i}}\begin{pmatrix} \exp(\frac{1}{2}\nu_i + \lambda_i) & \exp(-\frac{1}{2}\nu_i) \\ \exp(-\frac{1}{2}\nu_i) & \exp(\frac{1}{2}\nu_i - \lambda_i) \end{pmatrix}, \tag{18}$$

and

$$\tilde{e}_{i,3} = \tilde{e}^0_{i,3}\tilde{f}_{\tau_j} = \frac{1}{\sqrt{2\sinh\nu_i}}\begin{pmatrix} \exp(\frac{1}{2}\nu_i + \lambda_i - \tau_j) & \exp(-\frac{1}{2}\nu_i + \tau_j) \\ \exp(-\frac{1}{2}\nu_i - \tau_j) & \exp(\frac{1}{2}\nu_i - \lambda_i + \tau_j) \end{pmatrix}. \tag{19}$$

8. The algorithm

Assume we are given an explicit F-N system on a hyperbolic orbifold; this geometric information is given as the signature $(g, n, m; \alpha_1, \dots, \alpha_n)$ of S_0, and the *pairing table*, which has q rows, one for each pair of pants P_i, and three columns, one for each boundary element of P_i. The entry in the i-th row and k-th column identifies the boundary element $b_{i,k}$ as corresponding to either a coordinate geodesic L_j, or a boundary element b_j of S_0.

We also assume we are given the point Φ in the appropriate coordinate space.

Step 1. For each $i = 1, \dots, q$, and for $k = 1, 2, 3$, we read off from the pairing table whether $b_{i,k}$ corresponds to a coordinate geodesic or to a boundary element. If $b_{i,k}$ corresponds to L_j, then we read off the size of $a_{i,k}$ from the j-th entry in Φ. Each $b_{i,1}$ corresponds to an attaching geodesic; we write the size of this geodesic as λ_i. If $b_{i,k}$ corresponds to a boundary element, then we read off the type of this boundary element from the signature of S_0; if the type is hyperbolic, then we read off the size of $a_{i,k}$ from Φ; if the type is parabolic or elliptic, we read off the order from the signature.

We use the constructions of Section 5 to write down the matrices $\tilde{a}_{i,k}$. We remark that, in practice, we will not need all of these.

Step 2. The first conjugator c_1 is the identity. We have already constructed the matrix for the second conjugator; it is the elementary conjugator $\tilde{e}_{2,1}$.

Continuing inductively, assume that we have found matrices for the conjugators, $c_1, \dots, c_{j-1}$, $j \le q$. The attaching geodesic L_j appears in the pairing table once as $b_{j+1,1}$, and once as some $b_{i,k}$, $i < j+1$, $k > 1$. We have already constructed $\tilde{c}_i$; equation (9) then gives the formula for $\tilde{c}_j$.

Step 3. Write the matrices for the first $q-1$ generators. These are the generators corresponding to the attaching geodesics. We write:

$$\tilde{a}_1 = \tilde{a}_{1,1},\ \tilde{a}_2 = \tilde{c}_3 \tilde{a}_{3,1}^{-1} \tilde{c}_3^{-1}\ , \dots\ ,\ \tilde{a}_{q-1} = \tilde{c}_q \tilde{a}_{q,1}^{-1} \tilde{c}_q^{-1}.$$

Step 4. Find matrices for the generators corresponding to the handle geodesics; these are the generators $a_q, \dots, a_p$.

Each L_j, $q \le j \le p$, appears twice in the pairing table; either there is some i so that L_j appears as both $b_{i,2}$ and $b_{i,3}$, or there are two distinct rows, $i < i'$, so that L_j appears as $b_{i,k}$ and as $b_{i',k'}$.

In the first case, we write the matrix for the handle generator as $\tilde{a}_j = \tilde{c}_i \tilde{a}_{i,2} \tilde{c}_i^{-1}$.

In the second case, we write the matrix for the handle generator as $\tilde{a}_j = \tilde{c}_i \tilde{a}_{i,k} \tilde{c}_i^{-1}$.

Step 5. Find matrices for the handle closing generators, $a_{p+1}, \dots, a_{2p-q+1}$. We need to consider separately the same two possibilities as in the previous step.

If there is some j, $q \le j \le p$, so that L_j appears as both $b_{i,2}$ and $b_{i,3}$ in the pairing table, then we write $\tilde{a}_{p+j} = \tilde{d}_j$, where the formula for $\tilde{d}_j$ is given by equation (10).

For $q \le j \le p$, if the two entries of L_j in the pairing table appear as $b_{i,k}$ and $b_{i',k'}$, where $i < i'$, then we write $\tilde{a}_{p+j} = \tilde{d}_j$, where the formula for $\tilde{d}_j$ is given by equation (14). Note that, in order to use this equation, we need the quantities λ_i and $\lambda_{i'}$, defined in Step 1, and matrices for the untwisted elementary conjugators, $e^0_{i,k}$ and $e^0_{i',k'}$; these matrices are obtained by appropriate use of equations (16) and (18).

Step 6. Write down the matrices corresponding to the boundary elements of S_0. Each b_j, $j = 1, \dots, m+n$, appears exactly once in the pairing table. If b_j corresponds to $b_{i,k}$, then the matrix for the corresponding generator is given by $\tilde{a}_{2p+q+j} = \tilde{c}_i \tilde{a}_{i,k} \tilde{c}_i^{-1}$.

9. From matrices to F-N coordinates

In this section, we start with a finite set of matrices, $\tilde{a}_1, \dots, \tilde{a}_d \in \mathrm{SL}(2, \mathbf{R})$, which we regard as Möbius transformations. We assume that these are a fully normalized set of distinguished generators defined by a coordinate point Φ for some F-N system on some orbifold S. We give a procedure for determining the signature of S; the pairing table of the F-N system; and the coordinate point Φ that these generators represent.

Remark 9.1. We could start with an arbitrary finite set of matrices, $\tilde{a}_1, \dots, \tilde{a}_d$, and write down necessary and sufficient conditions for the corresponding Möbius transformations to be a (not necessarily normalized) distinguished set of generators corresponding to some coordinate point in some F-N system. The conditions are easy to derive from the rules in 2.1.1, together with the rules for directing the coordinate and boundary geodesics. This would yield a set of sufficient conditions for the corresponding transformations to generate a discrete group.

9.1. Recovering the signature and the pairing table. We know that the transformations $a_1, \dots, a_d$, are in non-increasing order, so the last n of them are not hyperbolic. Also, $d = 2q + 1$, so q is also determined. We write $S = \mathbf{H}^2/G$, where $G = \langle a_1, \dots, a_d \rangle$.

We find the endpoints of the axes of the hyperbolic generators, and compute which pairs of these axes cross each other, and which pairs are disjoint. If the axes of the hyperbolic generators are not all disjoint, then there is a largest index p so that $A_1, \dots, A_p$ are all disjoint. In this case, since $g = p - q + 1$, g is determined. We now have that the projections of $A_1, \dots, A_{q-1}$ are the attaching geodesics, and the projections of $A_q, \dots, A_p$ are the handle geodesics. Then $a_{p+1}, \dots, a_{2p-q+1}$ are the handle closers, and the remaining generators correspond to boundary elements of S. We can use the traces of these last $m + n$ generators to find m, and to find the orders of the elliptic and parabolic generators. Hence, in this case, we know the signature $(g, m, n; \alpha_1, \dots, \alpha_n)$.

If the axes of the hyperbolic generators are all disjoint, then $g = 0$, $p = n + m - 3$, and $q = n + m - 2$. Hence $p = q - 1$ and $m = q - n + 2$. So in this case as well, we know the signature of G.

We also need the (unordered) set of *basic generators*. These are the generators other than the handle-closers, together with the handle generators conjugated by the corresponding handle-closing generators; that is, the basic generators are:

$$a_1, \dots, a_p, a_{2p-q}, \dots, a_d, a_{p+1} a_q a_{p+1}^{-1}, \dots, a_{2p-q+1} a_{2p-q} a_{2p-q+1}^{-1}.$$

The axes of the hyperbolic basic generators divide the hyperbolic plane into regions. Each of these regions either contains three axes of distinguished generators in its closure, in which case the three corresponding basic generators generate one of the distinguished pants subgroups, or the region contains exactly one axis of a distinguished generator in its closure, in which case the corresponding generator is boundary hyperbolic. For $j = 1, \dots, q$, the attaching geodesic L_j appears in the pairing table as both $b_{j+1,1}$ and as some $b_{i,k}$, $i < j + 1$. Since we know that $a_j = \hat{a}_{j+1,1}$, $j = 1, \dots, q$, we can order the above regions as corresponding to $P_1, \dots, P_q$. If the other two axes on the boundary of the region corresponding to P_j are translates of A_i and $A_{i'}$, where $i < i'$, then L_i, respectively, $L_{i'}$, is the entry in the second, respectively, third, column of the j-th row of the pairing table. If $i = i'$, then L_i is the entry in both the second and third column of this row. This completes the reconstruction of the pairing table.

9.2. Recovering the F-N coordinates. The F-N coordinates

$$\Phi = (s_1, \dots, s_{p+m}, t_1, \dots, t_p)$$

can now be read off as follows.

The numbers $s_1, \dots, s_p$ are almost immediate. For $i = 1, \dots, p$, we have $s_i = \sinh \sigma_i$, where $\cosh \sigma_i = 2|\operatorname{tr}(\tilde{a}_i)|$. Similarly, for $i = p+1, \dots, p+m$, $s_i = 2\sinh \sigma_i$, where $\cosh \sigma_i = 2|\operatorname{tr}(\tilde{a}_{2p-q+1+i})|$.

Each of the twists about an attaching geodesic is of the following form. We have three basic generators, a, a' and a'', with disjoint axes, where A separates A' from A''. The twist 2τ is the distance between the point of intersection on A of the common orthogonal, N', of A with A', and the point of intersection on A of the common orthogonal, N'', of A with A''. We can now compute $2|\tau|$, for N' is the axis of the Fenchel–Jørgensen commutator, $r' = aa' - a'a$, and N'' is the axis of $r'' = aa'' - a''a$. These are both half-turns; their product, when represented by a matrix in $\mathrm{SL}(2, \mathbf{R})$, has trace equal to $\pm 2\cosh \tau$. The sign of τ is also determined; although we do not have an explicit formula for it, for $\tau > 0$ if and only if a and $r''r'$ have the same attracting fixed point.

Once we know the twists about the attaching geodesics, we can reconstruct the conjugators, for each conjugator is a product of primitive conjugators, and each primitive conjugator is determined by the pairing table, the lengths of the coordinate geodesics, and the twists about the attaching geodesics.

Once we have the conjugators and the pairing table, we can reconstruct the elementary conjugators. It is then an exercise to reconstruct the untwisted elementary conjugators from the twisted ones.

For the twist about a handle geodesic, we note (see equations (10) and (14)) that the handle-closer is of the form $cf_\tau(c')^{-1}$, where c and c' are products of primitive conjugators. Since we are given matrices for the handle closing generators, and we can reconstruct these products of primitive conjugators, we can compute f_τ; hence $\sinh \tau$ is determined.

10. Recapitulation—statements of results

In this section, we combine the results of the preceding sections, and formally state the theorems we have proven.

10.1. The algorithm. Given a hyperbolic orbifold S_0, with finitely generated fundamental group, given $L_1, \dots, L_p$, an F-N system on S_0, and given a point $\Phi = s_1, \dots, s_{p+m}, t_1, \dots, t_p \in (\mathbf{R}^+)^{p+m} \times \mathbf{R}^m$, we have given in Section 8 an algorithm yielding explicit formulae for a set of matrices $\tilde{a}_1, \dots, \tilde{a}_d$, so that $G = \langle a_1, \dots, a_d \rangle$ is the Fuchsian group uniformizing the deformation of S_0 determined by the coordinate point Φ. As part of our procedures, we have written the entries in these matrices as smooth algebraic functions of Φ.

10.2. The Fenchel–Nielsen theorem. Combining our algorithm with the results in Section 9, we have a new proof of a somewhat stronger version of the original Fenchel–Nielsen theorem.

Let G_0 be a finitely generated Fuchsian group. Let $\mathscr{DF}(G_0)$ be the identity component of the space of discrete faithful represention[s] of G_0 into $\mathrm{PSL}(2,\mathbf{R})$, modulo conjugation. It is well known, assuming that G_0 is of cofinite volume, that $\mathscr{DF}(G_0)$ is real-analytically equivalent to the Teichmüller space of $\mathbf{H}^2/G_0$ (see Abikoff [1]); if G_0 is not of cofinite volume, then $\mathscr{DF}(G_0)$ is real-analytically equivalent to the corresponding reduced Teichmüller space (see Earle [4]).

Theorem 10.1. *Let G_0 be a finitely generated non-elementary Fuchsian group, with an F-N system defined on $S_0 = \mathbf{H}^2/G_0$. Then the algorithm for constructing matrix generators for deformations of G_0 defines a canonical algebraic diffeomorphism of $\mathscr{DF}(G_0)$ onto $(\mathbf{R}^+)^{m+p} \times \mathbf{R}^p$.*

10.3. Algebraic orbifolds. We easily obtain the following corollaries.

Theorem 10.2. *Let S be a hyperbolic orbifold with finitely generated fundamental group, with an F-N system defined on it; S is algebraic if and only if its F-N coordinates in this system are algebraic.*

Theorem 10.3. *Let S be a hyperbolic orbifold with finitely generated fundamental group. If the F-N coordinates of S are algebraic in one F-N system, then they are algebraic in any F-N system.*

10.4. The action of the Teichmüller modular group. For our final observation, we use the well-known fact that if α is an element of the Teichmüller modular group, and if $L_1, \dots, L_p$ is an F-N system on $S_0 = \mathbf{H}^2/G_0$, then α either preserves this F-N system, in which case it maps some F-N coordinate for S_0 in this system onto another coordinate point for the same system, or α maps the given F-N system onto a different F-N system on S_0. We have proven the following.

Theorem 10.4. *Let G_0 be a finitely generated Fuchsian group, and let an F-N system on $S_0 = \mathbf{H}^2/G_0$ be given. Let* Mod *denote the action of the* (*reduced*) *Teichmüller modular group on $(\mathbf{R}^+)^{m+p} \times \mathbf{R}^p$, where this action is given using the canonical identification of Theorem* 10.1. *Then* Mod *acts as a group of algebraic diffeomorphisms.*

11. Special case of genus 3

There are five topologically distinct F-N systems on a closed surface of genus 3, and there are many possible ways to order the geodesics in each of them. We choose one particular case, in which there is one dividing geodesic, necessarily L_1, and five non-dividing geodesics. Then L_2 and L_3 are necessarily attaching geodesics, while L_4, L_5 and L_6 are handle geodesics.

The pairing table for our particular F-N system is given below.

P_1	L_1	L_2	L_4
P_2	L_1	L_5	L_5
P_3	L_2	L_3	L_6
P_4	L_3	L_4	L_6

We also assume that we are given the point in parameter space

$$\Phi = s_1, \dots, s_6, t_1, \dots, t_6.$$

Then the quantities σ_i and τ_i are given by $s_i = \sinh \sigma_i$, $\sigma_i > 0$, and $t_i = \sinh \tau_i$. We need to write down 9 matrices, $\tilde{a}_1, \dots, \tilde{a}_9$, so that, for $i = 1, \dots, 6$, A_i projects onto L_i, and for $i = 7, 8, 9$, a_i is the handle closer for L_{i-3}.

Step 1. The generating matrices for the four fully normalized pants groups, $H_1, \dots, H_4$, are as follows.

$$\tilde{a}_{1,1} = \begin{pmatrix} e^{\sigma_1} & 0 \\ 0 & e^{-\sigma_1} \end{pmatrix};$$
$$\tilde{a}_{1,2} = \frac{1}{\sinh \mu_1} \begin{pmatrix} \sinh(\mu_1 - \sigma_2) & \sinh \sigma_2 \\ -\sinh \sigma_2 & \sinh(\mu_1 + \sigma_2) \end{pmatrix},$$

where μ_1 is defined by

$$\coth \mu_1 = \frac{\cosh \sigma_1 \cosh \sigma_2 + \cosh \sigma_4}{\sinh \sigma_1 \sinh \sigma_2}, \qquad \mu_1 > 0;$$
$$\tilde{a}_{1,3} = -\tilde{a}_{1,2}^{-1} \tilde{a}_{1,1}^{-1} = \frac{1}{\sinh \nu_1} \begin{pmatrix} \sinh(\nu_1 - \sigma_4) & e^{\sigma_1} \sinh \sigma_4 \\ -e^{-\sigma_1} \sinh \sigma_4 & \sinh(\nu_1 + \sigma_4) \end{pmatrix},$$

where ν_1 is defined by

$$\coth \nu_1 = \frac{\cosh \sigma_1 \cosh \sigma_4 + \cosh \sigma_2}{\sinh \sigma_1 \sinh \sigma_4}, \qquad \nu_1 > 0.$$

For H_2, since both $b_{2,2}$ and $b_{2,3}$ correspond to the same geodesic, L_5, we have $\nu_2 = \mu_2$.

$$\tilde{a}_{2,1} = \begin{pmatrix} e^{\sigma_1} & 0 \\ 0 & e^{-\sigma_1} \end{pmatrix};$$
$$\tilde{a}_{2,2} = \frac{1}{\sinh \mu_2} \begin{pmatrix} \sinh(\mu_2 - \sigma_5) & \sinh \sigma_5 \\ -\sinh \sigma_5 & \sinh(\mu_2 + \sigma_5) \end{pmatrix};$$
$$\tilde{a}_{2,3} = \frac{1}{\sinh \mu_2} \begin{pmatrix} \sinh(\mu_2 - \sigma_5) & e^{\sigma_1} \sinh \sigma_5 \\ -e^{-\sigma_1} \sinh \sigma_5 & \sinh(\mu_2 + \sigma_5) \end{pmatrix},$$

where μ_2 is defined by

$$\coth\mu_2 = \coth\sigma_5 \frac{\cosh\sigma_1 + 1}{\sinh\sigma_1}, \qquad \mu_2 > 0.$$

The matrices for the generators of H_3 are as follows.

$$\tilde{a}_{3,1} = \begin{pmatrix} e^{\sigma_2} & 0 \\ 0 & e^{-\sigma_2} \end{pmatrix};$$
$$\tilde{a}_{3,2} = \frac{1}{\sinh\mu_3} \begin{pmatrix} \sinh(\mu_3 - \sigma_3) & \sinh\sigma_3 \\ -\sinh\sigma_3 & \sinh(\mu_3 + \sigma_3) \end{pmatrix},$$

where μ_3 is defined by

$$\coth\mu_3 = \frac{\cosh\sigma_2 \cosh\sigma_3 + \cosh\sigma_6}{\sinh\sigma_2 \sinh\sigma_3}, \qquad \mu_3 > 0;$$
$$\tilde{a}_{3,3} = -\tilde{a}_{3,2}^{-1}\tilde{a}_{3,1}^{-1} = \frac{1}{\sinh\nu_3} \begin{pmatrix} \sinh(\nu_3 - \sigma_6) & e^{\sigma_2}\sinh\sigma_6 \\ -e^{-\sigma_2}\sinh\sigma_6 & \sinh(\nu_3 + \sigma_6) \end{pmatrix},$$

where ν_3 is defined by

$$\coth\nu_3 = \frac{\cosh\sigma_2 \cosh\sigma_6 + \cosh\sigma_3}{\sinh\sigma_2 \sinh\sigma_6}, \qquad \nu_3 > 0.$$

Finally, the matrices for the generators of H_4 are as follows.

$$\tilde{a}_{4,1} = \begin{pmatrix} e^{\sigma_3} & 0 \\ 0 & e^{-\sigma_3} \end{pmatrix};$$
$$\tilde{a}_{4,2} = \frac{1}{\sinh\mu_4} \begin{pmatrix} \sinh(\mu_4 - \sigma_4) & \sinh\sigma_4 \\ -\sinh\sigma_4 & \sinh(\mu_4 + \sigma_4) \end{pmatrix},$$

where μ_4 is defined by

$$\coth\mu_4 = \frac{\cosh\sigma_3 \cosh\sigma_4 + \cosh\sigma_6}{\sinh\sigma_3 \sinh\sigma_4}, \qquad \mu_4 > 0;$$
$$\tilde{a}_{4,3} = -\tilde{a}_{4,2}^{-1}\tilde{a}_{4,1}^{-1} = \frac{1}{\sinh\nu_4} \begin{pmatrix} \sinh(\nu_4 - \sigma_6) & e^{\sigma_3}\sinh\sigma_6 \\ -e^{-\sigma_3}\sinh\sigma_6 & \sinh(\nu_4 + \sigma_6) \end{pmatrix},$$

where ν_4 is defined by

$$\coth\nu_4 = \frac{\cosh\sigma_3 \cosh\sigma_6 + \cosh\sigma_4}{\sinh\sigma_3 \sinh\sigma_6}, \qquad \nu_4 > 0.$$

Step 2. The second conjugator, c_2, is the primitive conjugator $e_{1,1}$; we obtain

$$\tilde{c}_2 = \tilde{e}_{1,1} = \begin{pmatrix} 0 & -e^{\tau_1} \\ e^{-\tau_1} & 0 \end{pmatrix}.$$

The third conjugator c_3 is the primitive conjugator $e_{1,2}$; we obtain

$$\tilde{c}_3 = \tilde{a}_{1,2} = \tilde{r}_{12}\tilde{r}_0\tilde{f}_{\tau_2} = \frac{1}{\sqrt{2\sinh\mu_1}}\begin{pmatrix} \exp(\frac{1}{2}\mu_1 - \tau_2) & \exp(-\frac{1}{2}\mu_1 + \tau_2) \\ \exp(-\frac{1}{2}\mu_1 - \tau_2) & \exp(\frac{1}{2}\mu_1 + \tau_2) \end{pmatrix}.$$

The fourth conjugator, c_4, which conjugates H_4 onto $\widehat{H}_4$, is a product of the primitive conjugator $c_3 = e_{1,2}$, and the primitive conjugator $e_{3,3}$, which maps the left half-plane onto the action half-plane of $a_{3,3}$. We first write the matrix for $e_{3,3}$:

$$\tilde{e}_{3,3} = \tilde{r}_{13}\tilde{r}_0\tilde{f}_{\tau_3} = \frac{1}{\sqrt{2\sinh\nu_3}}\begin{pmatrix} \exp(\frac{1}{2}\nu_3 - \tau_3) & \exp(-\frac{1}{2}\nu_3 + \sigma_2 + \tau_3) \\ \exp(\frac{1}{2}\nu_3 - \sigma_2 - \tau_3) & \exp(\frac{1}{2}\nu_2 + \tau_3) \end{pmatrix}.$$

Then we can write $\tilde{c}_4 = \tilde{c}_3\tilde{e}_{3,3}$.

Step 3. We write down the matrices corresponding to the attaching geodesics.

$$\tilde{a}_1 = \tilde{a}_{1,1}, \qquad \tilde{a}_2 = \tilde{a}_{1,2}, \qquad \tilde{a}_3 = \tilde{c}_3\tilde{a}_{3,2}\tilde{c}_3^{-1}.$$

Step 4. We write down the matrices corresponding to the handle geodesics.

$$\tilde{a}_4 = \tilde{a}_{1,3}, \qquad \tilde{a}_5 = \tilde{c}_2\tilde{a}_{2,2}\tilde{c}_2^{-1}, \qquad \tilde{a}_6 = \tilde{c}_3\tilde{a}_{3,3}\tilde{c}_3^{-1}.$$

Step 5. In order to write down the matrices for the handle closing geodesics, we need some untwisted elementary conjugators. For the first handle-closer, we need the untwisted elementary conjugators, $e^0_{1,3}$ and $e^0_{4,2}$. Matrices for these are as follows.

$$\tilde{e}^0_{1,3} = \frac{1}{\sqrt{2\sinh\nu_1}}\begin{pmatrix} \exp(\frac{1}{2}\nu_1) & \exp(-\frac{1}{2}\nu_1 + \sigma_1) \\ \exp(-\frac{1}{2}\nu_1 - \sigma_1) & \exp(\frac{1}{2}\nu_1) \end{pmatrix}, \quad \text{and}$$
$$\tilde{e}^0_{4,2} = \frac{1}{\sqrt{2\sinh\mu_4}}\begin{pmatrix} \exp(\frac{1}{2}\mu_4) & \exp(-\frac{1}{2}\mu_4) \\ \exp(-\frac{1}{2}\mu_4) & \exp(\frac{1}{2}\mu_4) \end{pmatrix}.$$

The matrix for the first handle-closer is then

$$\tilde{a}_7 = \tilde{c}_4\tilde{e}^0_{4,2}\tilde{f}_{\tau_4}(\tilde{e}^0_{1,3})^{-1}.$$

For the second handle-closer, we need the untwisted elementary conjugators, $e^0_{2,2}$ and $e^0_{2,3}$; we write

$$\tilde{e}^0_{2,2} = \frac{1}{\sqrt{2\sinh\mu_2}}\begin{pmatrix} \exp(\frac{1}{2}\mu_2) & \exp(-\frac{1}{2}\mu_2) \\ \exp(-\frac{1}{2}\mu_2) & \exp(\frac{1}{2}\mu_2) \end{pmatrix}, \quad \text{and}$$
$$\tilde{e}^0_{2,3} = \frac{1}{\sqrt{2\sinh\mu_2}}\begin{pmatrix} \exp(\frac{1}{2}\mu_2) & \exp(\frac{1}{2}\mu_2+\sigma_1) \\ \exp(-\frac{1}{2}\mu_2-\sigma_1) & \exp(\frac{1}{2}\mu_2) \end{pmatrix}.$$

Then we can write
$$\tilde{a}_8 = \tilde{c}_2\tilde{e}^0_{2,3}\tilde{f}_{\tau_5}(\tilde{e}^0_{2,2})^{-1}\tilde{c}_2^{-1}.$$

For the third handle-closer, we need matrices for the untwisted elementary conjugators $e^0_{3,3}$ and $e^0_{4,3}$; these are as follows.

$$\tilde{e}^0_{3,3} = \frac{1}{\sqrt{2\sinh\nu_3}}\begin{pmatrix} \exp(\frac{1}{2}\nu_3) & \exp(-\frac{1}{2}\nu_3+\sigma_2) \\ \exp(-\frac{1}{2}\nu_3-\sigma_2) & \exp(\frac{1}{2}\nu_3) \end{pmatrix}, \quad \text{and}$$
$$\tilde{e}^0_{4,3} = \frac{1}{\sqrt{2\sinh\nu_4}}\begin{pmatrix} \exp(\frac{1}{2}\nu_4) & \exp(-\frac{1}{2}\nu_4+\sigma_3) \\ \exp(-\frac{1}{2}\nu_4-\sigma_3) & \exp(\frac{1}{2}\nu_4) \end{pmatrix}.$$

Now we can write our final generator as
$$\tilde{a}_9 = \tilde{c}_4\tilde{e}^0_{4,3}\tilde{f}_{\tau_6}(\tilde{e}^0_{3,3})^{-1}\tilde{c}_3^{-1}.$$

12. The first variation

For our first variation, we distinguish between left and right half-planes. If H_i is a fully normalized pants group, with generators $a_{i,1}$, $a_{i,2}$ and $a_{i,3}$, then the corresponding *fully normalized right pants group* $H^R_i = H_i$, with generators $a^R_{i,k} = a_{i,k}$. The corresponding *fully normalized left pants group* H^L_i has generators $a^L_{i,1} = a^{-1}_{i,1}$, $a^L_{i,2} = ra^{-1}_{i,2}r^{-1}$, and $a^L_{i,3} = ra^{-1}_{i,3}r^{-1}$, where r denotes reflection in the imaginary axis. It is essentially immediate that $\mathbf{H}^2/H^L_i = \mathbf{H}^2/H^R_i$, but with reversed orientation. Since every pair of pants is symmetric; they are in fact indistinguishable.

By using both left and right pants groups, we can eliminate the elementary conjugators $e_{i,1}$ and $e^0_{i,1}$. That is, the fully normalized pants group corresponding to P_1 is H^R_1; the fully normalized pants group corresponding to P_2 is H^L_2; the fully normalized pants group corresponding to P_3 is H^R_3; the fully normalized pants group corresponding to P_4 depends on the entries in the pairing table. If the geodesic appearing as $b_{4,1}$ also appear in either the first or third row, then the corresponding fully normalized pants group is H^R_4; if this geodesic also appears in the second row, then the corresponding fully normalized pants group is H^L_4. Continuing inductively, the fully normalized pants group corresponding to P_j is H^R_j (H^L_j) if the fully normalized pants group corresponding to $P_{i(j)}$ is H^R_i (H^L_i), where $i(j)$ is the row in the pairing table containing the same coordinate geodesic as the entry in the $(j,1)$ place.

For those indices i for which the fully normalized pants group is H_j^R, the (right) conjugators are defined by $e_{i,k}^{0R} = e_{i,k}^0$ and $c_i^R = c_i$.

For those indices i for which the fully normalized pants group is H_i^L, the corresponding untwisted elementary left conjugators are defined by $e_{i,2}^{0L} = re_{i,2}^0 r^{-1}$, and $e_{i,3}^0 = re_{i,3}^0 r^{-1}$. As before, the conjugator $c_1 = c_1^R = 1$. However, for $i = 2$, the second fully normalized pants group is necessarily H_2^L, and the corresponding conjugator is defined by $c_2^L = f_{\tau_1}^{-1}$.

We define the left universal twist by $f_\tau^L = (f_\tau)^{-1}$. Then the twisted left elementary conjugators are given by $e_{i,k}^L = e_{i,k}^{0L} f_\tau^L$. Finally, the conjugator, c_i^L is defined inductively by $c_i^L = c_{j(i)}^L e_{j(i),k}^L$.

Now that we have defined the fully normalized pants groups corresponding to every P_i, and the corresponding twisted and untwisted conjugators, we can easily write down matrices for the generators and the conjugators, using the matrix,

$$\tilde{r} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

except that we will have to change the sign of some matrices so as to have positive traces. Once we have made these modifications, the algorithm proceeds as above, with the following modifications.

For Step 1, we need right and left fully normalized pants groups.

For Step 2, one has to modify equation (9) to appropriately reflect right and left conjugators.

Similarly, in Step 5, one has to replace (14) by appropriately replacing the conjugators by left and right conjugators, and one has to adjust the reversing transformation g. We define g^{XY} to be the identity if $X \neq Y$, and to be the reversing transformation $g(z) = -1/z$ if $X = Y$. Then the new equation for the handle closer is:

$$\tilde{d}_j = \tilde{c}_{j'}^X \tilde{e}_{j',k'}^{0X} g^{XY} \tilde{f}_\tau^Y (\tilde{c}_j^Y \tilde{e}_{j,k}^{0Y})^{-1}, \tag{20}$$

where X and Y both take on the values L and R, and are determined by the fact that, for every j, exactly one of c_j^L and c_j^R is defined.

13. Special case of genus 2

In this section, we illustrate the first variation of our algorithm in the case of a closed surface of genus 2, where all three coordinate geodesics of the F-N system are non-dividing.

In this case, the signature is $(2,0)$, and the pairing table is as follows:

P_1	L_1	L_2	L_3
P_2	L_1	L_2	L_3

We assume we are given $\Phi = (s_1, s_2, s_3, t_1, t_2, t_3)$.

Step 1. The two pairs of pants, P_1 and P_2, are necessarily isometric; we write down the generators for the fully normalized pants groups, $H_1 = H_1^R$ and $H_2 = H_2^L$, as follows.

$$\tilde{a}^R_{1,1} = \begin{pmatrix} e^{\sigma_1} & 0 \\ 0 & e^{-\sigma_1} \end{pmatrix};$$

$$\tilde{a}^R_{1,2} = \frac{1}{\sinh\mu} \begin{pmatrix} \sinh(\mu - \sigma_2) & \sinh\sigma_2 \\ -\sinh\sigma_2 & \sinh(\mu + \sigma_2) \end{pmatrix},$$

where μ is defined by

$$\coth\mu = \frac{\cosh\sigma_1 \cosh\sigma_2 + \cosh\sigma_3}{\sinh\sigma_1 \sinh\sigma_2}, \qquad \mu > 0;$$

$$\tilde{a}^R_{1,3} = \frac{1}{\sinh\nu} \begin{pmatrix} \sinh(\nu - \sigma_3) & e^{\sigma_1}\sinh\sigma_3 \\ -e^{-\sigma_1}\sinh\sigma_3 & \sinh(\nu + \sigma_3) \end{pmatrix},$$

where ν is defined by

$$\coth\nu = \frac{\cosh\sigma_1 \cosh\sigma_3 + \cosh\sigma_2}{\sinh\sigma_1 \sinh\sigma_3}, \qquad \nu_1 > 0;$$

$$\tilde{a}^L_{1,1} = \begin{pmatrix} e^{-\sigma_1} & 0 \\ 0 & e^{\sigma_1} \end{pmatrix};$$

$$\tilde{a}^L_{1,2} = \frac{1}{\sinh\mu} \begin{pmatrix} \sinh(\mu - \sigma_2) & \sinh\sigma_2 \\ -\sinh\sigma_2 & \sinh(\mu + \sigma_2) \end{pmatrix};$$

$$\tilde{a}^L_{1,3} = \frac{1}{\sinh\nu} \begin{pmatrix} \sinh(\nu - \sigma_3) & e^{\sigma_1}\sinh\sigma_3 \\ -e^{-\sigma_1}\sinh\sigma_3 & \sinh(\nu + \sigma_3) \end{pmatrix}.$$

Step 2. The first conjugator $c_1 = 1$; the second conjugator, $c_2^L = f_{\tau_1}^{-1}$. Then

$$\tilde{c}_2^L = \begin{pmatrix} e^{\tau_1} & 0 \\ 0 & e^{-\tau_1} \end{pmatrix}.$$

Step 3. We write down the matrix corresponding to the attaching geodesic; we obtain

$$\tilde{a}_1 = \tilde{a}^R_{1,1} = \begin{pmatrix} e^{\sigma_1} & 0 \\ 0 & e^{-\sigma_1} \end{pmatrix}.$$

Step 4. We write down the matrices corresponding to the handle geodesics.

$$\tilde a_2 = \tilde a^R_{1,2} = \frac{1}{\sinh\mu}\begin{pmatrix} \sinh(\mu-\sigma_2) & \sinh\sigma_2 \\ -\sinh\sigma_2 & \sinh(\mu+\sigma_2)\end{pmatrix},$$
$$\tilde a_3 = \tilde a^R_{1,3} = \frac{1}{\sinh\nu}\begin{pmatrix} \sinh(\nu-\sigma_3) & e^{\sigma_1}\sinh\sigma_3 \\ -e^{-\sigma_1}\sinh\sigma_3 & \sinh(\nu+\sigma_3)\end{pmatrix}.$$

Step 5. In order to write down the matrices for the handle closing geodesics, we need to note that $\lambda_1 = \lambda_2 = \sigma_1$, and we need matrices for the untwisted elementary conjugators, $e^{0R}_{1,2}$, $e^{0L}_{2,2}$, $e^{0R}_{1,3}$ and $e^{0L}_{2,3}$. We find from equations (16) and (18):

$$\tilde e^{0R}_{1,2} = \frac{1}{\sqrt{2\sinh\mu}}\begin{pmatrix} \exp(\frac12\mu) & \exp(-\frac12\mu) \\ \exp(-\frac12\mu) & \exp(\frac12\mu)\end{pmatrix},$$
$$\tilde e^{0L}_{2,2} = \frac{1}{\sqrt{2\sinh\mu}}\begin{pmatrix} \exp(\frac12\mu) & -\exp(-\frac12\mu) \\ -\exp(-\frac12\mu) & \exp(\frac12\mu)\end{pmatrix}.$$

For the first handle closer, equation (20) becomes

$$\tilde d_1 = \tilde c^L_2 \tilde e^{0L}_{2,2} \tilde f^R_{\tau_2} (\tilde c^R_1 \tilde e^{0R}_{1,2})^{-1}. \tag{21}$$

Since $c^R_1 = 1$, we obtain

$$\tilde a_4 = \tilde d_1 = \frac{1}{\sinh\mu}\begin{pmatrix} e^{\tau_1}\cosh(\mu-\tau_2) & -e^{\tau_1}\cosh\tau_2 \\ -e^{-\tau_1}\cosh\tau_2 & e^{-\tau_1}\cosh(\mu+\tau_2)\end{pmatrix}.$$

For the second handle-closer, we need the untwisted elementary conjugators, $e^{0R}_{1,3}$ and $e^{0L}_{2,3}$; we write

$$\tilde e^{0R}_{1,3} = \begin{pmatrix} \exp\big(\frac12(\nu+\sigma_1)\big) & \exp\big(\frac12(-\nu+\sigma_1)\big) \\ \exp\big(\frac12(-\nu-\sigma_1)\big) & \exp\big(\frac12(\nu-\sigma_1)\big)\end{pmatrix}, \qquad \text{and}$$
$$\tilde e^{0L}_{2,3} = \begin{pmatrix} \exp\big(\frac12(\nu+\sigma_1)\big) & -\exp\big(\frac12(-\nu+\sigma_1)\big) \\ -\exp\big(\frac12(-\nu-\sigma_1)\big) & \exp\big(\frac12(\nu-\sigma_1)\big)\end{pmatrix}.$$

We obtain

$$\tilde a_5 = \tilde c^L_2 \tilde e^{0L}_{2,3} \tilde f^R_{\tau_3} (\tilde e^{0R}_{1,3})^{-1} = \frac{1}{\sinh\nu}\begin{pmatrix} e^{\tau_1}\cosh(\nu-\tau_3) & -e^{(\tau_1+\sigma_1)}\cosh\tau_3 \\ -e^{(-\tau_1-\sigma_1)}\cosh\tau_3 & e^{-\tau_1}\cosh(\nu+\tau_3)\end{pmatrix}.$$

14. The second variation

In the second variation, which can start with either the original algorithm or with the first variation, we change the definitions of the conjugators, and of the handle closing generators.

The first two conjugators, $c_1 = 1$ and c_2, remain unchanged. Assume that we have constructed J_k representing Q'_k, $k < q$. Then there is some j so that A^k_{k+j} projects onto L_{k+1}. We redefine the untwisted conjugator c^0_{k+1}, so that it maps the left half-plane onto the action half-plane of a^k_{k+j}, while preserving the common orthogonal between the imaginary axis and A^k_{k+j}. The twisted conjugator c_{k+j} is then the composition of a twist by $2\tau_k$ in the positive direction along the imaginary axis, followed by the untwisted conjugator.

We follow the above procedure, until we reach the point where K^0 is defined. We assume that we have K^0, and we now redefine the base points and matrices for the handle closers. It suffices to describe this new procedure for the first handle-closer; the others are treated analogously.

The handle closer d_1 conjugates $a = a_{2p+1}$ onto the inverse of $a' = a_{2p+2}$. From the preceding step, we already have the matrices $\tilde{a}$ and $\tilde{a}'$. Using formula (3), we can write down the matrices for the reflections r_A and $r_{A'}$, in the axes A and A', respectively. Then $h = r_{A'} r_A$ is a hyperbolic element of $\mathrm{PSL}(2, \mathbf{R})$ whose axis is the common orthogonal between A and A', where the translation length of h is exactly twice the distance between these axes. It follows that the square root of h (i.e., the unique hyperbolic isometry whose square is h) maps A onto A', while preserving the common orthogonal between A and A'. We define the new untwisted handle-closer, d^0_1, to be this square root. Then the twisted handle-closer d_1 is given as: First twist along A by $2\tau_{p-q+1}$, then apply d^0_1. Since we can compute the fixed points of a and a', we can find a matrix for h, and then construct a matrix for the square root of h. Likewise, once we have the fixed points of a, and the trace, $2\cosh(\tau_{p-q+1})$, we can construct the matrix for the corresponding transformation. We then choose $\tilde{d}_1$ to have positive trace.

We remark that, while the entries in $\tilde{d}_1$ can be easily computed in each case, and are well defined algebraic functions of the entries in $\tilde{a}$ and $\tilde{a}'$, there are no easy formulae for these entries.

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