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# THE INTEGRAL MEANS SPECTRUM FOR LACUNARY SERIES

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**Abstract.** Asymptotically sharp bounds for the integral means spectrum of lacunary series are proved. In particular, we show that Rohde's estimates for lacunary series with positive coefficients are sharp and hold not only for the positive case. Moreover, a relation between the law of the iterated logarithm and the integral means spectrum is established. Using this we give a sharp version of the Makarov law of the iterated logarithm for lacunary series.

## 1. Introduction

Let $\log f'(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ be a lacunary series with bounded coefficients and $n_{k+1}/n_k \geq \lambda \geq 2$ and

$$\beta(t) = \overline{\lim}_{r\to 1-} \frac{\log \int_{|z|=r} |f'(z)|^t \mathrm{d}\theta}{\log \dfrac{1}{1-r}}$$

be the integral means spectrum.

In this paper we show that

$$\overline{\lim_{r\to 1}} \frac{|\log f'(r\zeta)|}{\sqrt{\log \dfrac{1}{1-r} \log\log\log \dfrac{1}{1-r}}} \leq 2 \lim_{t\to 0} \frac{\sqrt{\beta(t)}}{t}$$

for almost all $\zeta$ on $|\zeta| = 1$. The equality holds if there exists

$$\lim_{r\to 1} \frac{b^2(r)}{\log \dfrac{1}{1-r}} \quad \text{where } b^2(r) = \sum_{k=1}^{\infty} |a_k|^2 r^{2n_k}.$$

This result follows from the law of the iterated logarithm and the asymptotic formula

$$\beta(t) = \frac{t^2}{4} \overline{\lim_{r\to 1}} \frac{b^2(r)}{\log \dfrac{1}{1-r}} + O(t^{2.5}) \quad \text{as } t \to 0$$

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which will be proved later. The law of the iterated logarithm, proved by Weiss [11], states that

$$\overline{\lim_{r\to 1}}\frac{|\log f'(r\zeta)|}{b(r)\sqrt{\log\log b(r)}}=1 \quad \text{for almost all } |\zeta|=1.$$

A more general result was obtained by Makarov [5]:

$$\overline{\lim_{r\to 1}}\frac{|\log f'(r\zeta)|}{\sqrt{\log\dfrac{1}{1-r}\log\log\log\dfrac{1}{1-r}}}\le \|\log f'\|_{\mathbf{B}}$$

for almost all $\zeta$ on $|\zeta|=1$ and all functions $\log f'\in\mathbf{B}$. Here $\mathbf{B}$ is the class of Bloch functions, i.e., analytic in the unit disk $D=\{|z|<1\}$ functions $g$ with the norm

$$\|g\|_{\mathbf{B}}=\sup_{z\in D}|g'(z)|(1-|z|^2)<+\infty.$$

One of the main problems of the boundary behaviour of conformal maps is the investigation of the integral means spectrum $\beta_f(t)$ for univalent functions $f$ ([2], [3], [4], [7] and [9]). Since the properties of Bloch functions are similar to the properties of lacunary series it is helpful to study $\beta_f(t)$ for lacunary series because it is well known that, for any conformal mapping $f$, $\log f'$ is a Bloch function.

The first non-trivial result about the integral means spectrum for lacunary series was obtained by Makarov [6]. He showed that if $f$ is defined by

$$\log f'(z)=\frac{i}{5}\sum_{k=1}^{\infty}z^{2^k}$$

then $\beta_f(t)\ge 0.00035t^2$ for small $t$. Using Bessel functions Rohde [10], [8] improved this result. He obtained that $\beta_f(t)\ge\log I_0(at)/\log q$ for the lacunary series

$$\log f'(z)=a\sum_{k=1}^{\infty}z^{q^k},\quad a>0$$

where $I_0(x)$ is the modified Bessel function.

First we prove some auxiliary results. It is convenient to use the following abbreviation

$$\int h\,\mathrm{d}\theta\equiv\int_0^{2\pi}h(re^{i\theta})\,\mathrm{d}\theta.$$

**Lemma 1.** *Let $\log f',\log\varphi'\in B$. Then there exists $C>0$ such that*

$$\frac{1}{C}(1-r)^{t^{2.5}}\int|f'|^t\,d\theta\le\int|g'|\,d\theta\le C\left(\frac{1}{1-r}\right)^{Ct^{2.5}}\int|f'|^t\,d\theta,$$

*where ${f'}^t=g'{\varphi'}^{t^2}$ and $t<1$.*

*Proof.* It is known [8], [1] that there exists $C>0$ such that

$$|f'|^t,\ |g'|\le C\Big(\frac{1}{1-r}\Big)^{Ct},$$

$$\int |f'|^t\,\mathrm{d}\theta,\ \int|\varphi'|^{\pm t}\,\mathrm{d}\theta,\ \int|g'|\,\mathrm{d}\theta\le C\Big(\frac{1}{1-r}\Big)^{Ct^2}.$$

Using the Hölder inequality we have

$$\begin{aligned}\int |f'|^t\,\mathrm{d}\theta &= \int |g'|\,|\varphi'|^{t^2}\,\mathrm{d}\theta\\ &\le \Big(\int |g'|^{1/(1-t^{3/2})}\,\mathrm{d}\theta\Big)^{1-t^{3/2}}\Big(\int|\varphi'|^{t^{1/2}}\,\mathrm{d}\theta\Big)^{t^{3/2}}\\ &\le (\sup|g'|)^{t^{3/2}}\Big(\int|g'|\,\mathrm{d}\theta\Big)^{1-t^{3/2}}C\Big(\frac{1}{1-r}\Big)^{Ct^{5/2}}\\ &\le C^2\Big(\frac{1}{1-r}\Big)^{2Ct^{5/2}}\int|g'|\,\mathrm{d}\theta\Big(\int|g'|\,\mathrm{d}\theta\Big)^{t^{3/2}}\\ &\le C^3\Big(\frac{1}{1-r}\Big)^{3Ct^{5/2}}\int|g'|\,\mathrm{d}\theta.\end{aligned}$$

Analogously applying the Hölder inequality to $\int|f'|^t|\varphi'|^{-t^2}\,\mathrm{d}\theta$ we obtain

$$\int|g'|\,\mathrm{d}\theta=\int|f'|^t|\varphi'|^{-t^2}\,\mathrm{d}\theta\le C^3\Big(\frac{1}{1-r}\Big)^{3Ct^{5/2}}\int|f'|^t\,\mathrm{d}\theta.$$

Lemma 1 is proved.

**Lemma 2.** *Let $\log f'=\sum_{k=1}^\infty a_kz^k\in\mathbf{B}$. Then there exists $C>0$ such that*

$$\frac{1}{C}(1-r)^{Ct^{5/2}}I(r,t)\le\int|f'|^t\,d\theta\le C\Big(\frac{1}{1-r}\Big)^{Ct^{5/2}}I(r,t)$$

*where*

$$I(r,t)=\int\prod_{k=1}^\infty|1+a_ktz^k/2|^2\,d\theta.$$

*Proof.* We have

$$t\log f'=\sum_{k=1}^\infty ta_kz^k=\sum_{k=1}^\infty 2\log(1+ta_kz^k/2)+\sum_{k=1}^\infty t^2a_k^2z^{2k}/4+\sum_{k=1}^\infty O((ta_kr^k)^3).$$

From [8], [1] it follows that $\log\varphi'=\sum_{k=1}^\infty a_k^2z^{2k}\in\mathbf{B}$ and $\sum_{k=1}^\infty O\big((a_kr^k)^3\big)=O\left(\log\big(1/(1-r)\big)\right)$. Therefore, we can apply Lemma 1 and obtain the required inequalities.

Now we can prove our main result.

**Theorem 1.** *Let* $\log f' = \sum_{k=1}^{\infty} a_k z^{n_k}$ *be a lacunary series with bounded coefficients and* $n_{k+1}/n_k \ge \lambda \ge 2$.

*Then*

$$\beta_f(t) = \frac{t^2}{4} \varlimsup_{r\to 1} \frac{\sum_{k=1}^{\infty} |a_k|^2 r^{2n_k}}{\log \dfrac{1}{1-r}} + O(t^{5/2}) \quad \text{as } t \to 0.$$

*Proof.* By Lemma 2, to prove this result it is enough to estimate

$$\int \prod_{k=1}^{\infty} |1 + t a_k z^{n_k}/2|^2 \, d\theta = \int \prod_{k=1}^{\infty} \bigl(1 + t^2 |a_k|^2 r^{2n_k}/4 + t|a_k| r^{n_k} \cos(n_k\theta + \theta_k)\bigr) \, d\theta.$$

We multiply out and integrate term-by-term. Since

$$\cos\alpha \cos\beta = 0.5[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

then

$$\begin{aligned} &\cos(n_{p_1}\theta + \theta_{p_1}) \cos(n_{p_2}\theta + \theta_{p_2}) \times \cdots \times \cos(n_{p_j}\theta + \theta_{p_j}) \\ &\qquad = \frac{1}{2^j} \sum \cos((n_{p_1} \pm n_{p_2} \pm \cdots \pm n_{p_j})\theta + \gamma). \end{aligned} \tag{1}$$

The integral of (1) by $\theta$ is equal to zero because $\sum_{k=1}^{j-1} n_{p_k} \le \sum_{k=1}^{p_j - 1} n_k < n_{p_j}/(\lambda - 1) \le n_{p_j}$. Therefore

$$\begin{aligned} \int \prod_{k=1}^{\infty} |1 + a_k t z^{n_k}/2|^2 \, d\theta &= 2\pi \prod_{k=1}^{\infty} (1 + t^2 |a_k|^2 r^{2n_k}/4) \\ &= 2\pi \exp\left[\sum_{k=1}^{\infty} \log(1 + t^2|a_k|^2 r^{2n_k}/4)\right] \\ &= 2\pi \left(\frac{1}{1-r}\right)^{O(t^3)} \exp\left[0.25 t^2 \sum_{k=1}^{\infty} |a_k|^2 r^{2n_k}\right]. \end{aligned}$$

This concludes the proof.

Applying Theorem 1 to the law of the iterated logarithm we obtain

**Theorem 2.** *For almost all* $\zeta$ *on* $|\zeta| = 1$ *the following inequality holds*

$$\varlimsup_{r\to 1} \frac{|\log f'(r\zeta)|}{\sqrt{\log \dfrac{1}{1-r} \log\log\log \dfrac{1}{1-r}}} \le 2 \lim_{t\to 0} \frac{\sqrt{\beta(t)}}{t}.$$

*The inequality is sharp, i.e., if* $\lim_{r\to 1}\bigl(b^2(r)/\log(1/(1-r))\bigr)$ *exists then equality holds.*

Now we will prove an estimate for a partial case $n_k = q^k$, $q$ is an integer. To prove this result we need some lemmas.

**Lemma 3.** *Let*

$$I_n(x) = \left(\frac{x}{2}\right)^n \sum_{\nu=0}^{\infty} \frac{x^{2\nu}}{4^\nu \nu!(\nu+n)!} \quad (n = 0, 1, 2, \ldots)$$

*be modified Bessel functions. Then*

$$x\cos\theta - \log\left(I_0(x) + 2\sum_{n=1}^{q-1} I_n(x)\cos(n\theta)\right) = O(x^q) \quad \text{as } x \to 0.$$

*Proof.* It is known [8] that

$$\exp(x\cos\theta) = I_0(x) + 2\sum_{n=1}^{\infty} I_n(x)\cos(n\theta).$$

Hence

$$\exp(x\cos\theta) = I_0(x) + 2\sum_{n=1}^{q-1} I_n(x)\cos(n\theta) + O(x^q)$$

because $I_n(x) \le (|x|^n/2^n)\exp(|x|^2/4)/n!$. Therefore

$$\begin{aligned} x\cos\theta &= \log\left(I_0(x) + 2\sum_{n=1}^{q-1} I_n(x)\cos(n\theta)\right) \\ &\quad + \log\left(1 + \frac{O(x^q)}{I_0(x) + 2\sum_{n=1}^{q-1} I_n(x)\cos(n\theta)}\right) \\ &= \log\left(I_0(x) + 2\sum_{n=1}^{q-1} I_n(x)\cos(n\theta)\right) + O(x^q). \end{aligned}$$

**Lemma 4.** *Let $q > 1$, $s_{ij} \neq 0$, $p_{ij} > 0$ be some integers and $|s_{ij}| < q$, $p_{ij} \neq p_{ls}$, $(i,j) \neq (l,s)$. Then*

$$\int \prod_{i=1}^{m} \cos\left(\theta \sum_{j=1}^{n_i} s_{ij} q^{p_{ij}} + \theta_i\right) d\theta = 0. \tag{2}$$

*Proof.* The proof is by induction on $m$. Consider the case $m = 1$. Suppose that

$$\int \cos\left(\theta \sum_{j=1}^{n} s_j q^{p_j} + \theta_1\right) d\theta \neq 0.$$

Then $\sum_{j=1}^{n} s_j q^{p_j} = 0$. Without loss of generality we can suppose $p_1 < p_2 \cdots < p_n$. Multiplying our equation on $q^{-(p_1+1)}$ we conclude that $s_1/q$ is an integer number but this is impossible because $0 < |s_1| < q$.

Now suppose that $m \geq 2$ and (2) holds for $m-1$. Using $\cos\alpha\cos\beta = 0.5[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$ we can write our integral as the sum of two integrals with $m-1$ factors and hence our hypothesis is true.

**Theorem 3.** *We have*

$$\beta_f(t) = \overline{\lim_{r\to 1}} \frac{\sum_{k=1}^{\infty} \log I_0(t|a_k|r^{q^k})}{\log \dfrac{1}{1-r}} + O(t^q) \quad \text{as } t \to 0.$$

*Proof.* Since

$$\begin{aligned}\int |f'(z)|^t \,\mathrm{d}\theta &= \int \exp\left[t\sum_{k=1}^{\infty} |a_k| r^{q^k} \cos(q^k\theta + \theta_k)\right] \mathrm{d}\theta \\ &= \int \prod_{k=1}^{\infty} \exp[t|a_k|r^{q^k} \cos(q^k\theta+\theta_k)]\,\mathrm{d}\theta.\end{aligned}$$

it follows from Lemma 3 that

$$\begin{aligned}&\int \prod_{k=1}^{\infty} \exp[t|a_k|r^{q^k} \cos(q^k\theta+\theta_k)]\,\mathrm{d}\theta \\ &= \left(\frac{1}{1-r}\right)^{O(t^q)} \int \prod_{k=1}^{\infty} \left(I_0(t|a_k|r^{q^k}) + 2\sum_{j=1}^{q-1} I_j(t|a_k|r^{q^k}) \cos j(q^k\theta+\theta_k)\right) \mathrm{d}\theta\end{aligned}$$

where we have used that $\sum_{k=1}^{\infty} r^{q^k} = O\big(\log 1/(1-r)\big)$.

We multiply out and integrate term-by-term. By Lemma 4 our integral is

$$2\pi \prod_{k=1}^{\infty} I_0(t|a_k|r^{q^k}).$$

Theorem 3 is proved.

**Corollary.** *Let* $|a_k| = a > 0$ *then*

$$\beta_f(t) = \frac{\log I_0(at)}{\log q} + O(t^q) \quad \text{as } t \to 0.$$

*Proof.* From the proof of Rohde it follows easily that

$$\overline{\lim_{r\to 1}} \frac{\sum_{k=1}^{\infty} \log I_0(tar^{q^k})}{\log \dfrac{1}{1-r}} = \frac{\log I_0(at)}{\log q} + O(t^q) \quad \text{as } t \to 0.$$

Applying Theorem 3 we conclude that

$$\beta_f(t) = \overline{\lim_{r\to 1}} \frac{\sum_{k=1}^{\infty} \log I_0(atr^{q^k})}{\log \dfrac{1}{1-r}} + O(t^q) = \frac{\log I_0(at)}{\log q} + O(t^q) \quad \text{as } t \to 0.$$

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