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MODULUS AND CONTINUOUS CAPACITY

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Abstract. It is shown that if Ω is a domain in a metric measure space X such that X is proper, doubling, supports a $(1,p)$-Poincaré inequality, and is φ-convex, then $\operatorname{Mod}_p(E,F,\Omega)$ is equal to the locally Lipschitz p-capacity of the triple (E,F,Ω).

1. Introduction

Let X be a metric measure space equipped with a Borel measure, and Ω be a domain in X, that is, Ω is an open connected set in X. Assume also that for each ball $B\subset X$, $\mu(B)$ is finite and non-zero. If Γ is any collection of paths in Ω, for $1\le p<\infty$ the p-modulus of the collection is defined to be the (possibly infinite) number

$$\operatorname{Mod}_p(\Gamma) := \inf_{\varrho} \|\varrho\|^p_{L^p(\Omega)},$$

where the infimum is taken over all non-negative Borel-measurable functions ϱ such that for each locally rectifiable path γ in Γ, the integral $\int_\gamma \varrho\, ds$ is not less than 1. If E and F are disjoint non-empty compact sets in Ω, let $\operatorname{Mod}_p(E,F,\Omega)$ denote the p-modulus of the collection of all rectifiable paths γ in Ω with one endpoint in E and the other endpoint in F.

Following [HeK1], a non-negative Borel-measurable function ϱ is said to be an upper gradient of a real-valued function u if for all rectifiable paths γ

$$|u(x)-u(y)| \le \int_\gamma \varrho\, ds, \tag{1}$$

where x and y denote the endpoints of γ. If inequality (1) holds true only for the paths that are not in a fixed collection of p-modulus zero, then ϱ is said to be a p-weak upper gradient of u. In the rest of the paper let p be a fixed number such that $1<p<\infty$, and p-weak upper gradients are refered to as weak upper gradients. By [KM, Lemma 2.4], the existence of a weak upper gradient in $L^p(\Omega)$

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is equivalent to the existence of upper gradients in $L^p(\Omega)$ that converge in $L^p(\Omega)$ to the weak upper gradient. Hence in the following definitions of capacity one can consider weak upper gradients instead of only upper gradients to obtain the same number. The p-capacity of the triple (E,F,Ω) is defined to be the (possibly infinite) number

$$\mathrm{Cap}_p(E,F,\Omega)=\inf_{\varrho}\|\varrho\|^p_{L^p(\Omega)},$$

where the infimum is taken over all non-negative Borel-measurable functions ϱ that are upper gradients (or weak upper gradients) of some function u with the property that $u|_E\geq 1$ and $u|_F\leq 0$. It is easily seen that the same number is obtained if the above definition is modified to say that $u|_E=1$, $u|_F=0$, and $0\leq u\leq 1$. This definition does not assume any regularity on the functions u; it is not even required of u to be measurable. A more sensitive capacity is obtained if in the above definition it is also required that the functions u be continuous; denote this number $\text{Cont-Cap}_p(E,F,\Omega)$. If u is also required to be locally Lipschitz, then the corresponding number obtained is denoted $\text{locLip-Cap}_p(E,F,\Omega)$. It is immediate that

$$\mathrm{Mod}_p(E,F,\Omega)\leq \mathrm{Cap}_p(E,F,\Omega)\leq \text{Cont-Cap}_p(E,F,\Omega)\leq \text{locLip-Cap}_p(E,F,\Omega).$$

In [HeK1, Proposition 2.15] Heinonen and Koskela show that in arbitrary metric spaces $\mathrm{Mod}_p(E,F,\Omega)=\mathrm{Cap}_p(E,F,\Omega)$. Furthermore, they also prove that if Ω is a compact φ-convex metric measure space, then $\mathrm{Mod}_p(E,F,\Omega)=\text{Cont-Cap}_p(E,F,\Omega)$. A metric space X is said to be φ-convex if there is a cover of X by open sets $\{U_\alpha\}$ together with homeomorphisms $\{\varphi_\alpha\colon[0,\infty)\to[0,\infty)\}$ such that each pair of distinct points x and y in U_α can be joined by a curve whose length does not exceed $\varphi_\alpha\bigl(d(x,y)\bigr)$. For domains in $\mathbf{R}^n$ equipped with the Lebesgue measure Hesse, Shlyk, and Ziemer proved that $\mathrm{Mod}_p(E,F,\Omega)=\text{Cont-Cap}_p(E,F,\Omega)$, while Shlyk, Aikawa and Ohtsuka prove this for bounded Euclidean domains with strong A_∞ weights, see [Z1], [Z2], [Shl1], [Shl2], [AO], [H], and [F]. In [HeK1, Remark 2.17] Heinonen and Koskela pose the question whether it is true that $\mathrm{Mod}_p(E,F,\Omega)=\text{Cont-Cap}_p(E,F,\Omega)$ if Ω is a general domain in a metric measure space, not necessarily bounded. This paper answers this question in the affirmative in the case that Ω is a domain in a φ-convex metric measure space X that is proper, doubling, and supports a $(1,p)$-Poincaré inequality. The arguments used in this paper are different from those in the above citations. The machinery of Newtonian spaces developed in [Sh] is used here. Properness means that closed balls are compact and the measure μ is doubling if there is a constant $C_\mu\geq 1$ such that $\mu(2B)\leq C_\mu\mu(B)$ for all balls B in the space; $2B$ is the ball with the same center as B but with twice the radius of B. Following [HeK1], a metric measure space X is said to support a $(1,p)$-Poincaré inequality if there are constants $C>0$ and $\tau\geq 1$ so that for each ball B in X and each continuous function u in τB with upper gradient ϱ in τB the following inequality holds:

$$\fint_B |u-u_B| \le C\big(\operatorname{diam}(B)\big)\bigg(\fint_{\tau B} \varrho^p\bigg)^{1/p}, \tag{2}$$

where f_B denotes the mean value integral of f over B. In the rest of the paper it is assumed that $\tau = 1$. Minor modifications would yield the results when $\tau > 1$.

The main theorem of this paper is:

Theorem 1.1. *If X is a proper φ-convex metric measure space equipped with a doubling measure and supporting a $(1,p)$-Poincaré inequality with $1 < p < \infty$, and Ω is a domain in X, then for all disjoint compact non-empty subsets E and F of Ω,*

$$\operatorname{Mod}_p(E,F,\Omega) = \text{Cont-Cap}_p(E,F,\Omega) = \text{locLip-Cap}_p(E,F,\Omega),$$

where $\text{locLip-Cap}_p(E,F,\Omega)$ is defined similarly to $\text{Cont-Cap}_p(E,F,\Omega)$, with the test functions being required to be locally Lipschitz.

The authors do not know whether the $(1,p)$-Poincaré inequality assumption is essential for the conclusion in Theorem 1.1. This theorem has a wide range of applications. For example, in [KST, Corollary C], Koskela, Shanmugalingam and Tuominen show that certain types of porous sets are removable for the Loewner condition by using this theorem. The idea here is to show that these porous sets are removable for continuous capacitary estimates, and then as a consequence of Theorem 1.1 they note that the Loewner property is preserved when such porous sets are removed.

Following the notation in [Sh], the space $N^{1,p}_{\mathrm{loc}}(\Omega)$ is defined to be the collection of all functions in $L^p_{\mathrm{loc}}(\Omega)$ which have upper gradients in $L^p(\Omega)$. Note that the upper gradients are required to be in $L^p(\Omega)$, not merely in $L^p_{\mathrm{loc}}(\Omega)$. The collection $L^p_{\mathrm{loc}}(\Omega)$ is the collection of all functions in Ω that are p-integrable on every bounded subset of Ω. The proof of Theorem 1.1 uses the fact that in the setting of Theorem 1.1 locally Lipschitz functions are dense in $N^{1,p}_{\mathrm{loc}}(\Omega)$.

Proposition 1.2. *If X is a metric measure space equipped with a doubling measure and supporting a $(1,p)$-Poincaré inequality with $1 < p < \infty$, and Ω is a domain in X, then there exists a constant $C > 0$ such that for each function u in $N^{1,p}_{\mathrm{loc}}(\Omega)$ and for all $\varepsilon > 0$ there is a locally Lipschitz function u_ε in $N^{1,p}_{\mathrm{loc}}(\Omega)$ with the property that there exists an upper gradient g_ε of $u_\varepsilon - u$ so that*

$$\|g_\varepsilon\|_{L^p(\Omega)} < \varepsilon \qquad \text{and} \qquad \|u_\varepsilon - u\|_{L^p(\Omega)} < C\varepsilon.$$

Proposition 1.2 is interesting in its own right as it proves a local analogue of the classical Sobolev space theory result $H^{1,p} = W^{1,p}$. Using an idea of Semmes [S1], the result is proved for the case $\Omega = X$ in [Sh, Theorem 4.1].

Section 2 is devoted to proving Proposition 1.2, and in Section 3 Theorem 1.1 is proved.

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2. Proof of Proposition 1.2

The idea behind the proof is to approximate u in balls in Ω by applying [Sh, Theorem 4.1] in these balls, and then paste the approximations together using a partition of unity to obtain an approximation in Ω.

The following covering lemma is needed for the proof of Proposition 1.2. For a proof of this lemma see [S2] and [KST].

If B is a ball of radius r, then for $k>0$ the ball with the same center as B and radius kr is denoted kB.

Lemma 2.1. *Let X be a metric measure space equipped with a doubling measure, and Ω a domain in X. Then there is a collection $\{B_i\}_{i\in\mathbf{N}}$ of balls in X such that*

(i) $\bigcup_i B_i = \Omega$,

(ii) *there exists a constant $C_1 \ge 1$ such that if $2B_i \cap 2B_j \ne \emptyset$, then*

$$\frac{1}{C_1}\mathrm{rad}\,(B_i) \le \mathrm{rad}\,(B_j) \le C_1 \mathrm{rad}\,(B_i)$$

where $\mathrm{rad}\,(B_i)$ *is a pre-assigned radius of B_i,*

(iii) *with $C_2 = 12(C_1+1)$, $C_2 B_i \Subset \Omega$, and*

(iv) *there exists a constant $D \ge 1$ so that $\sum_i \chi_{2B_i} \le D$.*

Note that a ball in X may have more than one center and more than one radius. Hence in this lemma it may be necessary to consider the balls B_i to have pre-assigned centers and radii.

If $\{B_i\}_{i\in\mathbf{N}}$ is the collection obtained by Lemma 2.1, then for each positive integer i there exists a $(C/\mathrm{diam}\,(B_i))$-Lipschitz function φ_i so that $0<\varphi_i\le 1$, $\sum_{i\in\mathbf{N}}\varphi_i = \chi_\Omega$, and $\varphi_i|_{\Omega\setminus 2B_i} = 0$. The constant C in the Lipschitz constant estimate depends only on the constants C_1 and D of Lemma 2.1.

The following lemma also is needed in the proof of Proposition 1.2. Its proof is a modification of a technique in [Ha, Proposition 1], and will be omitted here. Let u be a real-valued function on a metric measure space Y. Following the notation in [Sh], the function u is said to be ACC_p in Y if on all rectifiable paths $\gamma\subset Y$ outside a family of p-modulus zero the function $u\circ\gamma$ is absolutely continuous. By [Sh, Proposition 3.1], functions in $N^{1,p}_{\mathrm{loc}}(\Omega)$ are ACC_p in Ω.

Lemma 2.2. *Let Y be a metric measure space, and u be a real-valued ACC_p function on Y. If there exist two non-negative Borel measurable functions g and h on Y such that for all rectifiable paths γ connecting x to y it is true that*

$$|u(x)-u(y)| \le \int_\gamma g + d(x,y)\big(h(x)+h(y)\big),$$

then $g+4h$ is a weak upper gradient of u.

Proof of Proposition 1.2. By [HeK2, Theorem A], the $(1,p)$-Poincaré inequality (2) is satisfied by any real-valued function u in $L^1_{\mathrm{loc}}(X)$ and its upper gradient g, since X is proper, φ-convex, doubling, and supports a $(1,p)$-Poincaré inequality.

Let $\{B_i\}_{i\in\mathbf{N}}$ be as in Lemma 2.1, and $\{\varphi_i\}_{i\in\mathbf{N}}$ be the corresponding partition of unity constructed above. For each i the proof of [Sh, Theorem 4.1] can be applied on the space $N^{1,p}(C_2B_i)$ to obtain Lipschitz approximations to $u|_{C_2B_i}$. The proof of [Sh, Theorem 4.1] uses the doubling property of the measure and the Poincaré inequality. In this general situation of an arbitrary domain Ω, we can still use these two properties on the balls B_i because of the properties from Lemma 2.1. Hence we can apply the proof of [Sh, Theorem 4.1] here.

Let $v_{i,\lambda}$ be such a $C\lambda$-Lipschitz approximation of $u|_{C_2B_i}$. Fix $\varepsilon>0$. Choosing λ sufficiently large, an approximation $v_i=v_{i,\lambda}$ can be obtained for each ball $2C_2B_i$ so that in addition,

$$\|v_i-u\|_{N^{1,p}(2C_2B_i)}<2^{-i}\varepsilon, \tag{3}$$

and

$$\|v_i-u\|_{L^p((2C_2B_i))}<2^{-i}\mathrm{diam}\,(B_i)\varepsilon. \tag{4}$$

By (3), we have an upper gradient g_i of v_i-u with $\|g_i\|_{L^p((2C_2B_i)}<2^{-i}\varepsilon$. Now let $u_\varepsilon=\sum_i v_i\varphi_i$. While $v_i\varphi_i$ is defined in all of Ω, since it has its non-zero values only inside $2B_i$, by (iv) of Lemma 2.1 the above sum is a finite sum on Ω. Furthermore, it is easily seen that u_ε is Lipschitz on each ball B_i, and hence is locally Lipschitz. Since $u=\sum_i u\varphi_i$, we have $u_\varepsilon-u=\sum_i(v_i-u)\varphi_i$. Just as in the papers [S2] and [KST] it can be shown that the function

$$g_\varepsilon=\sum_i\left(g_i+\frac{4C}{\mathrm{diam}\,(B_i)}|v_i-u|\right)\chi_{2B_i}$$

is an upper gradient of $u_\varepsilon-u$. Now just as in [KST], it can be easily seen by the bounded overlap property (iv) of Lemma 2.1 and by inequalities (3) and (4) that $\int_\Omega g_\varepsilon(x)^p\le C\varepsilon^p$. Therefore $\|g_\varepsilon\|_{L^p(\Omega)}\le C\varepsilon$, and the function u_ε has $g+g_\varepsilon\in L^p(\Omega)$ as an upper gradient. Furthermore, by inequality (4),

$$\|u_\varepsilon-u\|_{L^p(\Omega)}\le\sum_i\|v_i-u\|_{L^p((4C_1+2)B_i)}\le\left(\sum_i 2^{-i}\right)\varepsilon.$$

Hence u_ε is a locally Lipschitz function that approximates u as required. □

3. Proof of Theorem 1.1

The idea behind the proof of Theorem 1.1 is as follows. It is first shown that if A, B are non-empty compact subsets of Ω, then

$$\operatorname{Mod}_p(A,B,\Omega) = N^{1,p}_{\mathrm{loc}} - \operatorname{Cap}_p(A,B,\Omega)$$

where $N^{1,p}_{\mathrm{loc}} - \operatorname{Cap}_p(A,B,\Omega)$ is defined similarly to the other capacity definitions, with the test functions u being required to be in $N^{1,p}_{\mathrm{loc}}(\Omega)$. Theorem 1.1 is then proved if $N^{1,p}_{\mathrm{loc}} - \operatorname{Cap}_p(A,B,\Omega)$ can be replaced by $\text{Cont-Cap}_p(E,F,\Omega)$. This is done as follows: If E and F are non-empty, disjoint compact subsets of Ω, compact neighbourhoods A_ε and B_ε of E and F are considered. The test functions u used to calculate the number $N^{1,p}_{\mathrm{loc}} - \operatorname{Cap}_p(A_\varepsilon,B_\varepsilon,\Omega)$ are "smoothed out" by locally Lipschitz approximations. These approximations take on the same value as u at the "centers" E and F of A_ε and B_ε, and hence are functions that can be used to calculate upper bounds for $\text{Cont-Cap}_p(E,F,\Omega)$. The argument is completed by using a continuity property of $N^{1,p}_{\mathrm{loc}} - \operatorname{Cap}_p(A_\varepsilon,B_\varepsilon,\Omega)$, that is, as the sets A_ε and B_ε shrink to the sets E and F respectively $N^{1,p}_{\mathrm{loc}} - \operatorname{Cap}_p(A_\varepsilon,B_\varepsilon,\Omega)$ tends to the number $N^{1,p}_{\mathrm{loc}} - \operatorname{Cap}_p(E,F,\Omega)$.

By the proof in [HeK1, Proposition 2.15], it is true that if A and B are non-empty disjoint compact subsets of Ω with X φ-convex and Ω_j is a subdomain of Ω such that $A\cup B\subset\Omega_j\Subset\Omega$, then

$$\text{Cont-Cap}_p(A,B,\overline{\Omega}_j) \le \operatorname{Cap}_p(A,B,\Omega) = \operatorname{Mod}_p(A,B,\Omega). \tag{5}$$

Here $\text{Cont-Cap}_p(A,B,\overline{\Omega}_j)$ is defined similarly to $\text{Cont-Cap}_p(A,B,\Omega)$. In order to prove that $\operatorname{Mod}_p(A,B,\Omega)$ is equal to $N^{1,p}_{\mathrm{loc}} - \operatorname{Cap}_p(A,B,\Omega)$, the domain Ω is exhausted by such subdomains Ω_j (which is possible since X is proper), and it is shown that

$$\lim_{j\to\infty} \text{Cont-Cap}_p(A,B,\overline{\Omega}_j) \ge N^{1,p}_{\mathrm{loc}} - \operatorname{Cap}_p(A,B,\Omega).$$

In order to do so, it is necessary to build up $N^{1,p}_{\mathrm{loc}}(\Omega)$-test functions admissible for calculating $N^{1,p}_{\mathrm{loc}} - \operatorname{Cap}_p(A,B,\Omega)$ from the test functions used in calculating $\text{Cont-Cap}_p(A,B,\Omega_j)$. To overcome the restrictions imposed by not knowing whether $N^{1,p}(\Omega_j)$ is reflexive or not in this general setting, it is necessary to obtain the $N^{1,p}_{\mathrm{loc}}$-test function via Mazur's lemma applied to convex combinations of functions and upper gradients in $L^p(\Omega)$.

Lemma 3.1. *Let Y be a metric measure space. If $\{f_j\}_{j\in\mathbf{N}}$ is a sequence of functions in $L^p(Y)$ with upper gradients $\{g_j\}_{j\in\mathbf{N}}$ in $L^p(Y)$, such that f_j weakly converges to f in L^p and g_j weakly converges to g in L^p, then g is a weak upper gradient of f after modifying f on a set of measure zero, and there is a convex combination sequence $\tilde{f}_j = \sum_{k=j}^{n_j}\lambda_{kj}f_k$ and $\tilde{g}_j = \sum_{k=j}^{n_j}\lambda_{kj}g_k$ with $\sum_{k=j}^{n_j}\lambda_{kj}=1$, $\lambda_{kj}>0$, so that $\tilde{f}_j$ converges in L^p to f and $\tilde{g}_j$ converges in L^p to g.*

Proof. Applying Mazur's lemma (see [Y]) to each sequence $\{f_j\}_{j=1}^{\infty}$ and $\{g_j\}_{j=1}^{\infty}$ simultaneously, a sequence of convex combinations of f_j's and g_j's that converge in the $L^p(Y)$-norm to f and some function g can be formed. Denote these convex combination sequences $\{\tilde{f}_i\}$ and $\{\tilde{g}_i\}$. It is easy to see that $\tilde{g}_i$ is a weak upper gradient of $\tilde{f}_i$.

To see that g is an upper gradient of f in Y, note by a theorem of Fuglede [F] that if $\tilde{g}_i$ is a sequence of Borel-measurable non-negative functions in $L^p(Y)$ converging in $L^p(Y)$ to a function g, then there exists a subsequence of $\{\tilde{g}_i\}$, also denoted $\{\tilde{g}_i\}$ for brevity, and a collection Γ of rectifiable paths in Y with $\mathrm{Mod}_p(\Gamma) = 0$, so that whenever γ is a rectifiable path not in Γ, then $\int_\gamma \tilde{g}_i \to \int_\gamma g$, and $\int_\gamma g < \infty$. One can redefine f on a set of measure zero in Y so that

$$f(x) = \tfrac{1}{2}\Big\{\limsup_{i\to\infty} \tilde{f}_i(x) + \liminf_{i\to\infty} \tilde{f}_i(x)\Big\}, \tag{6}$$

wherever it makes sense; see [Sh]. Now just as in [Sh], it can be shown that g is a weak upper gradient of f. □

Lemma 3.2. *Let Ω_j be a sequence of bounded domains such that $\Omega_j \subset \Omega_{j+1}$, $\Omega_j \Subset \Omega$, $\Omega = \bigcup_j \Omega_j$, and E, F are compact subsets of Ω_j for each j. Then*

$$\lim_{j\to\infty} \text{Cont-Cap}_p(E, F, \overline{\Omega}_j) \ge N^{1,p}_{\mathrm{loc}} - \mathrm{Cap}_p(E, F, \Omega).$$

Proof. If $\lim_{j\to\infty} \text{Cont-Cap}_p(E, F, \overline{\Omega}_j) = \infty$, then there is nothing to prove, and because $\text{Cont-Cap}_p(E, F, \overline{\Omega}_j)$ is an increasing function of j, it can be assumed that the limit of $\text{Cont-Cap}_p(E, F, \overline{\Omega}_j)$ exists and is equal to $M < \infty$. Now $\text{Cont-Cap}_p(E, F, \overline{\Omega}_j) \le M$ for each j. For each positive integer j, a continuous function f_j can be chosen, together with its upper gradient g_j in $\overline{\Omega}_j$, so that $0 \le f_j \le 1$, $\|g_j\|^p_{L^p(\Omega)} \le \text{Cont-Cap}_p(E, F, \overline{\Omega}_j) + \varepsilon$, and $f_j|_A = 1$, $f_j|_B = 0$.

Fix $j \in \mathbf{N}$. For each integer $k \ge j$, f_k is defined on $\overline{\Omega}_j$ and has g_k as an upper gradient on Ω_j, $\|f_k\|_{L^p(\overline{\Omega}_j)} \le \mu(\overline{\Omega}_j)^{1/p}$, and $\|g_k\|_{L^p(\overline{\Omega}_j)} \le \|g_k\|_{L^p(\overline{\Omega}_k)} \le (M+\varepsilon)^{1/p}$. Thus both sequences $\{f_k\}_{k\ge j}$ and $\{g_k\}_{k\ge j}$ are bounded sequences in $L^p(\overline{\Omega}_j)$, and by the weak compactness property of $L^p(\overline{\Omega}_j)$, there exist functions f^j and g^j in $L^p(\overline{\Omega}_j)$ to which subsequences of the two sequences weakly converge respectively. By Lemma 3.1, there is a convex combination $\tilde{f}_{k,j}$ and the corresponding convex combination $\tilde{g}_{k,j}$ converging in $L^p(\overline{\Omega}_j)$ to f^j and g^j respectively, and g^j is a weak upper gradient of f^j in $\overline{\Omega}_j$. Moreover, by definition (6), $f^j|_E = 1$, $f^j|_F = 0$, and $0 \le f^j \le 1$. Now for every $k \in \mathbf{N}$, by a diagonalization argument it can be ensured that $f^k|_{\Omega_{k-1}} = f^{k-1}$ outside of a set of zero p-capacity in Ω_{k-1} (see [Sh, Corollary 3.3]), and $g^k|_{\Omega_{k-1}} = g^{k-1}$ almost everywhere in Ω_{k-1}. Hence the function f can be defined by $f(x) = f^k(x)$ whenever $x \in \overline{\Omega}_k$ and g can be defined by $g(x) = g^k(x)$ whenever $x \in \overline{\Omega}_k \setminus \overline{\Omega}_{k-1}$. Then

$f \in L^p_{\text{loc}}(\Omega)$, and g is a weak upper gradient of f since every rectifiable curve in Ω lies in some Ω_k and $g = g^k$ almost everywhere in Ω_k. Since $\|g_k\|^p_{L^p(\Omega_j)} \le M + \varepsilon$, extending each g_i by zero to all of Ω,

$$\left(\int_\Omega |g^k|^p\right)^{1/p} = \left(\int_\Omega \left(\sum_{s=j}^{n_j} \lambda_{js} g_s\right)^p\right)^{1/p} \le (M+\varepsilon)^{1/p}.$$

Hence $\|g\|^p_{L^p(\Omega)} \le M + \varepsilon$ and therefore $f \in N^{1,p}_{\text{loc}}(\Omega)$ is an admissible function in calculating $N^{1,p}_{\text{loc}} - \text{Cap}_p(E, F, \Omega)$. Hence,

$$M + \varepsilon \ge N^{1,p}_{\text{loc}} - \text{Cap}_p(E, F, \Omega),$$

where we use the fact that $M = \lim_{j\to\infty} \text{Cont-Cap}_p(E, F, \overline{\Omega}_j)$. □

Remark 3.3. By Lemma 3.2 and by inequality (5), if $p > 1$ then

$$\begin{aligned} \text{Mod}_p(E, F, \Omega) &= \text{Cap}_p(E, F, \Omega) \\ &\ge \text{Cont-Cap}_p(E, F, \overline{\Omega}_j) \to M \ge N^{1,p}_{\text{loc}} - \text{Cap}_p(E, F, \Omega) \end{aligned}$$

as $j \to \infty$. Since $N^{1,p}_{\text{loc}} - \text{Cap}_p(E, F, \Omega) \ge \text{Cap}_p(E, F, \Omega)$, it follows that

$$\text{Mod}_p(E, F, \Omega) = N^{1,p}_{\text{loc}} - \text{Cap}_p(E, F, \Omega) = \text{Cap}_p(E, F, \Omega) \tag{7}$$

whenever E and F are non-empty disjoint compact subsets of a domain Ω in a proper φ-convex metric measure space.

Lemma 3.4. *If E and F are non-empty disjoint compact subsets of Ω, there exist for each $0 < \varepsilon < \frac{1}{2}$ disjoint compact sets A_ε, B_ε of Ω such that for some $\delta > 0$*

(i) $\overline{\bigcup_{x\in E} B(x,\delta)} = A_\varepsilon$,
(ii) $\overline{\bigcup_{x\in F} B(x,\delta)} = B_\varepsilon$, *and*
(iii) $N^{1,p}_{\text{loc}} - \text{Cap}_p(A_\varepsilon, B_\varepsilon, \Omega) \le (1-2\varepsilon)^{-p} N^{1,p}_{\text{loc}} - \text{Cap}_p(E, F, \Omega) + \varepsilon(1-2\varepsilon)^{-p} + 2\varepsilon$.

Proof. We just give a brief sketch of the proof here. The idea is to use the test functions considered in calculating $N^{1,p}_{\text{loc}} - \text{Cap}_p(E, F, \Omega)$ to construct test functions for calculating $N^{1,p}_{\text{loc}} - \text{Cap}_p(A_\varepsilon, B_\varepsilon, \Omega)$. A priori we only know that the test functions u used in calculating $N^{1,p}_{\text{loc}} - \text{Cap}_p(E, F, \Omega)$ take on the value of 1 on E and the value of 0 on F. We need test functions that take on the value of 1 in a neighborhood of E and the value of 0 in a neighborhood of F. This is done by noting that by Proposition 1.2 we have that u is p-quasi-continuous, and hence by modifying u on a set of small p-capacity we obtain the required test function for calculating $N^{1,p}_{\text{loc}} - \text{Cap}_p(A_\varepsilon, B_\varepsilon, \Omega)$. □

By Lemma 3.4 and the equality (7), for each $0<\varepsilon<\frac{1}{2}$ there exist non-empty disjoint compact sets A_ε, B_ε as in Lemma 3.4 so that

$$\frac{1}{(1-2\varepsilon)^p}\mathrm{Mod}_p(E,F,\Omega)+\eta(\varepsilon)\geq N^{1,p}_{\mathrm{loc}}-\mathrm{Cap}_p(A_\varepsilon,B_\varepsilon,\Omega), \tag{8}$$

where $\lim_{\varepsilon\to 0}\eta(\varepsilon)=0$.

Lemma 3.5. *With X and Ω as in Theorem* 1.1 *and ε, δ, A_ε, B_ε, E, F, and Ω as in Lemma* 3.4, *it is true that*

$$N^{1,p}_{\mathrm{loc}}-\mathrm{Cap}_p(A_\varepsilon,B_\varepsilon,\Omega)\geq \mathrm{locLip\text{-}Cap}_p(E,F,\Omega).$$

Here the definition of $\mathrm{locLip\text{-}Cap}_p(E,F,\Omega)$ is the same as the definition of $\mathrm{Cont\text{-}Cap}_p(E,F,\Omega)$, but with functions u being required to be locally Lipschitz in Ω rather than merely continuous.

Proof. If u is an admissible function for calculating $N^{1,p}_{\mathrm{loc}}-\mathrm{Cap}_p(A_\varepsilon,B_\varepsilon,\Omega)$, then $u|_{\overline{\bigcup_{x\in E}B(x,\delta)}}=1$ and $u|_{\overline{\bigcup_{x\in F}B(x,\delta)}}=0$. Therefore if g is an upper gradient of u, then so is

$$\tilde g = g\chi_{\Omega\setminus\overline{\bigcup_{x\in E\cup F}B(x,\delta)}}.$$

Then if $x\in E\cup F$, denoting $U=\Omega\setminus\overline{\bigcup_{x\in E\cup F}B(x,\delta)}$,

$$M\tilde g^p(x)=\sup_{r>0}\fint_{B(x,r)}g^p\chi_U=\sup_{r\geq\delta}\fint_{B(x,r)}g^p\chi_U\leq\frac{1}{\mu\bigl(B(x,\delta)\bigr)}\|g\|^p_{L^p(\Omega)}.$$

Since the measure μ is doubling, for all $0<r\leq R$ and for all $x_1\in X$ and $x_0\in B(x_1,R)$,

$$\frac{\mu\bigl(B(x_0,r)\bigr)}{\mu\bigl(B(x_1,R)\bigr)}\geq C\Bigl(\frac{r}{R}\Bigr)^s$$

where C is a constant independent of x_0, x_1, r, and R, and $s=\log_2 C_2$ (C_2 being the doubling constant associated with μ). As $E\cup F$ is compact, there exists a positive number $\varrho>\delta>0$ and $x_1\in X$ so that $E\cup F\subset B(x_1,\varrho)$. Hence if x_0 is a point in $E\cup F$, then

$$\mu\bigl(B(x_0,\delta)\bigr)\geq C\Bigl(\frac{\delta}{\varrho}\Bigr)^s\mu\bigl(B(x_1,\varrho)\bigr)=:C_{E\cup F}.$$

Therefore for $x\in E\cup F$,

$$M\tilde g^p(x)\leq\frac{1}{C_{E\cup F}}\|g\|^p_{L^p(\Omega)}.$$

In the use of [Sh, Theorem 4.1] in the proof of Proposition 1.2, we can choose the Lipschitz constant λ for the approximating function to be sufficiently large, namely, choose λ so that $\lambda>\|g\|^p_{L^p(\Omega)}/C_{E\cup F}$, so that the approximating function u_ε agrees with the function u being approximated on $E\cup F$. Hence u_ε is an admissible function for calculating $\mathrm{locLip\text{-}Cap}_p(E,F,\Omega)$:

$$\mathrm{locLip\text{-}Cap}_p(E,F,\Omega)\leq\|g+g_\varepsilon\|^p_{L^p(\Omega)}\leq\|g\|^p_{L^p(\Omega)}+\varepsilon.$$

Thus $\mathrm{locLip\text{-}Cap}_p(E,F,\Omega)\leq N^{1,p}_{\mathrm{loc}}-\mathrm{Cap}_p(A_\varepsilon,B_\varepsilon,\Omega)$. □

Now combining inequalities (8) and Lemma 3.5 and then letting ε tend to zero, we obtain a proof of Theorem 1.1.

Remark 3.6. In the event that the domain Ω is all of X, using Semmes' idea [S1] and [Sh, Theorem 4.1], globally Lipschitz approximations can be obtained in Proposition 1.2 instead of merely locally Lipschitz approximations. Hence a better result is obtained: $\text{Cont-Cap}_p(E,F,\Omega) = \text{Lip} - \text{Cap}_p(E,F,\Omega) = \text{Mod}_p(E,F,\Omega)$.

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