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ANTICONFORMAL AUTOMORPHISMS AND SCHOTTKY COVERINGS

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Abstract. In this work, we consider anticonformal automorphisms of closed Riemann surfaces and Schottky groups. We study the problem of deciding when an anticonformal automorphism can be lifted for some Schottky covering (Schottky type automorphisms). This can be seen as generalization of the results due to Sibner [19], Heltai [8] and Natanzon [15] on anticonformal involutions. Also, for the conformal automorphisms, we study the relation between the condition of being the square of an anticonformal automorphism and of being of Schottky type.

1. Introduction and main results

The retrosection theorem (see [1], [12] and [13]) asserts that for every closed Riemann surface S there is a Schottky group G (a purely loxodromic Kleinian group isomorphic to a free group of finite rank), with region of discontinuity Ω, and a holomorphic covering $P\colon \Omega \to S$ with G as covering group. We say that $(\Omega, G, P\colon \Omega \to S)$ is a Schottky uniformization of S.

A conformal or anticonformal automorphism $f\colon S \to S$ of a closed Riemann surface S is called of Schottky type if there is a Schottky uniformization of S, say $(\Omega, G, P\colon \Omega \to S)$, such that f can be lifted by P (then such a lifting is the restriction of a Möbius transformation or the composition of a Möbius transformation with the conjugation).

A necessary and sufficient condition for a conformal automorphism to be of Schottky type, called condition (A), was given in [9].

Theorem 1. *Let $f\colon S \to S$ be a finite order conformal automorphism of a closed Riemann surface S. Then f is of Schottky type if and only if it satisfies condition* (A).

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The condition (A). Let $f\colon S \to S$ be a conformal automorphism of finite order and H be the cyclic group generated by f. Denote by $\mathscr{F}$ the set of fixed-points of the non-identity elements of H. For each $p \in \mathscr{F}$ and $h \in H - \{I\}$ with $h(p) = p$, we have a well defined number $\alpha(h,p) \in [-\pi,\pi)$, called the rotation number of h at p. Set $H(p) = \{h \in H : h(p) = p\}$. We say that f satisfies the condition (A) if there is a collection $\mathscr{C} = \{A_\alpha = \{p_\alpha, q_\alpha\} : \alpha \in \mathscr{A}\}$ of pairwise disjoint subsets of $\mathscr{F}$ such that:

(1) $p_\alpha \neq q_\alpha$, for all $\alpha \in \mathscr{A}$;
(2) $\bigcup_{\alpha\in\mathscr{A}} A_\alpha = \mathscr{F}$;
(3) For each $\alpha \in \mathscr{A}$, we have:
  (3.1) $H(p_\alpha) = H_\alpha = H(q_\alpha)$,
  (3.2) $\alpha(h,p) = -\alpha(h,q)$ for each $h \in H_\alpha$ of order greater than two,
  (3.3) there is no $h \in H$ satisfying $h(p_\alpha) = q_\alpha$.

A first approach to the study of anticonformal automorphisms is using its conformal square. If an anticonformal automorphism is of Schottky type then its square is also of Schottky type and then it satisfies condition (A).

In Section 2 we study the relation between the fact of being of Schottky type and to be the square of an anticonformal automorphism. The following results hold:

Theorem 2. *Let $f\colon S \to S$ be a conformal automorphism of a closed Riemann surface S. If there is an anticonformal automorphism $g\colon S \to S$ such that $g^2 = f$, then f is of Schottky type, equivalently, f satisfies the condition* (A).

Corollary 1. *Let $f\colon S \to S$ be a conformal automorphism of some closed Riemann surface S. If f does not satisfy the condition* (A), *then there is no orientation-reversing homeomorphism $\sigma\colon S \to S$ such that $f = \sigma^2$.*

Theorem 3. *Let $f\colon S \to S$ be a conformal automorphism of Schottky type of odd order. Then there is an orientation-reversing homeomorphism $g\colon S \to S$ such that $g^2 = f$.*

As application we provide a proof of a result of [5]: every embeddable (as the restriction of a rotation of the Euclidean 3-space) fixed point free automorphism of odd order is the square of an orientation-reversing homeomorphism.

There is an interesting complementation between the considerations in [18] and the above results. For example, the square of anticonformal automorphisms have unique liftings to matrices (cf. [18]) and they are of Schottky type by Theorem 2.

Next, we are interested in finding Schottky uniformizations for anticonformal automorphisms. In the case of anticonformal automorphisms of order two, results due to Heltai [8], Sibner [19] and Natanzon [15] assert the following:

Theorem 4. *Let $\sigma\colon S \to S$ be an anticonformal automorphism of order two of a closed Riemann surface S. Then σ is of Schottky type.*

The above result has been applied in the study of Schroedinger operators, by S.M. Natanzon in [15]. Now we study the anticonformal automorphisms with order > 2, we prove the following complete results for special situations in Sections 3 and 4:

Theorem 5. *Let $\sigma\colon S \to S$ be an anticonformal automorphism of order four of a closed Riemann surface S. Then σ is of Schottky type.*

Theorem 6. *Let $\sigma\colon S \to S$ be an anticonformal automorphism of a closed Riemann surface S. Set R the quotient surface obtained by the action of $f = \sigma^2$, and τ the anticonformal involution induced by σ on R. If τ has fixed points, then σ is of Schottky type.*

Remark. If the order of σ is $2N$, then the condition for τ to have fixed points (in the above theorem) implies that N is necessarily odd. The fixed point condition on τ is equivalent to the condition for σ^N to have fixed points.

At this point, we have that any anticonformal automorphism σ of order $2N$ (with N odd) such that σ^N has fixed points is necessarily of Schottky type.

In Section 5 we give a necessary condition, called condition (B), for an anticonformal automorphism to be of Schottky type. We proceed to see that there are anticonformal automorphisms which are not of Schottky type. In fact, we obtain a necessary condition to be satisfied by an anticonformal automorphism to be of Schottky type, called the condition (B).

The condition (B). Let $\sigma\colon S \to S$ be an anticonformal automorphism of order $2r$, H the cyclic group generated by $f = \sigma^2$, and $\mathscr{F}$ the set of fixed points of the non-trivial elements of H. We say that σ satisfies the condition (B) if there is a collection $\mathscr{C} = \{A_\alpha = \{p_\alpha, q_\alpha\} : \alpha \in \mathscr{A}\}$ of pairwise disjoint subsets of $\mathscr{F}$ satisfying the properties of condition (A) for the group H, and

$(*)$ If there is some δ in the group generated by σ and a pair $A_\alpha = \{p_\alpha, q_\alpha\} \in \mathscr{C}$ such that $\delta(p_\alpha) = q_\alpha$, then δ is an odd power of σ and $\delta(q_\alpha) = p_\alpha$.

Remarks. (1) If no non-trivial power of f has fixed points, then σ satisfies trivially the condition (B).

(2) Example 1 in Section 5 shows that every anticonformal automorphism of order $2N$, where N is odd, satisfies condition (B).

Theorem 7 (Condition (B) is necessary). *Let S be a closed Riemann surface and $\sigma\colon S \to S$ be an anticonformal automorphism of finite order. If σ is of Schottky type, then it satisfies condition* (B).

As a consequence of Theorem 7, we have that anticonformal automorphisms as in Theorems 4, 5 and 6, condition (B) is equivalent to the Schottky type condition. The actual question is if condition (B) is sufficient for the case when $\tau\colon R \to R$ has no fixed points. Example 3, at the end of Section 5, shows that in general condition (B) is not sufficient in this case. In Section 6 we establish a sufficient condition, condition (B1), in order for σ to be of Schottky type.

Condition (B1). Assuming the same notation as in the definition of condition (B), we say that σ satisfies the condition (B1) if there is a collection $\mathscr{C} = \{A_\alpha = \{p_\alpha, q_\alpha\} : \alpha \in \mathscr{A}\}$ satisfying condition (B) and the following extra property:

$(**)$ For all $\alpha \in \mathscr{A}$ and for all odd powers δ of σ, $\delta(p_\alpha) \neq q_\alpha$.

Theorem 8. *Let $\sigma\colon S \to S$ be an anticonformal automorphism of finite order. Set R the quotient Riemann surface obtained by the action of σ^2 on S, and let $\tau\colon R \to R$ the anticonformal involution induced by σ. If τ has no fixed points, then condition* (B1) *is sufficient for σ to be of Schottky type.*

In Example 4 of Section 6 we give an automorphism satisfying condition (B1) and being of Schottky type. Also the description of a Schottky uniformization is given.

Corollary 2. *Let $\sigma\colon S \to S$ be a fixed point free anticonformal automorphism of finite order of the closed Riemann surface S. Then σ is of Schottky type.*

Remark. In [6] it is shown that there are fixed point free anticonformal automorphisms that are non-embeddable and, by Corollary 2, of Schottky type.

Theorem 8 can be also written in the following way:

Theorem 8′. *Let $\sigma\colon S \to S$ be an anticonformal automorphism of finite order of a closed Riemann surface S. Condition* (B1) *is sufficient for σ to be of Schottky type if S/σ is a closed and non-orientable surface.*

Finally, using Theorems 5, 7 and 8, we can give a complete answer to the main problem of this work for automorphisms of order 6:

Theorem 9. *Let $\sigma\colon S \to S$ be an anticonformal automorphism of order 6 of the closed Riemann surface S. Then σ is of Schottky type.*

Remark. Conditions (B) and (B1) can be rewritten for any finite group of conformal and anticonformal automorphisms of a closed Riemann surface, giving necessary conditions for lifting it to some Schottky covering.

2. The squares of anticonformal automorphisms are of Schottky type

Theorem 2. *Let $f\colon S \to S$ be a conformal automorphism, $f \neq I$ of a closed Riemann surface S. If there is an anticonformal automorphism $g\colon S \to S$ such that $g^2 = f$, then f is of Schottky type, equivalently, f satisfies the condition* (A).

Proof. From the results in [9], we only need to see that the fixed points of the non-trivial powers of $f = g^2$ can be arranged into pairs satisfying the condition (A). Since the trivial automorphism is always of Schottky type, we assume $f \neq I$.

The idea is the same given in the proof of Theorem 1 in [5]. We will use the same notation $\alpha(f^k, p) \in [-\pi, \pi)$ given there.

Assume that $x \in S$ is the fixed point of some non-trivial power f^k and $\{f^r \in \langle f \rangle : f^r(x) = x\} = \langle f^k \rangle$.

We have that k necessarily divides the order r of f, and the equality

$$(*) \qquad gf^l g^{-1} = gg^{2l} g^{-1} = g^{2l} = f^l,$$

asserts that for $g(x) = y$ we have $\{f^r \in \langle f \rangle : f^r(y) = y\} = \langle f^k \rangle$.

We claim that $x \neq y$. In fact, if we have $x = y$, that is, $g(x) = x$, then g has order two (see [2]) and, as a consequence, $f = I$ a contradiction.

The equality $(*)$ also shows that

$$(**) \qquad \alpha(f^k, x) = -\alpha(f^k, y)$$

if f^k has order greater than two, and $\alpha(f^k, x) = \alpha(f^k, y) = -\pi$ if f^k has order two.

There is no element f^l with $f^l(x) = y$ with $l \leq$ order of f. In fact, if this happens, then $\alpha(f^k, x) = \alpha(f^k, y)$. This together with $(**)$ asserts that either f^k has order two or $f^k = I$. Since we have k less than the order of f, it follows that f^k has order two. Now we have that the order of g is $4k$.

The orbit of the point x is $\{f^m(x) : m = 0, 1, 2, \ldots, k-1\}$. In particular, we may assume $l \in \{1, 2, \ldots, k-1\}$. We have then the equality $g(x) = f^l(x)$ or equivalently $g^{2l-1}(x) = x$. This gives us two possibilities for g^{2l-1}: either it is the identity or has order two. But $2(2l-1) < 4k$ implies it cannot have order two and, for the other case, $2l - 1 = 0$ (since $2l - 1 < 4k$) is a contradiction. □

Corollary 1. *Let f: $S \to S$ be a conformal automorphism of some closed Riemann surface S. If f does not satisfy the condition* (A)*, then there is no orientation-reversing homeomorphism σ: $S \to S$ such that $f = \sigma^2$.*

Proof. In fact, if there is some orientation-reversing homeomorphism σ: $S \to S$ such that $\sigma^2 = f$, then there is a Riemann surface R and a quasiconformal homeomorphism h: $S \to R$ so that $h\sigma h^{-1}$ is an anticonformal automorphism of R. By Theorem 2, we have that hfh^{-1} is of Schottky type and, in particular, it satisfies the condition (A). It forces f to satisfy the condition (A). □

It is well known (see [7] and [16]) that every Riemann surface S is embeddable in $\mathbf{R}^3$, that is, there is a smooth surface $X \subset \mathbf{R}^3$ and a conformal homeomorphism T: $S \to X$. A conformal or anticonformal automorphism f: $S \to S$ is called embeddable if there are X and T as above so that TfT^{-1}: $X \to X$ is the restriction of a rigid motion. Necessary and sufficient conditions for a conformal automorphism to be embeddable were given by R. Rüedy in [17], the conditions for an anticonformal automorphism to be embeddable were given in [6].

Rüedy's conditions for an automorphism to be embeddable are a particular case of condition (A) and, in particular, every embeddable conformal automorphism is of Schottky type. The reverse is not true (in [5] there is an example of a Schottky type automorphism that is not embeddable, see also [20]).

Necessary and sufficient conditions for the square of an anticonformal automorphism of order $2r$, r a prime, to be embeddable are given in [5]. It is not hard to see that one may remove the condition r to be a prime, but then adding the following condition: if $p \in S$ is a fixed point of a non-trivial power of f, then p is also fixed by f.

Every involution is both embeddable [17] and of Schottky type [9]. In [5] it is shown that not every involution is the square of an orientation-reversing homeomorphism. In fact, necessary and sufficient conditions are given on the genus of the surface and the number of fixed points of the involution to have such a property.

For an embeddable automorphism $f\colon S \to S$ of odd order, Theorem 2 in [5] asserts the existence of an orientation-reversing homeomorphism $g\colon S \to S$ such that $g^2 = f$. The proof given in that paper requires the existence of fixed points of the automorphism, but with a suitable modification of the argument the fixed point free case follows. In the following, we give another proof of the case when the automorphism acts without fixed points by using Schottky groups.

Let S be a closed Riemann surface of genus g and $f\colon S \to S$ be a fixed point free conformal automorphism, of odd order $n > 1$. The Riemann–Hurwitz formula asserts that $g = n(\gamma - 1) + 1$, where γ denotes the genus of the quotient surface R.

From the results in [9], there is a Schottky group G of genus g, a set of generators of G, say $A_1, \dots, A_g$, and an elliptic transformation F of order n such that:

(1) $FA_g = A_g F$,

(2) $FA_{kn+i}F^{-1} = A_{kn+i+1}$, $i = 1, \dots, n-1$, and $FA_{(k+1)n}F^{-1} = A_{kn+1}$, for $k = 0, 1, \dots, \gamma - 2$.

Conjugating the above group by a suitable quasiconformal homeomorphism $W\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$, we may assume that there is a reflection J with fixed points so that

(3) $JWA_jW^{-1}J = WA_j^{-1}W^{-1}$, all $j = 1, \dots, g$, and

(4) $JWFW^{-1}J = WFW^{-1}$.

If we take $L = WF^{(n-1)/2}W^{-1}J$, then

$$\begin{aligned} L^2 &= WF^{(n+1)/2}W^{-1}JWF^{(n+1)/2}W^{-1}J \\ &= WF^{n+1}W^{-1} = WFW^{-1}. \end{aligned}$$

The orientation-reversing homeomorphism $W^{-1}LW$ descends to an orientation-reversing homeomorphism on S with square equal to f, obtaining in this way the desired result.

Similar arguments as above permit us to show that Theorem 2 in [5] is also valid for the category of Schottky type automorphisms. At this point, we remark that there are conformal automorphisms of Schottky type that are not embeddable.

Theorem 3. *Let $f\colon S \to S$ be an automorphism of Schottky type of odd order. Then there is an orientation-reversing homeomorphism $g\colon S \to S$ such that $g^2 = f$.*

Proof. Let us denote by R the Riemann surface quotient by the action of the cyclic group generated by f, of odd order r.

We have a natural quotient map $P\colon S \to R$ induced by the action of f. If we consider a pairing $B = \{(p_1,q_1),\dots,(p_n,q_n)\}$ obtained from the condition (A), then the surface R can be uniformized, from the results in [9], by a Kleinian group K with:

(i) K has as generators loxodromic elements $A_1,\dots,A_\gamma$, and elliptic elements $E_1,\dots,E_n$, F, where F has order r, and E_i has order l_i a divisor of r (the branching number of $P(p_i)$), with a fundamental domain as shown in Figure 1,

(ii) the group G given by the smallest normal subgroup of K containing the elements $A_1,\dots,A_\gamma$, $F^{r/l_i}E_i^{-1}$, is a Schottky group uniformizing S, and

(iii) the transformation F is a lifting of the automorphism f.

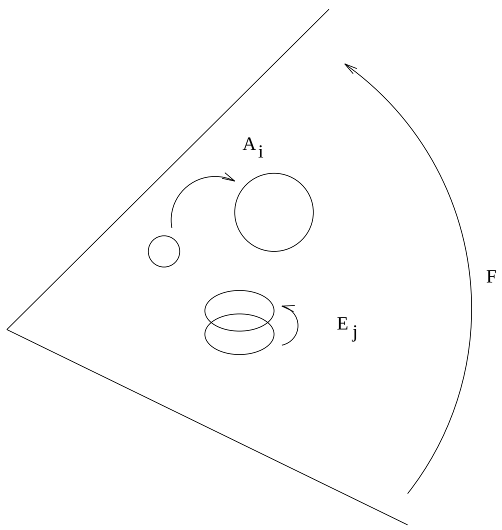

Figure 1.

We can conjugate K by a quasiconformal homeomorphism $W\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ so that, if J denotes the reflection on the unit circle, the following hold (see Figure 2):

(1) $JWE_iW^{-1} = E_i$,
(2) $JWA_jW^{-1} = WA_j^{-1}W^{-1}$,
(3) $JWFW^{-1}J = WFW^{-1}$.

Let us consider $L = JWF^{(r+1)/2}W^{-1}$. In this case, $W^{-1}LW$ gives an orientation-reversing homeomorphism with square equal to F. The projection of it to S gives the desired orientation-reversing homeomorphism. □

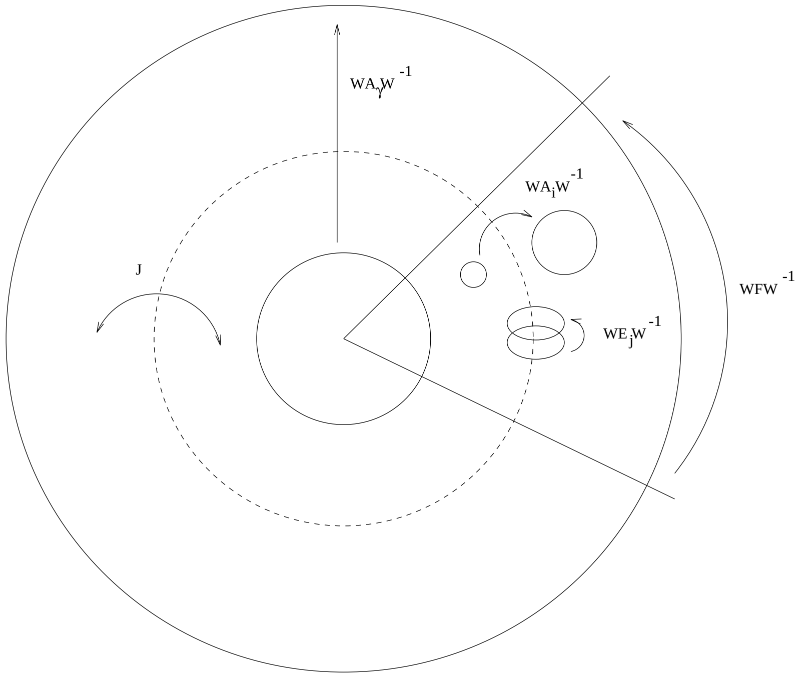

Figure 2.

3. Anticonformal automorphisms of order four

Theorem 5. *Let $\sigma\colon S \to S$ be an anticonformal automorphism of order four of a closed Riemann surface S. Then σ is of Schottky type.*

Proof. We can observe in this case, since $f = \sigma^2$ has order two, that σ^2 satisfies condition (A) and that property (B) holds trivially.

We assume the genus g of S to be at least two (the low genera cases can be checked by direct inspection).

Let us denote by $P\colon S \to R$ the holomorphic (possible branched) 2-fold covering, induced by the action of the cyclic group of order two H.

On R we have a natural anticonformal automorphism of order two $\tau\colon R \to R$ with $\tau P = P\sigma$.

Since σ has order four then τ is fixed point free (see for instance [2]). From Harnack's theorem and its extension in [14], there is a set of pairwise disjoint simple curves in R, $\alpha_i : i = 1, \dots, r$, such that

(1) $\tau(\alpha_i) = \alpha_i$, all $i = 1, \dots, r$, the curves α_i are invariant but not fixed by τ, in fact τ acts as a rotation of angle π.

(2) $R - \bigcup_{i=1}^{n} \alpha_i$ consists of two surfaces, R_+ and R_-, each one of genus γ and exactly n boundary components, and

(3) τ permutes the two surfaces R_+ and R_-.

If we consider a simple closed curve β, for instance in R_+, such that divides R in two components, one of them containing all the singular points and all the curves α_i, then such a curve lifts by P to a closed curve (it is the boundary of a surface without branch points). Hence the following formula holds: $\#\operatorname{Fix}(\sigma^2)/2 + r = 0 \pmod 2$. We have that the square of each one of the curves α_i lifts to a loop.

The fact that the loops $\alpha_1, \dots, \alpha_r$, is a collection of pairwise disjoint simple loops on R asserts that $\mathscr{F} = P^{-1}(\{\alpha_i : i = 1, \dots, n\})$ is a collection of pairwise disjoint simple loops on S.

Following the results in [10], we are able to construct a set of 2γ disjoint homologically independent simple loops on R_+, say $\eta_1, \dots, \eta_{2\gamma}$, each one disjoint from β, such that each loop lifts to a loop on $\widetilde{S}$.

Now consider the simple closed curves δ_i, $i = 1, \dots, \lfloor (r+1)/2 \rfloor$ enclosing in R_+ two curves α_{2i}, α_{2i+1}, and the curves ε_i, $i = 1, \dots, \lfloor \#\operatorname{Fix}(\sigma^2)/4 \rfloor$ enclosing two singular points in R_+; see Figure 3(i). Finally, if r is odd, we construct one more closed curve ω enclosing the last singular point and α_r; see Figure 3(ii).

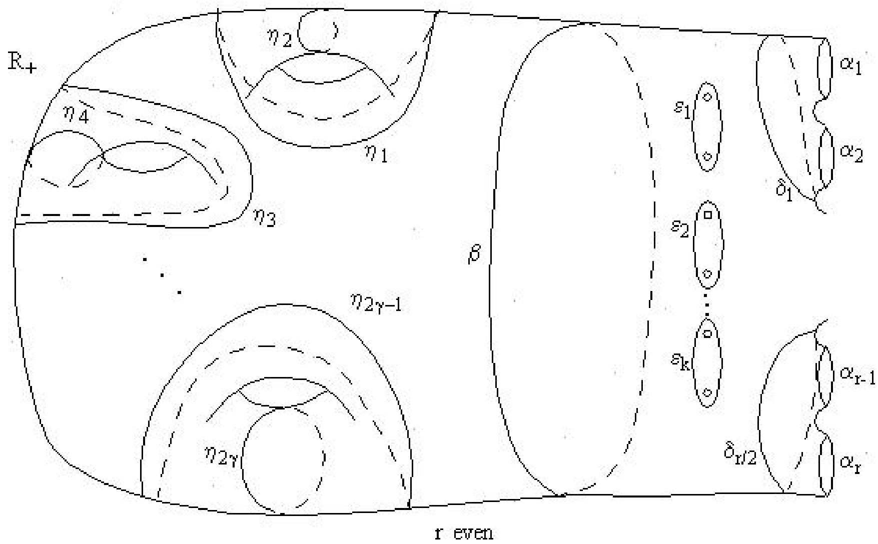

Figure 3(i).

Now the collection of curves on S, obtained by lifting all of the loops $\alpha_1, \dots, \alpha_r$, β, $\eta_1, \dots, \eta_{2\gamma}$, δ_i, $i = 1, \dots, \lfloor (r+1)/2 \rfloor$, ε_i, $i = 1, \dots, \lfloor \#\operatorname{Fix}(\sigma^2)/4 \rfloor$, ω above and their images by τ, is invariant under the action of σ and cut-off S into genus zero surfaces. This is enough to imply that σ is of Schottky type. □

4. Anticonformal automorphisms with a power that is an anticonformal involution with fixed curves

Theorem 6. *Let $\sigma\colon S \to S$ be an anticonformal automorphism of a closed Riemann surface S. Set R the quotient surface obtained by the action of $f = \sigma^2$, and τ the anticonformal involution induced by σ on R. If τ has fixed points, then σ is of Schottky type.*

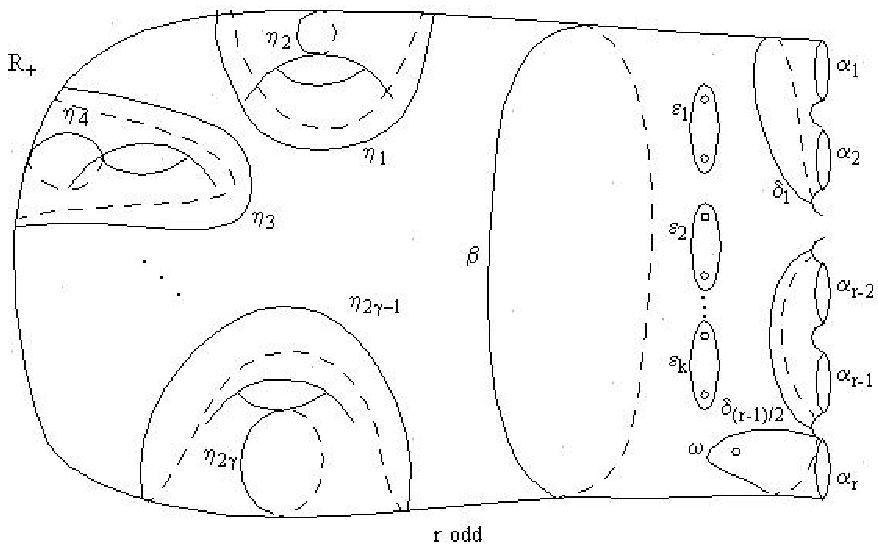

Figure 3(ii).

Proof. Harnack's theorem and its extension in [14], asserts that on R we have a set of pairwise disjoint simple loops $\theta_1, \ldots, \theta_k$, $\eta_1, \ldots, \eta_l$, such that

(i) $\mathrm{Fix}(\tau) = \theta_1 \cup \cdots \theta_k$, and $k \geq 1$;
(ii) $\tau(\eta_j) = \eta_j$;
(iii) R is divided by the above loops into two connected components, say R_1 and R_2, which are permuted by σ.

The surface R_1 has genus t and $k + l$ holes. We draw a set of simple loops and arcs $\alpha_1, \ldots, \alpha_t$, $\beta_1, \ldots, \beta_t$, $\gamma_1, \ldots, \gamma_t, \gamma$, $\delta_1, \ldots, \delta_r$, $\kappa_{\theta_2}, \ldots, \kappa_{\eta_l}$, as shown in Figure 4.

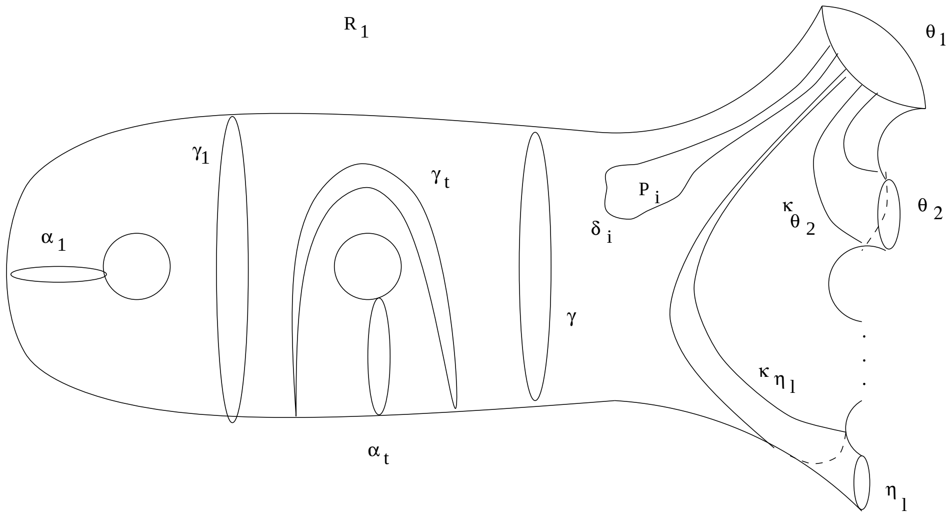

Figure 4.

Since we have a regular cyclic covering $\pi: S \to R$, we have that the loops

$\gamma_1, \dots, \gamma_t$ and γ must lift to a loop (this is simple consequence of the fact that each γ_i is free homotopic to the commutator between α_i and β_i, and γ is free homotopic to the product $\gamma_1 \cdots \gamma_t$). Moreover, we have that any connected component of the lifting of any of the three-holed sphere determined by α_i and γ_i is a sphere with holes. The sphere with holes bounded by $\gamma_1, \dots, \gamma_t$ and γ lifts homeomorphically to a sphere with same number of holes.

Each loop $\tilde{\delta}_j = \delta_j \cup \tau(\delta_j)$ also lifts to loops (this is the condition (A) assumption, which holds trivially in our case), and the bounded topological disc lifts to spheres with holes.

In a similar way, each loop $\tilde{\kappa}_M = \kappa_M \cup \tau(\kappa_M)$ determines a torus (τ invariant) with a hole. Since this loop is a commutator, it lifts to a loop on S, and the three-holed sphere determined by this loop and M lifts to spheres with holes, where $M \in \{\theta_2, \dots, \eta_l\}$.

The sphere with holes determined by γ, $\tau(\gamma)$ and the loops $\tilde{\delta}_1, \dots, \tilde{\delta}_r$, $\tilde{\kappa}_{\theta_2}, \dots, \tilde{\kappa}_{\eta_l}$, lifts homeomorphically to spheres with the same number of holes.

The family $\mathscr{F}$ of pairwise disjoint simple loops on S, obtained by lifting the loops $\alpha_1, \dots, \alpha_t$, $\gamma_1, \dots, \gamma_t$, γ, $\tau(\alpha_1), \dots, \tau(\alpha_t)$, $\tau(\gamma_1), \dots, \tau(\gamma_t)$, $\tau(\gamma)$, $\theta_2, \dots, \theta_k$, $\eta_1, \dots, \eta_l$, $\tilde{\delta}_1, \dots, \tilde{\delta}_r$, $\tilde{\kappa}_{\theta_2}, \dots, \tilde{\kappa}_{\eta_l}$, is invariant under the action of σ, and the connected components of $S - \mathscr{F}$ are all of genus zero.

It follows that on $\mathscr{F}$ there is a subfamily $\mathscr{G}$ consisting of g homologically independent pairwise disjoint simple loops (g is the genus of S) determining a Schottky uniformization of S for which σ lifts. □

5. A necessary condition: Condition (B)

As consequence of Theorem 6, the case we need to consider from now on is when the induced anticonformal involution $\tau: R \to R$ is fixed point free. We start this section giving a necessary condition for an anticonformal automorphism of finite order (with or without fixed points) to be of Schottky type.

The condition (B). Let $\sigma: S \to S$ be an anticonformal automorphism of order $2r$, H the cyclic group generated by $f = \sigma^2$, and $\mathscr{F}$ the set of fixed points of the non-trivial elements of H. We say that σ satisfies the condition (B) if there is a collection $\mathscr{C} = \{A_\alpha = \{p_\alpha, q_\alpha\} : \alpha \in \mathscr{A}\}$ of pairwise disjoint subsets of $\mathscr{F}$ satisfying the properties of condition (A) for the group H, and

($*$) if there is some δ in the group generated by σ and a pair $A_\alpha = \{p_\alpha, q_\alpha\} \in \mathscr{C}$ such that $\delta(p_\alpha) = q_\alpha$, then δ is an odd power of σ and $\delta(q_\alpha) = p_\alpha$.

Theorem 7 (Condition (B) is necessary). *Let S be a closed Riemann surface and $\sigma: S \to S$ be an anticonformal automorphism of finite order. If σ is of Schottky type, then it satisfies condition* (B).

Proof. We follow the same ideas as in [9]. Let us consider a closed Riemann surface S and an anticonformal automorphism $\sigma: S \to S$ of order $2r$. Denote by

$\widetilde{H}$ the cyclic group generated by σ, by H the cyclic group generated by $f=\sigma^2$, and by $\eta=\sigma^r$.

Assume there is a Schottky covering $(\Omega, G, P\colon \Omega \to S)$ for which σ lifts. Let $\Sigma\colon \Omega \to \Omega$ be a lifting of σ (which is the restriction of an anticonformal automorphism of the Riemann sphere since Ω is of type O_{AD}). Set J the group obtained by lifting the cyclic group $\widetilde{H}$. We have that J is a finite normal extension of index $2r$ of G (in particular, J cannot have parabolic elements).

Let $p\in S$ be a fixed point of some non-trivial power $h=f^l$ in H. We assume that h generates the cyclic group $H(p)$. The transformation Σ^{2l} is a lifting of h.

Take some $x\in\Omega$ with $P(x)=p$. Then there is some $g\in G$ with $\Sigma^{2l}(x)=g(x)$. Let us consider the transformation $T=g^{-1}\Sigma^{2l}$ which belongs to the index two subgroup of orientation-preserving transformations of J (a finite normal extension of index r of G).

Since $h\neq I$, we have that T is an elliptic transformation, different from the identity, with $x\in\Omega$ as a fixed point. The results in [11] assert that its other fixed point y must also be in Ω.

We claim that y cannot be equivalent by G to x. In fact, if $P(x)=P(y)$, then there is some $k\in G-\{I\}$ with $k(x)=y$. The elliptic transformations kTk^{-1} and T both fix the point x. The absence of parabolic transformations in J asserts that both have the same fixed points. It follows that $k(y)=x$, and $k^2=I$. This is a contradiction to the fact that Schottky groups have no elliptic transformations (different from the identity).

Set $q=P(y)$ and assume that there is some $\delta\in\widetilde{H}-\{I\}$ so that $\delta(p)=q$, and let us consider a lifting $\theta\in J$ of δ. We have that there is some $k\in G$ with $\theta(x)=k(y)$ or, equivalently, $k^{-1}\theta(x)=y$. Again we argue as above to obtain that the absence of parabolic transformations forces $L=k^{-1}\theta$ to permute x and y and, in particular, we have that δ permutes p and q.

Since $\delta(p)=q$ and $\delta h\delta^{-1}=h$, we must have either

(a) $\alpha(h,p)=\alpha(h,q)$, if $\delta\in H$; or

(b) $\alpha(h,p)=-\alpha(h,q)$, if $\delta\notin H$.

If $\delta\in H$, then (a) asserts that $H(p)$ has order two. From this, we must have r even and, in particular, that $\delta\in H$. It follows that L is an orientation-preserving Möbius transformation of order two permuting the points x and y, and it follows that L and T generate the abelian dihedral group of order 4. But this is a contradiction to the fact that δ and h commute.

Now the construction of a collection $\mathscr{C}$ satisfying the properties of condition (B) follows from the above. ▫

Remarks. (1) As a consequence of Theorems 6 and 7, we have that any anticonformal automorphism σ of order $2N$, with N odd, and for which σ^N has fixed points, must satisfy the condition (B). In general, the only property for condition (B) to hold automatically is to have N odd (see Example 1 below).

(2) Theorem 7 gives us a restriction on the class of anticonformal automorphisms to be of Schottky type. Example 2 below shows that there are in fact anticonformal automorphisms that cannot satisfy the condition (B) and, in particular, cannot be of Schottky type.

(3) Condition (B) may be translated in terms of the branch values of the orbifold S/σ. Let us assume Γ is a NEC group that uniformizes S/σ. Let us denote by F the finite index normal subgroup of Γ that uniformizes S/f, where $f=\sigma^2$. Let us assume that Γ has signature $\big(g,\pm;[m_1,\dots,m_r];\{(-),\dots,(-)\}\big)$. If $\{(-),\dots,(-)\}$ is different from $\{-\}$ (that is, S/σ has boundary), then the order N of $f=\sigma^2$ is odd. In this case, condition (B) is nothing but condition (A) as already observed in (1). In the other situation, that is $N=2^a s$, with s odd, the signature of Γ has the form $(g,\pm;[m_1,\dots,m_r])$. The branch values associated with $m_i \neq 2^{a-1}t$ with t odd are paired so that condition (A) holds.

Example 1. Every anticonformal automorphism of order $2N$, with N odd, satisfies trivially the condition (B). In effect, let us assume we have $\sigma\colon S\to S$ an anticonformal automorphism of order $2N$, with N odd. Set R the quotient Riemann surface obtained by the action of $f=\sigma^2$ on S, and by $\tau\colon R\to R$ the anticonformal involution induced by σ. If $p\in R$ is a branch value, then set $q=\tau(p)$. If we denote by $\pi\colon S\to R$ the natural holomorphic branched covering, then $\pi^{-1}(p)=\{x_1,\dots,x_l\}$ and $\pi^{-1}(q)=\{y_1,\dots,y_l\}$, where l is odd. We may assume that $f(x_i)=x_{i+1}$, $f(y_i)=y_{i+1}$, i modulo l, and $\sigma(x_1)=y_1$. If we proceed to pair the point $f^k(x_1)$ with $f^k(y_{(1+l)/2})$, with $k=0,1,\dots,l-1$, then we get a pairing satisfying the condition (B).

In particular, assume we have an anticonformal automorphism $\sigma\colon S\to S$ of finite order such that the induced anticonformal involution $\tau\colon R\to R$ has fixed points. In this case, necessarily the order of σ must be of the form $2N$, with N odd. It follows from the above that σ satisfies the condition (B).

Example 2. The above example shows that if the order of the anticonformal automorphism σ is $2N$, with N odd, then σ always satisfies the condition (B). On the other hand, if the order of σ is $2N$, with N even, the situation is different as can be seen in the following.

Consider the NEC group

$$\Gamma=\langle A,B,C : A^3=B^4=I,\ CBC=A\rangle,$$

uniformizing a projective plane with two singular points of order 3 and 4, respectively.

Let us consider the surjective homomorphism $\Phi\colon\Gamma\to\langle x\rangle\cong\mathbf{Z}/24\mathbf{Z}$ defined by $\Phi(A)=x^8$, $\Phi(B)=x^6$ and $\Phi(C)=x$.

Denote by F the kernel of Φ. Then F is a torsion-free normal subgroup of Γ containing only orientation-preserving isometries.

By Riemann–Hurwitz, $\mathbf{H}^2/F$ is a closed Riemann surface of genus 6, and C descends to an anticonformal automorphism σ of order 24.

If we set $f = \sigma^2$, then we obtain that f and f^2 do not leave fixed points, f^3 has exactly 6 fixed points and f^4 has exactly 8 fixed points.

Let us look at the 8 fixed points of f^4. We must have that they are given by $\{a, f(a), f^2(a), f^3(a)\}$ and $\{b, f(b), f^2(b), f^3(b)\}$.

We cannot have $\sigma(a) \in \{a, f(a), f^2(a), f^3(a)\}$ since the rotation number of f^4 (an element of order three) is the same at each of these points and σ would reverse it. It follows that $\sigma(a) \in \{b, f(b), f^2(b), f^3(b)\}$.

Without loss of generality we may assume that $\sigma(a) = b$. Now, if we want to have a pairing satisfying the condition (B), we must pair a with some of the points in $\{b, f(b), f^2(b), f^3(b)\}$.

(1) If we pair a and b, then $b = \sigma(a)$ and condition (B) imply that $a = \sigma(b) = f(a)$, a contradiction.

(2) If we pair a and $f(b)$, then $f(b) = \sigma^3(a)$ and condition (B) imply that $a = \sigma^3\bigl(f(b)\bigr) = f^3(a)$, a contradiction.

(3) If we pair a and $f^2(b)$, then $f^2(b) = \sigma^5(a)$ and condition (B) imply that $a = \sigma^5\bigl(f^2(b)\bigr) = \sigma^{10}(a) = f^5(a) = f(a)$, a contradiction.

(4) If we pair a with $f^3(b)$, then $f^3(b) = \sigma^7(a)$ and condition (B) imply that $a = \sigma^7\bigl(f^3(b)\bigr) = \sigma^{14}(a) = f^7(a) = f^3(a)$, a contradiction.

The above asserts the impossibility to pair the fixed points of f^4 to satisfy the condition (B). We can also use part (3) of the remark after Theorem 7 to obtain that σ cannot satisfy condition (B).

Example 3. The following is an example of an anticonformal automorphism of order 30 acting on a Riemann surface of genus 8, with induced anticonformal involution acting fixed point free, satisfying condition (B) but not of Schottky type. This shows that condition (B) is not sufficient under this assumption.

Let us consider the NEC group Γ generated by A, B and C, with $A^3 = B^5 = B^{-1}CAC = I$ (C is a glide reflection), acting on the hyperbolic plane $\mathbf{H}^2$ with quotient the projective plane having exactly two singular points p and q of orders 3 and 5, respectively.

We consider the surjective homomorphism $\Phi\colon \Gamma \to \langle x : x^{30} = 1\rangle$, defined by $\Phi(A) = x^{10}$, $\Phi(B) = x^{12}$ and $\Phi(C) = x$.

Set G the kernel of Φ. Then G is a torsion-free Fuchsian group of genus 8. On the closed Riemann surface $S = \mathbf{H}^2/G$ there is an anticonformal automorphism (induced by C) of order 30, say σ.

We denote by $\pi\colon S \to M = \mathbf{H}^2/\Gamma$ the natural di-analytic branched covering induced by the action of σ on S.

Set $\pi^{-1}(p) = \{x_1, \dots, x_5, y_1, \dots, y_5\}$, and $\pi^{-1}(q) = \{z_1, \dots, z_3, w_1, \dots, w_3\}$, where $\sigma(x_i) = y_i$, $\sigma(y_i) = x_{i+1}$, $\sigma(z_i) = w_i$ and $\sigma(w_i) = z_{i+1}$.

As seen in Example 1, this anticonformal automorphism trivially satisfies the condition (B). More precisely, if we pair x_1 with y_3 and z_1 with w_2, and we translate them by the powers of σ, then we obtain a pairing satisfying the condition (B).

Now we proceed to see that σ cannot be of Schottky type. Assume this is the case. Then we have a Schottky uniformization $(\Omega, G, P\colon \Omega \to S)$ for which σ lifts. The existence of an anticonformal Möbius transformation T follows, satisfying:

(1) $PT = \sigma P$;

(2) $TGT^{-1} = G$;

(3) $T^{30} \in G$.

The group $\widehat{G}$, generated by G and T, uniformizes the projective plane $\mathbf{RP}_2$ with exactly two singular points of order 3 and 5, respectively. The index two subgroup $\widehat{G}^+$ of orientation-preserving transformations uniformizes the Riemann sphere $\widehat{\mathbf{C}}$ with four singular points of order 3, 3, 5 and 5. Let us denote by τ the anticonformal involution induced by σ on the Riemann sphere.

We have a simple loop $\gamma \subset \widehat{\mathbf{C}}$ invariant by τ, so that each of the two discs determined by γ contains exactly one singular point of order 3 and other of order 5. Denote by D one of these two discs, that is, $\partial D = \gamma$, and by $\widehat{D}$ a connected component of the lifting of D on Ω.

Assume that γ does not lift to a loop on Ω. In this case let us consider the subgroup $\widehat{G}^+(\widehat{D}) = \{g \in \widehat{G}^+ : g(\widehat{D}) = \widehat{D}\}$. Since $\widehat{G}^+$ is a subgroup of $\widehat{G}$, whose limit set is a totally disconnected set, we have that it is a function group (also with totally disconnected limit set).

We have that $\widehat{G}^+$ uniformizes a sphere with exactly three singular points (two of them of order 3 and 5 and the other of order $2 \le n \le \infty$, where n is either the minimal positive power so that γ^n lifts to a loop on Ω or ∞ if it does not exist). This asserts that $\widehat{G}^+(\widehat{D})$ is a finitely generated subgroup of the geometrically finite Kleinian group $\widehat{G}^+$ and, by a result due to Thurston, it must also be geometrically finite.

In resume, $\widehat{G}^+(\widehat{D})$ is a geometrically finite function group uniformizing a sphere with three branch values. It follows by results due to I. Kra that this group must be a triangular Fuchsian group. This is a contradiction to the fact that the limit set is totally disconnected as observed above.

The above arguments then ensure that γ must lift to a loop on Ω and, in particular, on S. Let us consider a connected component $X \subset S$ of a lifting of D. We have that X must be a closed Riemann surface of genus $g_1 < 8$ with some holes.

Denote by H the cyclic group, of order 15, generated by σ^2. Set $H(X) = \{h \in H : h(X) = X\}$ and N the order of it. It follows that $N \in \{1, 3, 5, 15\}$ and that the number of holes of X is exactly N. We must have the equality $8 = 2g_1 + N - 1$. The fact that in D we have singular points of orders 3 and 5, asserts that $N = 15$, a contradiction to the above equality.

It follows that σ cannot be of Schottky type.

6. A sufficient condition: Condition (B1)

Let $\sigma\colon S \to S$ be an anticonformal automorphism of order $2N$ so that the induced anticonformal involution $\tau\colon R \to R$, where $R = S/\sigma^2$, does not have fixed points. Example 3 shows that condition (B) is not always sufficient to ensure σ to be of Schottky type. In this section, we add an extra property to condition (B) to obtain condition (B1) which turns out to be sufficient in this case.

Condition (B1). Assuming the same notation as in the definition of condition (B), we say that σ satisfies the condition (B1) if there is a collection $\mathscr{C} = \{A_\alpha = \{p_\alpha, q_\alpha\} : \alpha \in \mathscr{A}\}$ satisfying condition (B) and the following extra property:

$(**)$ For all $\alpha \in \mathscr{A}$ and for all odd powers δ of σ, $\delta(p_\alpha) \neq q_\alpha$.

Theorem 8. *Let $\sigma\colon S \to S$ be an anticonformal automorphism of finite order. Set R the quotient Riemann surface obtained by the action of σ^2 on S, and let $\tau\colon R \to R$ the anticonformal involution induced by σ. If τ has no fixed points, then condition* (B1) *is sufficient for σ to be of Schottky type.*

Proof. We have that $\tau\colon R \to R$ acts without fixed points. In this case, we have two possibilities:

Case (a) There is a dividing simple loop α on R, which is τ invariant, so that R is divided by α into two surfaces, say R_1 and R_2. On R_1 we can, by condition (B1), construct a set of pairwise disjoint simple loops, each one surrounding exactly two branch points coming from a pair. Each of these loops lifts to a loop.

We construct (as in the proof of Theorem 6) a set of pairwise disjoint simple loops, each one lifting to a loop on S, so that they cut off R_1 into a sphere bounded by α and these loops. Denote it by E.

This new surface E has no branch values, in which case α lifts to a loop.

The stabilizer (in the cyclic group generated by σ) of any of the lifts of α is always trivial or always a cyclic group of order two (generated by $\eta = \sigma^r$). Proceeding as in the proof of Theorem 6, we get the desired result.

Case (b) There are two disjoint non-dividing simple loops, each one τ invariant, so that both together divide R into two components. Let us denote by α_1 and α_2 the above two disjoint simple loops. Following as in case (a), we have that the stabilizer of any lifts of α_1 is the same as for any lift of α_2.

We consider a simple loop α on R_1 (one of the two surfaces obtained from R after cutting along α_1 and α_2) so that these three loops bound a three-holed sphere and no branch point is in it.

We have that α must lift to a loop. The liftings of the three-holed sphere are spheres with holes.

The other part of R_1 is bounded by α and we can proceed as in case (a). □

Example 4. Let us consider the closed Riemann surface S of genus two given by the following algebraic equation

$$y^2 = (x-a)\left(x - \frac{w}{a}\right)(x - w^2 a)\left(x + \frac{1}{a}\right)(x + wa)\left(x + \frac{w^2}{a}\right),$$

where $a > 1$ and $w = e^{2\pi i/6}$. This one has an anticonformal automorphism σ of order six given by

$$\sigma : \begin{cases} x \longmapsto \dfrac{w}{\overline{x}}, \\ y \longmapsto -\overline{y}. \end{cases}$$

The quotient Klein surface obtained by the action of σ on S is the projective plane with exactly two branch points of order 3. The quotient Riemann surface obtained by the action of $f = \sigma^2$ is the sphere with four points of order 3. Theorem 7 asserts that σ is of Schottky type. In effect, consider the Kleinian group J freely generated by two transformations B and C, with $B^3 = C^2 = I$, where C is a symmetry with no fixed points. The Schottky group G, with generators $B^{-1}CBC$, $B^{-2}CBCB$ and $CBCB^{-1}$ (which are not free generators of G), uniformizes S with the property that σ lifts. The lifting of σ is given by BC and a lifting of f is given by B^2.

Corollary 2. *Let σ: $S \to S$ be a fixed point free anticonformal automorphism of finite order of the closed Riemann surface S. Then σ is of Schottky type.*

Proof. Let us denote by R the quotient Riemann surface, of genus γ, obtained by the action of σ^2 on S and by τ: $R \to R$ the anticonformal involution induced by σ. The fact that σ is fixed point free asserts that (i) σ satisfies trivially condition (B1) and (ii) τ has no fixed points. Now we are in the hypothesis of Theorem 8. $\square$

Example 5. Assume in Corollary 2 that the order of σ is $2q$, with q an odd positive integer, and the genus γ of S/σ to be at least two. We proceed to describe explicitly the Schottky uniformization that lifts σ. Let us consider the group J generated by a glide-reflection B and $(\gamma - 2)$ fixed point free anticonformal involutions $\tau_1, \dots, \tau_{\gamma-2}$, so that a fundamental domain is as shown in Figure 5. We have that J is isomorphic to the free product of $\mathbf{Z}$ and $(\gamma - 2)$ copies of $\mathbf{Z}_2$.

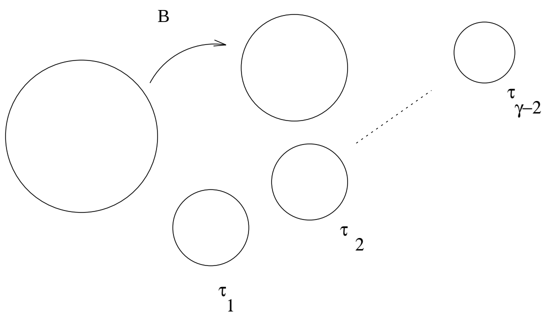

Figure 5.

The index two subgroup J^+ of orientation preserving elements of J is free generated by the loxodromic transformations B^2, $B\tau_1, \dots, B\tau_{\gamma-2}$. In fact, J^+ is a Schottky group of genus $\gamma - 1$.

We consider G the subgroup generated by the transformations B^q, $B^q\tau_1, \ldots,$ $B^q\tau_{\gamma-2}$, $B^{q+1}\tau_1 B^{-1}, \ldots, B^{q+1}\tau_{\gamma-2}B^{-1}, \ldots, B^{2q-1}\tau_1 B^{1-q}, \ldots, B^{2q-1}\tau_{\gamma-2}B^{1-q}$.

Then we have that G is a normal subgroup of J of index $2q$, so that J/G is isomorphic to the cyclic group $\mathbf{Z}_{2q}$, and it is a Schottky group of genus $g = q(\gamma-2)+1$.

Using quasiconformal deformation theory and a theorem in [3], which asserts that the topological action of σ is unique, we get that the above group is the desired (up to quasiconformal conjugation) Schottky group. This is also an argument for the proof in the restricted case that the order of σ is $2q$, with q odd.

In the case that the order of σ is $2q$, with q even, we must necessarily have that γ is even. This fact is a consequence of the following. We have a surjective homomorphism from the (orbifold) fundamental group Γ that uniformizes S/σ onto the cyclic group $\mathbf{Z}_{2q}$, say $\Psi\colon \Gamma \to \mathbf{Z}_{2q}$. If γ is odd, then Γ is generated by $a_1, \ldots, a_g$, $b_1, \ldots, b_g$ and c, with $2g = \gamma - 1$, and defining relation $c^2\Pi_{i=1}^{g}[a_i, b_i] = 1$, where $[a,b]$ denotes the commutator between a and b. From this, we have that $\Psi(c)^2 = 1$. The only possibilities are that $\Psi(c) \in \{1, [q]\}$. In either case, we will have that c must preserve orientation (because q is even) and, this is a contradiction.

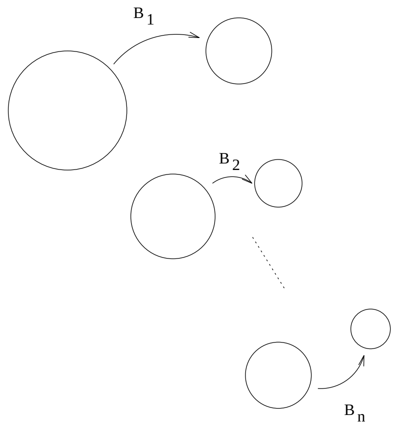

Figure 6.

Now, we write $\gamma = 2n$. We consider the group J generated by the glide-reflections $B_1, \ldots, B_n$, with fundamental domain as shown in Figure 6. The index two subgroup J^+ of orientation preserving elements of J is a Schottky group of genus $\gamma - 1$, free generated by the loxodromic transformations B_1^2, $B_1B_2, \ldots, B_1B_n$, $B_2B_1^{-1}, \ldots, B_nB_1^{-1}$. To get a Schottky group G as desired (up to quasiconformal conjugation) we need to consider all possible surjective homomorphisms $\Phi\colon J \to \mathbf{Z}_{2q}$ so that $\Phi(B_i) = [t_i]$ and

(i) each t_i is odd, and

(ii) $\mathbf{Z}_{2q}$ is generated by $[t_1], \ldots, [t_n]$.

For then the kernel G of each isomorphism Φ as above gives the desired Schottky group.

7. Anticonformal automorphisms of order six

We want to end with the following result concernig anticonformal automorphisms of order six. The particularity in order 6 is that the branch values on the quotient surface R all are of order 3.

Theorem 9. *Let $\sigma: S \to S$ be an anticonformal automorphism of order 6 of the closed Riemann surface S. Then σ is of Schottky type.*

Proof. If the induced anticonformal involution τ on the quotient surface $R = S/f$, where $f = \sigma^2$ acts with fixed points, then the result follows from Theorem 5. Let us assume now that τ is fixed point free.

If the genus of R is even, then we choose a dividing simple loop γ, invariant under τ. If the genus of R is odd, then we choose two non-dividing simple loops γ_1 and γ_2, each one invariant under τ. In either case, denote by R_1 and R_2 the two components of $R - \gamma$ or $R - (\gamma_1 \cup \gamma_2)$, respectively.

We know from Theorem 2 that the number of branch values in R must be even. If that number is a multiple of four, it is possible to see that we can choose the above loops in such a way that all branch values in R_1 (respectively, R_2) can be paired in order that each pair consists of projections of a fixed point of f and one of f^{-1} (here it is important that f has order three). In this situation we have that σ then satisfies condition (B1) and the result follows from Theorem 7.

Let us assume that the number of branch values has the form $4q+2$. In this case, we can arrange the loops γ (or γ_1 and γ_2) in order to have in R_1 (respectively, R_2) $2q$ of these values paired as above. In particular, we can proceed to choose simple loops around each of these pairs (together with their images under τ) pairwise disjoint. The lifting of each of them consists exactly of three loops. We can proceed as in the proof of Theorem 8, together with the following lemma to complete the proof. ▫

Lemma. *Let $\sigma: S \to S$ be an anticonformal automorphism of order 6 acting on a Riemann surface of genus 3 so that $R = S/\sigma^2$ has genus one with exactly two branch values and σ^3 is fixed point free. Then σ is of Schottky type.*

Proof. There is only one topological type satisfying the conditions of the lemma (see Theorem 0.2 of [4]). We only need to find a Schottky group of genus 3, say G, and an anticonformal Möbius transformation T such that $T^6 = I$, $TGT^{-1} = G$, the group generated by G and T uniformizes the connected sum of two projective planes and exactly one branch value of order 3.

For this, we choose $T(z) = e^{\pi i/3}/\bar{z}$ and a loxodromic transformation A_1 with fixed points of the form r and $-1/r$, for some $r > 1$. We can choose r close enough to 1 in order to have that the group G generated by A_1, $A_2 = TA_1T^{-1}$ and $A_3 = TA_2T^{-1}$ is a Schottky group of rank three. ▫

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