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HIGHER ORDER VARIATIONAL INEQUALITIES WITH NON-STANDARD GROWTH CONDITIONS IN DIMENSION TWO: PLATES WITH OBSTACLES

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Abstract. For a domain $\Omega \subset \mathbf{R}^2$ we consider the second order variational problem of minimizing $J(w) = \int_\Omega f(\nabla^2 w)\,dx$ among functions $w\colon \Omega \to \mathbf{R}$ with zero trace respecting a side condition of the form $w \geq \Psi$ on Ω. Here f is a smooth convex integrand with non-standard growth, a typical example is given by $f(\nabla^2 w) = |\nabla^2 w| \ln(1 + |\nabla^2 w|)$. We prove that—under suitable assumptions on Ψ—the unique minimizer is of class $C^{1,\alpha}(\Omega)$ for any $\alpha < 1$. Our results provide a kind of interpolation between elastic and plastic plates with obstacles.

1. Introduction and main result

Let Ω denote a bounded, star-shaped Lipschitz domain in $\mathbf{R}^2$ and suppose we are given an N-function A having the Δ_2-property, precisely (see, e.g. [A] for details) the function A: $[0,\infty) \to [0,\infty)$ satisfies

(N1) A is continuous, strictly increasing and convex;

(N2) $$\lim_{t\downarrow 0} \frac{A(t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{A(t)}{t} = +\infty;$$

(N3) there exist $k,\ t_0 \geq 0:\ A(2t) \leq kA(t)$ for all $t \geq t_0$.

The function A generates the Orlicz space $L_A(\Omega)$ equipped with the Luxemburg norm

$$\|u\|_{L_A(\Omega)} := \inf\left\{ l > 0 : \int_\Omega A\left(\frac{1}{l}|u|\right) dx \leq 1 \right\},$$

the Orlicz–Sobolev space $W^l_A(\Omega)$ is defined in a standard way (see again [A]), finally, we let

$$\overset{\circ}{W}{}^l_A(\Omega) := \text{ closure of } C^\infty_0(\Omega) \text{ in } W^l_A(\Omega).$$

For local spaces we use symbols like $\overset{\circ}{W}{}^l_{A,\mathrm{loc}}(\Omega)$, $L^p_{\mathrm{loc}}(\Omega)$ etc. Suppose further that we are given a function $\Psi \in W^3_2(\Omega)$ $(\subset C^{1,\alpha}(\overline{\Omega}))$ which satisfies

$$\Psi_{|\partial\Omega} < 0, \qquad \max_{\overline{\Omega}} \Psi > 0$$

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and let

$$\mathbf{K} := \Big\{ v \in \overset{\circ}{W}{}^2_A(\Omega) : v \geq \Psi \text{ a.e. on } \Omega \Big\}.$$

It is easy to see that $\mathbf{K}$ contains a function Ψ_0 of class $C_0^\infty(\Omega)$: let $\Omega^+ := [\Psi \geq 0]$ and choose $\eta \in C_0^\infty(\Omega)$ such that $\eta \equiv 1$ on Ω^+ and $0 \leq \eta \leq 1$ on Ω. Then $\Psi_0 := \eta \max\big\{0, \max_{\overline{\Omega}} \Psi\big\}$ has the desired properties.

Next we formulate the hypotheses imposed on the integrand: $f\colon \mathbf{R}^{2\times 2} \to [0,\infty)$ is of class C^2 satisfying

$$c_1\big\{A\big(|\xi|\big) - 1\big\} \leq f(\xi) \leq c_2\big\{A\big(|\xi|\big) + 1\big\}; \tag{1.1}$$

$$\lambda\big(1 + |\xi|^2\big)^{-\mu/2} |\eta|^2 \leq D^2 f(\xi)(\eta,\eta); \tag{1.2}$$

$$|D^2 f(\xi)| \leq \Lambda < +\infty; \tag{1.3}$$

$$|D^2 f(\xi)|\, |\xi|^2 \leq c_3\big\{f(\xi) + 1\big\}; \tag{1.4}$$

$$A^*\big(|Df(\xi)|\big) \leq c_4\big\{A\big(|\xi|\big) + 1\big\} \tag{1.5}$$

for all ξ, $\eta \in \mathbf{R}^{2\times 2}$. Here c_1, c_2, c_3, c_4, λ and Λ denote positive constants, μ is some parameter in $[0,2)$, and A^* is the Young transform of A. From (1.3) we see that f is of subquadratic growth, i.e. $\limsup_{|\xi|\to\infty} f(\xi)/|\xi|^2 < +\infty$, (1.4) is the so-called balancing condition being of importance also in the papers [FO], [FM] and [BFM]. As shown for example in [FO] we can take $f(\xi) := |\xi| \ln(1+|\xi|)$ or its iterated version $f_l(\xi) := |\xi| \tilde{f}_l(\xi)$ with $\tilde{f}_1(\xi) = \ln(1+|\xi|)$, $\tilde{f}_{l+1}(\xi) = \ln\big(1+\tilde{f}_l(\xi)\big)$. But also power growth $(1+|\xi|^2)^{p/2}$, $1 < p \leq 2$, is included. Moreover, we can consider integrands f such that $c|\xi|^p \leq f(\xi) \leq C|\xi|^p$, $|\xi| \gg 1$, $1 < p \leq 2$, and which are elliptic in the sense of (1.2) for any given $0 \leq \mu < 2$ (compare [BFM] for a concrete construction). Let us now state our main result.

Theorem 1.1. *Let* (1.1)–(1.5) *hold. Then the obstacle problem*

$$J(w) := \int_\Omega f(\nabla^2 w)\, dx \rightsquigarrow \min \ \textit{in } \mathbf{K} \tag{V}$$

admits a unique solution u which is of class $W^2_{p,\mathrm{loc}}(\Omega)$ for any finite p, in particular we have $u \in C^{1,\alpha}(\Omega)$ for any $\alpha < 1$, thus u belongs—at least locally—to the same Hölder class as the obstacle Ψ.

Remark 1.2. The statement clearly extends to the vectorial setting of functions $v\colon \Omega \to \mathbf{R}^M$ and componentwise constraints $v^i \geq \Psi^i$ provided $\Psi^1, \dots, \Psi^M$ are as above.

First of all, let us remark that Theorem 1.1 extends the power-growth case studied in [FLM] to the whole scale of arbitrary subquadratic growth which is described in terms of the N-function A. The main difficulty here is that we have no analogue to the density property of smooth functions with compact support in the class $\{v \in \overset{\circ}{W}{}^2_p(\Omega) : v \geq \Psi\}$ stated in Lemma 2.3 of [FLM] which in turn is based on the deep result Theorem 9.1.3 of [AH]. In place of this we now use a more elaborate approximation procedure involving not only the functional J but also the obstacle Ψ which has the advantage that the density result (see Lemma 2.2 for a precise statement) becomes more or less evident. Of course, this strategy is also applicable in the setting of [FLM] which is included as a subcase.

The problem under consideration is of some physical interest: consider a plate which is clamped at the boundary and whose undeformed state is represented by the region Ω. If some outer forces are applied acting in vertical direction, then the equilibrium configuration can be found as a minimizer of the energy

$$I(w) := \int_\Omega g(\nabla^2 w)\, dx + \text{potential terms.}$$

The physical properties of the plate are characterized in terms of the given convex function g: $\mathbf{R}^{2\times 2} \to \mathbf{R}$. In the case of elastic plates we have $g(\xi) = |\xi|^2$ (up to physical constants), for perfectly plastic plates (treated for the unconstrained case e.g. in [S] with the help of duality methods) g is of linear growth near infinity. Since we describe g in terms of the arbitrary N-function A, we can construct any kind of interpolation between the limit cases of linear and quadratic growth. Let us also mention that for elastic plates with obstacles the minimizer is of class $C^2(\overline{\Omega})$ (see [FR]) provided that Ψ is sufficiently regular. For unconstrained plates with logarithmic hardening law it was shown in [FS, Theorem 5.1], that u is of class $C^{2,\alpha}(\Omega)$ for any $0 < \alpha < 1$.

Our paper is organized as follows: in Section 2 we introduce suitable regularisations of problem (V) and prove some convergence properties. Moreover, a density result is established. Section 3 is devoted to the proof of Theorem 1.1: we show that for the approximative solutions u^ε the quantities $(1 + |\nabla^2 u^\varepsilon|^2)^{(2-\mu)/4}$ are locally uniformly bounded in $W^1_{2,\mathrm{loc}}(\Omega)$ which gives the claim with the help of Sobolev's embedding theorem.

2. Regularisation and a density result

From now on assume that all the hypotheses stated in and before Theorem 1.1 hold. Without loss of generality we may also assume that

$$\Psi > -1 \quad \text{on} \quad \partial\Omega \qquad \text{and} \qquad \Omega = D_1 = \big\{z \in \mathbf{R}^2 : |z| < 1\big\}.$$

Proceeding exactly as in [FO, Theorem 3.1], we find that (V) has a unique solution u (which of course holds for any strictly convex f with property (1.1)). For the reader's convenience we remark that the trace theorem 2.1 of [FO] used during the existence proof has now to be replaced by the statement that $\overset{\circ}{W}{}^2_A(\Omega) = W^2_A(\Omega) \cap \overset{\circ}{W}{}^2_1(\Omega)$ which can be obtained with the same arguments as used in [FO, Theorem 2.1].

Since the statement of Theorem 1.1 is local, we fix some disc $D \Subset \Omega$. Let us introduce a sequence $\{\Psi^\varepsilon\}_\varepsilon$ such that

$$\begin{aligned}&\Psi^\varepsilon \in W_2^3(\Omega),\\ &\Psi^\varepsilon = \Psi \text{ in a neighborhood of } D,\\ &\Psi^\varepsilon \equiv -1 \text{ on } D_1 - D_{1-\varepsilon} \quad \text{and}\\ &\Psi^\varepsilon \to \Psi \quad \text{a.e. on } D_1 \quad \text{as } \varepsilon \downarrow 0.\end{aligned}$$

Of course we can also arrange $\Psi_0 \geq \Psi \geq \Psi^\varepsilon$. Consider now the problems

$$(\mathrm{V}^\varepsilon) \qquad J(w) \rightsquigarrow \min \quad \text{in} \quad \mathbf{K}^\varepsilon := \big\{v \in \overset{\circ}{W}{}_A^2(\Omega) : v \geq \Psi^\varepsilon \text{ a.e.}\big\}$$

with unique solution u^ε and its quadratic regularisation

$$(\mathrm{V}_\delta^\varepsilon) \qquad \begin{aligned}J_\delta(w) := \frac{\delta}{2}\int_\Omega |\nabla^2 w|^2\,dx + J(w) \rightsquigarrow \min\\ \text{in } \mathbf{K}^{\varepsilon\prime} := \big\{v \in \overset{\circ}{W}{}_2^2(\Omega) : v \geq \Psi^\varepsilon \text{ a.e.}\big\}.\end{aligned}$$

Note that $\Psi_0 \in \mathbf{K}^{\varepsilon\prime}$, hence $\mathbf{K}^{\varepsilon\prime} \neq \emptyset$, and $(\mathrm{V}_\delta^\varepsilon)$ has a unique solution u_δ^ε. We have

$$J_\delta(u_\delta^\varepsilon) \leq J_\delta(\Psi_0) \leq J_1(\Psi_0) < +\infty, \quad \text{thus} \quad \int_\Omega A\big(|\nabla^2 u_\delta^\varepsilon|\big)\,dx \leq \text{ const } < +\infty$$

and similar to [FO, Lemma 3.1], or [FLM, Lemma 2.4], we deduce

Lemma 2.1. *For any fixed* $\varepsilon > 0$ *we have*

$$\text{(i)} \qquad u_\delta^\varepsilon \overset{\delta\downarrow 0}{\rightharpoonup} u^\varepsilon \qquad \textit{in } W_1^2(\Omega),$$

$$\text{(ii)} \qquad \delta\int_\Omega |\nabla^2 u_\delta^\varepsilon|^2\,dx \overset{\delta\downarrow 0}{\to} 0,$$

$$\text{(iii)} \qquad J^\delta(u_\delta^\varepsilon) \overset{\delta\downarrow 0}{\to} J(u^\varepsilon).$$

Proof. Clearly $u_\delta^\varepsilon \rightharpoonup \tilde{u}^\varepsilon$ as $\delta \downarrow 0$ in $W_1^2(\Omega)$ for some function $\tilde{u}^\varepsilon$ which is easily seen (compare [FO]) to belong to the class $\mathbf{K}^\varepsilon$ (obviously $u_\delta^\varepsilon \to \tilde{u}^\varepsilon$ a.e. on Ω as $\delta \downarrow 0$). For $w \in \mathbf{K}^{\varepsilon\prime}$ we have

$$J_\delta(\tilde{u}^\varepsilon) \leq J_\delta(w) \overset{\delta\downarrow 0}{\to} J(w) \qquad \text{and} \qquad J(\tilde{u}^\varepsilon) \leq \liminf_{\delta\downarrow 0} J(u_\delta^\varepsilon) \leq \liminf_{\delta\downarrow 0} J_\delta(u_\delta^\varepsilon);$$

thus it is proved for all $w \in \mathbf{K}^{\varepsilon\prime}$

$$(2.1) \qquad J(\tilde{u}^\varepsilon) \leq J(w).$$

By Lemma 2.2 we also know that $\mathbf{K}^{\varepsilon\prime}$ is dense in $\mathbf{K}^\varepsilon$, hence (2.1) holds for any $w \in \mathbf{K}^\varepsilon$ and $\tilde{u}^\varepsilon = u^\varepsilon$ follows. The other statements of Lemma 2.1 are obvious. □

Lemma 2.2. *The class* $\mathbf{K}^{\varepsilon\prime}$ *is dense in* $\mathbf{K}^{\varepsilon}$.

Proof. Consider $v \in \mathbf{K}^{\varepsilon}$ and define $(0 < \varrho < 1)$

$$v_\varrho(x) := \begin{cases} v\left(\dfrac{1}{\varrho}x\right), & \text{if } |x| \le \varrho, \\ 0, & \text{if } \varrho \le |x|, \end{cases}$$

for $x \in \Omega$; v_ϱ is of class $\mathring{W}^2_A(\Omega)$ and

$$\|v_\varrho - v\|_{W^2_A(\Omega)} \to 0 \qquad \text{as } \varrho \uparrow 1. \tag{2.2}$$

According to Poincaré's inequality (see, for example, [FO, Lemma 2.4]) (2.2) is a consequence of

$$\|\nabla^2 v_\varrho - \nabla^2 v\|_{L_A}(\Omega) \to 0 \qquad \text{as } \varrho \uparrow 1, \tag{2.3}$$

and (2.3) is established as soon as we can show (compare, e.g. [FO, Lemma 2.1])

$$\int_\Omega A\big(|\nabla^2 v_\varrho - \nabla^2 v|\big)\, dx \to 0 \qquad \text{as } \varrho \uparrow 1. \tag{2.4}$$

To this end observe that

$$\nabla^2 v_\varrho - \nabla^2 v \overset{\varrho\uparrow 1}{\to} 0 \qquad \text{a.e. on } \Omega.$$

Moreover

$$A\big(|\nabla^2 v_\varrho - \nabla^2 v|\big) \le A\big(|\nabla^2 v_\varrho| + |\nabla^2 v|\big) \le \tfrac{1}{2}\big(A\big(2|\nabla^2 v_\varrho|\big) + A\big(2|\nabla^2 v|\big)\big)$$

by convexity and monotonicity of A. The Δ_2-condition yields (see [FO, inequality (2.1)])

$$A(mt) \le A(mt_0) + \big(1 + k^{(\ln m/\ln 2)+1}\big)A(t)$$

for all m, $t \ge 0$. This implies for a.a. $|x| \le \varrho$

$$\begin{aligned} A\big(2|\nabla^2 v_\varrho(x)|\big) &= A\big(2\varrho^{-2}|\nabla^2 v(x/\varrho)|\big) \\ &\le A\big(2\varrho^{-2}t_0\big) + \big(1 + k^{(\ln 2\varrho^{-2}/\ln 2)+1}\big)A\big(|\nabla^2 v(x/\varrho)|\big) := \tilde{g}_\varrho(x), \end{aligned}$$

hence

$$A\big(|\nabla^2 v_\varrho - \nabla^2 v|\big) \le \tfrac{1}{2}\big(A\big(2|\nabla^2 v|\big) + \tilde{g}_\varrho(x)\big) =: g_\varrho(x)$$

being valid for a.a. $x \in \Omega$ if we define $\tilde{g}_\varrho(x) = 0$ for $|x| > \varrho$. We have

$$g_\varrho(x) \overset{\varrho\uparrow 1}{\to} \tfrac{1}{2}\big(A\big(2|\nabla^2 v(x)|\big) + A(2t_0) + \big(1 + k^2\big)A\big(|\nabla^2 v(x)|\big)\big) =: g(x)$$

a.e. and also $\int_\Omega g_\varrho \, dx \to \int_\Omega g \, dx$ as $\varrho \uparrow 1$. The version of the dominated convergence theorem given in [EG, Theorem 4, p. 21], implies (2.4).

For small enough $h > 0$ let $(\varphi)_h$ denote the mollification of a function φ with radius h. Let us define

$$w := (v_\varrho)_h + \Psi^\varepsilon - \big([\Psi^\varepsilon]_\varrho\big)_h, \qquad \text{where}$$

$$[\Psi^\varepsilon]_\varrho(x) := \begin{cases} \Psi^\varepsilon\left(\dfrac{1}{\varrho}x\right), & \text{if } |x| \le \varrho, \\ -1, & \text{if } |x| \ge \varrho, \end{cases}$$

for $x \in \Omega$. Of course we assume $1 - \varrho \le \frac{1}{2}\varepsilon$ and $h \le \frac{1}{2}(1-\varrho)$ (note that we can define the mollified functions for any $x \in \Omega$ since v_ϱ and $[\Psi^\varepsilon]_\varrho$ are constant near the boundary and therefore can be extended by the same value to the whole plane). Then

$$(v_\varrho)_h - \big([\Psi^\varepsilon]_\varrho\big)_h \ge 0$$

which is a consequence of $v_\varrho - [\Psi^\varepsilon]_\varrho \ge 0$, thus $w \ge \Psi^\varepsilon$. Since $\Psi^\varepsilon \equiv -1$ on $D_1 - D_{1-\varepsilon}$ we also have $w = 0$ near $\partial\Omega$, moreover, $w \in W^3_2(\Omega)$, and $\|w - v\|_{W^2_A(\Omega)}$ becomes as small as we want if we first choose ϱ close to 1 and then let h go to zero. □

Lemma 2.3. *We have the following convergence properties*

(i) $$u^\varepsilon \overset{\varepsilon\downarrow 0}{\rightharpoondown} u \qquad \text{in } W^2_1(\Omega),$$

(ii) $$J(u^\varepsilon) \overset{\varepsilon\downarrow 0}{\to} J(u).$$

Proof. From $\Psi_0 \in \mathbf{K}^\varepsilon$ we get $J(u^\varepsilon) \le J(\Psi_0) < +\infty$; as usual this implies that $u^\varepsilon \rightharpoondown: \tilde{u}$ in $W^2_1(\Omega)$ as $\varepsilon \downarrow 0$ and that $\tilde{u}$ is in the space $\mathring{W}^2_1(\Omega)$. We may assume that $u^\varepsilon \to \tilde{u}$ a.e. as $\varepsilon \downarrow 0$, hence $\Psi = \lim_{\varepsilon\downarrow 0} \Psi^\varepsilon \le \lim_{\varepsilon\downarrow 0} u^\varepsilon = \tilde{u}$ a.e. Thus $\tilde{u} \in \mathbf{K}$ and in conclusion

$$J(u) \le J(\tilde{u}).$$

On the other hand

$$u \ge \Psi \ge \Psi^\varepsilon$$

implies $u \in \mathbf{K}^\varepsilon$, hence

$$J(u^\varepsilon) \le J(u) \quad \text{and in conclusion} \quad J(\tilde{u}) \le \liminf_{\varepsilon\downarrow 0} J(u^\varepsilon) \le J(u).$$

By strict convexity $J(u) = J(\tilde{u})$ implies $u = \tilde{u}$. □

3. Proof of Theorem 1.1

Consider now $\eta \in C_0^\infty(D)$, $0 \le \eta \le 1$. Following the lines of [FLM] we get estimate (3.6) of [FLM] with g_δ replaced by $f_\delta(\xi) = \frac{1}{2}\delta|\xi|^2 + f(\xi)$ and u_δ^ε, Ψ^ε in place of u_δ, Φ, i.e. (summation with respect to $\gamma = 1, 2$)

$$\int_D \eta^6 D^2 f_\delta\big(\nabla^2 u_\delta^\varepsilon\big)\big(\partial_\gamma \nabla^2 u_\delta^\varepsilon, \partial_\gamma \nabla^2 u_\delta^\varepsilon\big)\, dx$$

$$\le c\int_D \big|D^2 f_\delta\big(\nabla^2 u_\delta^\varepsilon\big)\big|\,\big(|\nabla u_\delta^\varepsilon|^2 + |\nabla^2 u_\delta^\varepsilon|^2 + |\nabla \Psi^\varepsilon|^2 + |\nabla^2 \Psi^\varepsilon|^2 + |\nabla^3 \Psi^\varepsilon|^2\big)\, dx. \tag{3.1}$$

By construction, $\Psi^\varepsilon = \Psi$ in a neighborhood of D, hence we may write Ψ in place of Ψ^ε on the right-hand side of (3.1). Note also that the constant c appearing in (3.1) is independent of ε and δ. (1.3) together with the remark that $\Psi = \Psi^\varepsilon$ on D implies

$$\int_D \big|D^2 f_\delta(\nabla^2 u_\delta^\varepsilon)\big|\big(|\nabla \Psi^\varepsilon|^2 + |\nabla^2 \Psi^\varepsilon|^2 + |\nabla^3 \Psi^\varepsilon|^2\big)\, dx \le c \qquad \text{(independent of } \varepsilon,\ \delta).$$

From

$$J_\delta(u_\delta^\varepsilon) \le J_1(\Psi_0) < +\infty$$

we deduce

$$\delta \int_D |\nabla^2 u_\delta^\varepsilon|^2\, dx \le c \qquad \text{(independent of } \varepsilon,\ \delta).$$

From (1.4) we get

$$\begin{aligned}\int_D \big|D^2 f(\nabla^2 u_\delta^\varepsilon)\big|\,|\nabla^2 u_\delta^\varepsilon|^2\, dx &\le c\int_D \big(f\big(\nabla^2 u_\delta^\varepsilon\big) + 1\big)\, dx \\ &\le c\big(J(u_\delta^\varepsilon) + 1\big) \le c\big(J(\Psi_0) + 1\big).\end{aligned}$$

From the uniform bound on $J(u_\delta^\varepsilon)$ we deduce a uniform bound for the quantity $\|u_\delta^\varepsilon\|_{W_1^2(\Omega)}$, and since $n = 2$, we see that $\|\nabla u_\delta^\varepsilon\|_{L^2(\Omega)}$ is bounded independent of ε and δ. Inserting these estimates in (3.1) we end up with

$$\int_D \eta^6 D^2 f_\delta\big(\nabla^2 u_\delta^\varepsilon\big)\big(\partial_\gamma \nabla^2 u_\delta^\varepsilon, \partial_\gamma \nabla^2 u_\delta^\varepsilon\big)\, dx \le c(\eta) < +\infty \tag{3.2}$$

being valid for all sufficiently small ε and δ. Consider now the auxiliary function

$$h_\delta^\varepsilon := \big(1 + |\nabla^2 u_\delta^\varepsilon|^2\big)^{(2-\mu)/4}$$

which is of class $W^1_{2,\mathrm{loc}}(\Omega)$ (note that $\mu < 2$ and that $u_\delta^\varepsilon \in W^3_{2,\mathrm{loc}}(\Omega)$, the last statement following exactly along the lines of [FLM]). (3.2) implies

$$\int_D |\nabla h_\delta^\varepsilon|^2 \eta^6\, dx \le c(\eta) < +\infty, \tag{3.3}$$

and from $\mu \geq 0$ we get

$$h_\delta^\varepsilon \leq \big(1 + |\nabla^2 u_\delta^\varepsilon|^2\big)^{1/2}.$$

$J_\delta(u_\delta^\varepsilon) \leq$ const implies $\int_\Omega h_\delta^\varepsilon \, dx \leq$ const $< +\infty$ and together with (3.3) we find $h_\delta^\varepsilon \in W^1_{2,\text{loc}}(D)$ with local bound independent of ε and δ. We claim

$$h_\delta^\varepsilon \overset{\delta\downarrow 0}{\rightharpoondown} \big(1 + |\nabla^2 u^\varepsilon|^2\big)^{(2-\mu)/4} \tag{3.4}$$

weakly in $W^1_{2,\text{loc}}(D)$. First of all, for any fixed $\varepsilon > 0$, we find a subsequence $\delta \downarrow 0$ and a function h_ε in $W^1_{2,\text{loc}}(D)$ such that

$$\begin{aligned} h_\delta^\varepsilon &\rightharpoondown h^\varepsilon \qquad \text{in } W^1_{2,\text{loc}}(D),\\ h_\delta^\varepsilon &\to h^\varepsilon \qquad \text{a.e. as } \delta \downarrow 0. \end{aligned}$$

For proving (3.4) let us write (observe (1.5))

$$\begin{aligned} J_\delta(u_\delta^\varepsilon) - J(u^\varepsilon) &= \frac{\delta}{2}\int_\Omega |\nabla^2 u_\delta^\varepsilon|^2 \, dx + J(u_\delta^\varepsilon) - J(u^\varepsilon)\\ &= \frac{\delta}{2}\int_\Omega |\nabla^2 u_\delta^\varepsilon|^2 \, dx + \int_\Omega Df(\nabla^2 u^\varepsilon) : (\nabla^2 u_\delta^\varepsilon - \nabla^2 u^\varepsilon) \, dx\\ &\quad + \int_\Omega \int_0^1 D^2 f\big((1-t)\nabla^2 u^\varepsilon + t\nabla^2 u_\delta^\varepsilon\big)\big(\nabla^2 u_\delta^\varepsilon - \nabla^2 u^\varepsilon, \nabla^2 u_\delta^\varepsilon - \nabla^2 u^\varepsilon\big)(1-t)\, dt \, dx. \end{aligned}$$

The minimality of u^ε together with $u_\delta^\varepsilon \in \mathbf{K}^\varepsilon$ implies

$$\int_\Omega Df(\nabla^2 u^\varepsilon) : (\nabla^2 u_\delta^\varepsilon - \nabla^2 u^\varepsilon) \, dx \geq 0$$

so that by Lemma 2.1

$$\lim_{\delta\downarrow 0} \int_\Omega \int_0^1 D^2 f\big((1-t)\nabla^2 u^\varepsilon + t\nabla^2 u_\delta^\varepsilon\big)\big(\nabla^2 u_\delta^\varepsilon - \nabla^2 u^\varepsilon, \nabla^2 u_\delta^\varepsilon - \nabla^2 u^\varepsilon\big)(1-t)\, dt \, dx = 0.$$

From the ellipticity condition (1.2) we get

$$\begin{aligned} &\int_0^1 D^2 f\big((1-t)\nabla^2 u^\varepsilon + t\nabla^2 u_\delta^\varepsilon\big)\big(\nabla^2 u_\delta^\varepsilon - \nabla^2 u^\varepsilon, \nabla^2 u_\delta^\varepsilon - \nabla^2 u^\varepsilon\big)(1-t)\, dt\\ &\qquad \geq \lambda \int_0^1 \Big(1 + \big|\nabla^2 u^\varepsilon + t\big(\nabla^2 u_\delta^\varepsilon - \nabla^2 u^\varepsilon\big)\big|^2\Big)^{-\mu/2} |\nabla^2 u_\delta^\varepsilon - \nabla^2 u^\varepsilon|^2 (1-t)\, dt\\ &\qquad \geq c(\mu,\lambda)\big(1 + |\nabla^2 u^\varepsilon|^2 + |\nabla^2 u_\delta^\varepsilon|^2\big)^{-\mu/2} |\nabla^2 u_\delta^\varepsilon - \nabla^2 u^\varepsilon|^2, \end{aligned}$$

hence

$$\big(1+|\nabla^2 u^\varepsilon|^2+|\nabla^2 u^\varepsilon_\delta|^2\big)^{-\mu/2}|\nabla^2 u^\varepsilon_\delta-\nabla^2 u^\varepsilon|^2 \overset{\delta\downarrow 0}{\to} 0 \tag{3.5}$$

in $L^1(\Omega)$ and a.e. for a subsequence. $h^\varepsilon_\delta \to h^\varepsilon$ a.e. on D implies

$$|\nabla^2 u^\varepsilon_\delta|^2 \overset{\delta\downarrow 0}{\to} \{h^\varepsilon\}^{4/(2-\mu)}-1 \qquad \text{a.e.,}$$

$\{h^\varepsilon\}^{4/(2-\mu)}-1$ being finite a.e. Returning to (3.5) and observing that $(1+|\nabla^2 u^\varepsilon|^2+|\nabla^2 u^\varepsilon_\delta|^2)^{-\mu/2}$ has a pointwise limit a.e. on D as $\delta\downarrow 0$ which is not zero we get

$$\nabla^2 u^\varepsilon_\delta \overset{\delta\downarrow 0}{\to} \nabla^2 u^\varepsilon \qquad \text{a.e. on } D$$

and in conclusion (3.4) is established at least for a subsequence of $\delta\downarrow 0$. But since the limit is unique, the statement is true for any sequence $\delta\downarrow 0$. Recall that

$$\|h^\varepsilon_\delta\|_{W^1_2(\widetilde D)} \le c(\widetilde D) < +\infty$$

for any subdomain $\widetilde D \Subset D$. Combining this with (3.4) we get

$$\big\|\big(1+|\nabla^2 u^\varepsilon|^2\big)^{(2-\mu)/2}\big\|_{W^1_2(\widetilde D)} \le \liminf_{\delta\downarrow 0}\|h^\varepsilon_\delta\|_{W^1_2(\widetilde D)} \le c(\widetilde D)$$

so that by Sobolev's embedding theorem

$$\|\nabla^2 u^\varepsilon\|_{L^p(\widetilde D)} \le c(p,\widetilde D) \le +\infty$$

for any finite p. Therefore $u^\varepsilon \in W^2_{p,\mathrm{loc}}(D)$ uniformly for any finite p and Lemma 2.3 implies $u\in W^2_{p,\mathrm{loc}}(D)$ (u^ε converges weakly as $\varepsilon\downarrow 0$ to some function in $W^2_{p,\mathrm{loc}}(D)$, by Lemma 2.3 the limit is just u). □

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