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ON THE DERIVATIVE OF HYPERBOLICALLY CONVEX FUNCTIONS

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Abstract. A conformal mapping f of the unit disk $\mathbf{D}$ into itself is called hyperbolically convex if the non-euclidean segment between any two points of $f(\mathbf{D})$ also belongs to $f(\mathbf{D})$. In this paper we obtain the exact order of growth for the derivative of these functions and investigate their boundary behaviour.

1. The growth of the derivative

Let $\mathbf{D}$ be the unit disk and $\mathbf{T} = \partial\mathbf{D}$. The analytic univalent function $f\colon \mathbf{D} \to \mathbf{D}$ is called *hyperbolically convex* (or simply h-convex) if the non-euclidean segment between any two points of $f(\mathbf{D})$ also belongs to $f(\mathbf{D})$. An h-convex function is continuous in $\overline{D}$.

Hyperbolically convex functions were first systematically studied by William Ma and David Minda [MM1]. Among many other results they obtained the characterization

$$\operatorname{Re}\left[1 + z\frac{f''(z)}{f'(z)} + \frac{2zf'(z)\overline{f(z)}}{1-|f(z)|^2}\right] > 0 \qquad (z \in \mathbf{D}); \tag{1.1}$$

see also [MM2]. The present authors [MP1], [MP2] and Alexandre Vasil′ev [MPV] derived a number of estimates for h-convex functions. The upper bound for the derivative remained an open problem and it was conjectured [MP2] that

$$f'(z) = O\left(\frac{1}{1-|z|}\left(\log\frac{1}{1-|z|}\right)^{-2}\right) \qquad (|z| \to 1); \tag{1.2}$$

this was proved with the exponent -1 instead of -2.

The property of being h-convex is invariant under Möbius transformations of $\mathbf{D}$ onto itself. This fact can be used to achieve the normalization

$$f(z) = \alpha z + a_2 z^2 + \cdots, \qquad 0 < \alpha \le 1. \tag{1.3}$$

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The h-convex function

$$k_\alpha(z) = 2\alpha z/\big(1 - z + \sqrt{(1-z)^2 + 4\alpha^2 z}\,\big) = \alpha z + \cdots \tag{1.4}$$

often plays the role of the extremal function and it was shown in [MPV] that

$$|f'(z)| \le k'_\alpha(|z|) \qquad \text{for } |z| \le \sqrt{2} - 1, \tag{1.5}$$

however not for $|z|$ close to 1.

Let $\lambda(z,\varsigma)$ denote the non-euclidean distance in $\mathbf{D}$ normalized such that

$$\lambda(z,0) = \frac{1}{2}\log\frac{1+|z|}{1-|z|} \qquad (z \in \mathbf{D}). \tag{1.6}$$

Theorem 1. *Let f be hyperbolically convex. Then*

$$\frac{(1-|z|^2)|f'(z)|(1-|\varsigma|^2)|f'(\varsigma)|}{|f(z)-f(\varsigma)|^2}\lambda(z,\varsigma)^2 \le c_1 \qquad \text{for } z,\varsigma \in \mathbf{D}, \tag{1.7}$$

where c_1 is an absolute constant. In particular, if $f(z) = \alpha z + \cdots$, then

$$(1-|z|^2)|f'(z)|\lambda(z,0)^2 \le \frac{c_1}{\alpha}|f(z)|^2 \qquad \text{for } z \in \mathbf{D}. \tag{1.8}$$

The proof will be very geometric. It follows from (1.8) that the conjecture (1.2) is true. The stronger conjecture [MP1]

$$a_n = O\big(n^{-1}(\log n)^{-2}\big) \qquad (n \to \infty)$$

however remains open.

Theorem 2. *Let f be h-convex. Then*

$$b(\varsigma) = \lim_{r\to 1}(1-r^2)|f'(r\varsigma)|\lambda(r,0)^2 < \infty \qquad (\varsigma \in \mathbf{T}) \tag{1.9}$$

exists. If $b(\varsigma) > 0$ then $f(\varsigma) \in \mathbf{T}$ and

$$\big(f(\varsigma) - f(z)\big)L(\bar\varsigma z) \to f(\varsigma)b(\varsigma) \qquad \text{as } z \to \varsigma,\ z \in \Delta, \tag{1.10}$$

$$(1-\bar\varsigma^2 z^2)f'(z)L(\bar\varsigma z)^2 \to \bar\varsigma f(\varsigma)b(\varsigma) \qquad \text{as } z \to \varsigma,\ z \in \Delta, \tag{1.11}$$

where $L(z) = \frac{1}{2}\log\big[(1+z)/(1-z)\big]$. Here Δ is any Stolz angle at ς.

Martin Chuaqui and the second author [CP] have proved that a meromorphic function $f\colon \mathbf{D} \to \widehat{\mathbf{C}}$ satisfies

$$\frac{(1-|z|^2)|f'(z)|(1-|\varsigma|^2)|f'(\varsigma)|}{|f(z)-f(\varsigma)|^2}\lambda(z,\varsigma)^2 \le 1 \qquad (z,\varsigma \in \mathbf{D}), \tag{1.12}$$

if and only if its Schwarzian derivative S_f satisfies

$$(1-|z|^2)^2|S_f(z)| \le 2 \qquad (z \in \mathbf{D}). \tag{1.13}$$

It was shown in [MP2, Example 5.3] that (1.13) does not hold for all h-convex functions. Hence (1.12) does not hold for all h-convex functions and it follows that the constant in Theorem 1 satisfies $c_1 > 1$.

See [MP2] and [MM2] for bounds for the Schwarzian derivative of h-convex functions. In [MP2] it is conjectured that

$$\sup_f \sup_z (1-|z|^2)^2|S_f(z)| \approx 2.384.$$

2. Boundary points on the unit circle

We now study the closed set $\mathbf{T} \cap f(\mathbf{T})$ where $f(\mathbf{D})$ meets the unit circle. It follows from the McMillan twist theorem [Mc], [P, p. 142] that f has a finite angular derivative for almost all $\varsigma \in \mathbf{T}$ with $f(\varsigma) \in \mathbf{T}$, which implies that $f(\mathbf{T})$ is tangential to $\mathbf{T}$ at $f(\varsigma)$.

Let $c_2, c_3, \dots$ denote suitable positive absolute constants.

Theorem 3. *Let $f(z) = \alpha z + \cdots$ be h-convex and let $\varsigma \in \mathbf{T}$, $f(\varsigma) \in \mathbf{T}$. If there exists $r_0 < 1$ such that*

$$(1-r^2)|f'(r\varsigma)| \le c_2|f(\varsigma) - f(r\varsigma)| \le c_3\alpha \qquad (r_0 \le r < 1), \tag{2.1}$$

then, for $\omega \in \mathbf{T} \cap f(\mathbf{T})$, $\omega \ne f(\varsigma)$,

$$|\omega - f(\varsigma)| \ge c_4|f(\varsigma) - f(r\varsigma)| \qquad (r_0 \le r < 1). \tag{2.2}$$

The assumption (2.1) roughly says that, near $f(\varsigma)$, the image domain lies in a narrow sector. The assertion (2.2) implies that $f(\varsigma)$ is an isolated point of $\mathbf{T} \cap f(\mathbf{T})$, of which there can be only countably many.

Theorem 4. *Let $f(z) = \alpha z + \cdots$ be h-convex and let $b(\varsigma)$ be defined by (1.9). Then $b(\varsigma) = 0$ except possibly for countably many ς and moreover*

$$\sum_{b(\varsigma)>0} b(\varsigma) \le \frac{c_5}{\alpha^2}. \tag{2.3}$$

3. Proof of Theorem 1

(a) First we prove inequality (1.8) for h-convex functions of the form $f(z) = a_1 z + \cdots$ with $|a_1| = \alpha$. Let $z \in \mathbf{D}$ be given. By rotational invariance, we may assume that

$$u = f(z) > 0.$$

We write

$$d(s) = \operatorname{dist}\big(s, \partial f(\mathbf{D})\big) \qquad \text{for } 0 \le s \le u. \tag{3.1}$$

Let $C^{\pm}$ be the circles orthogonal to $\mathbf{T}$ that are tangential to $\{w : |w| < \alpha/2\}$ and $\{|w-u| < d(u)\}$ and let $A^{\pm}$ be the arcs of $C^{\pm}$ between the points of contact. The centers ω and $\overline{\omega}$ and the radius ϱ of $C^{\pm}$ satisfy

$$|\omega| = (1+\alpha^2/4)/\alpha, \qquad \varrho = (1-\alpha^2/4)/\alpha, \tag{3.2}$$

in particular $|\omega|^2 - \varrho^2 = 1$. We write $\xi = \operatorname{Re}\omega$.

Let G be the domain between A^+ and A^- including the disks $\{|w| < \alpha/2\}$ and $\{|w-u| < d(u)\}$ touched by $A^{\pm}$. We have $\{|w-u| < d(u)\} \subset f(\mathbf{D})$ by (3.1) and $\{|w| < \alpha/2\} \subset f(D)$ by [MM1, Theorem 2]. Hence $G \subset f(\mathbf{D})$ because f is h-convex.

Now let $0 < s < u$. Geometric considerations show that the points of ∂G nearest to s lie on the arcs $A^{\pm}$ of $C^{\pm}$. Hence

$$d(s) \ge |\omega - s| - \varrho = \frac{|\omega - s|^2 - \varrho^2}{|\omega - s| + \varrho} \ge \frac{1 - 2\xi s + s^2}{2 + 2\varrho} > \frac{\alpha}{4}(1 - 2\xi s + s^2). \tag{3.3}$$

The construction of $C^{\pm}$ shows that $|\omega - u| = d(u) + \varrho$, which implies

$$d(u) < \frac{d(u)^2 + 2\varrho d(u)}{2\varrho} = \frac{1 - 2\xi u + u^2}{2\varrho} < \alpha(1 - 2\xi u + u^2) \tag{3.4}$$

by (3.2).

We will show in part (b) that

$$y \equiv \sqrt{1 - 2\xi u + u^2} \int_0^u \frac{ds}{1 - 2\xi s + s^2} < c_5 u. \tag{3.5}$$

It follows from (3.3), (3.4) and (3.5) that

$$\sqrt{d(u)} \int_0^u \frac{ds}{d(s)} < \frac{4c_5}{\sqrt{\alpha}}\, u.$$

Since $u = f(z)$ and $d(u) \le (1 - |z|^2)|f'(z)| \le 4d(u)$ by (3.1), we conclude that

$$\begin{aligned} \sqrt{(1-|z|^2)|f'(z)|}\,\lambda(z,0) &\le 2\sqrt{d(u)} \int_{f^{-1}([0,u])} \frac{|f'(\varsigma)|}{(1-|\varsigma|^2)|f'(\varsigma)|}\,|d\varsigma| \\ &\le 2\sqrt{d(u)} \int_0^u \frac{ds}{d(s)} \le \frac{8c_5}{\sqrt{\alpha}}\,|f(z)| \end{aligned}$$

which implies (1.8).

(b) Now we prove (3.5). We distinguish three cases.

Case I. Let $-\infty < \xi \leq -1$. We write $x = -\xi u$ and obtain

$$y \leq \sqrt{1+2x+u^2}\int_0^u \frac{ds}{1+2xs/u} \leq \sqrt{2+2x}\,\frac{u}{2x}\log(1+2x) \leq c_6 u.$$

Case II. Let $-1 < \xi < 1$. We can write

$$1-2\xi s+s^2 = (1-s)^2+2(1-\xi)s \geq \tfrac{1}{2}(1-s)^2+\tfrac{1}{2}(1-\xi). \tag{3.6}$$

First let $(1-u)^2 < 1-\xi$. Then, by (3.6),

$$1-2\xi u+u^2 = (1-u)^2+2(1-\xi)u < 3(1-\xi). \tag{3.7}$$

If $u \geq \frac{1}{2}$ then, with y defined in (3.5), we see from (3.6) that

$$y < \int_0^u \frac{2\sqrt{3(1-\xi)}\,ds}{(1-s)^2+(1-\xi)} < \int_0^\infty \frac{2\sqrt{3}}{t^2+1}\,dt \leq 2\pi\sqrt{3}\,u.$$

If $u < \frac{1}{2}$ then $\sqrt{1-\xi} > 1-u > \frac{1}{2}$ and thus, by (3.6),

$$y < \int_0^u \frac{2\sqrt{3(1-\xi)}\,ds}{1-\xi} < 4\sqrt{3}\,u.$$

Now let $(1-u)^2 \geq 1-\xi$. Then $1-2\xi u+u^2 < 3(1-u)^2$ as in (3.7) and thus, by (3.6),

$$y < \int_0^u \frac{2\sqrt{3}\,(1-u)}{(1-s)^2}\,ds = 2\sqrt{3}\,u.$$

Case III. Let $1 \leq \xi < +\infty$. We define

$$\eta = \sqrt{\xi^2-1}, \qquad v = \xi-\eta = 1/(\xi+\eta). \tag{3.8}$$

Then $1-2\xi v+v^2 = 0$ and therefore $u < v$ by (3.4). This time we can write

$$1-2\xi s+s^2 = (v-s)^2+2\eta(v-s). \tag{3.9}$$

First let $v-u \leq \eta$. Then, by (3.9),

$$1-2\xi u+u^2 = (v-u)^2+2\eta(v-u) \leq 3\eta(v-u). \tag{3.10}$$

If $u \geq \frac{1}{2}$ then, by (3.9),

$$\int_0^u \frac{ds}{1-2\xi s+s^2} \leq \int_0^{v-\eta} \frac{ds}{(v-s)^2} + \int_{v-\eta}^u \frac{ds}{2\eta(v-s)} \leq \frac{1}{\eta}+\frac{1}{2\eta}\log\frac{\eta}{v-u};$$

if $v-\eta<0$ then the first integral is omitted. We deduce from (3.10) and $v-u\le\eta$ that

$$y\le\sqrt{\frac{v-u}{\eta}}\left(\sqrt{3}+\frac{\sqrt{3}}{2}\log\frac{\eta}{v-u}\right)\le c_7\le 2c_7u.$$

On the other hand, if $u<\frac{1}{2}$ then, by (3.8),

$$2\sqrt{\xi^2-1}=2\eta=\xi-(v-\eta)\ge\xi-u>\xi-\tfrac{1}{2}\ge\tfrac{1}{2}$$

which implies $\xi\ge\xi_1=\sqrt{17}/4>1$. Hence (3.9) and (3.10) show that

$$y\le\frac{\sqrt{3\eta(v-u)}}{2\eta}\int_0^u\frac{ds}{v-s}\le\frac{\sqrt{3}}{2\sqrt{\eta}}\int_0^u\frac{ds}{\sqrt{v-s}}=\sqrt{\frac{3}{\eta}}\,\frac{u}{\sqrt{v}+\sqrt{v-u}}<\sqrt{\frac{3}{v\eta}}\,u$$

and (3.5) follows because, by (3.8),

$$\frac{1}{v\eta}=\frac{\xi}{\sqrt{\xi^2-1}}+1\le\frac{\xi_1}{\sqrt{\xi_1^2-1}}+1.$$

Now let $v-u>\eta$. Then (3.9) shows that $1-2\xi u+u^2<3(v-u)^2$ and thus, by (3.9),

$$y<\sqrt{3}\,(v-u)\int_0^u\frac{ds}{(v-s)^2}=\sqrt{3}\,\frac{u}{v}.$$

We obtain from (3.8) that $\xi-\eta=v>\eta+u\ge\eta$ and therefore $\xi>2\eta$ which implies $\xi\le\sqrt{4/3}$. Hence we have

$$y\le\sqrt{3}\,\frac{u}{v}=\sqrt{3}\left(\xi+\sqrt{\xi^2-1}\,\right)u<4u.$$

This completes the proof of (3.5) and thus of assertion (1.8) of Theorem 1.

(c) Now we prove assertion (1.7). Let $z\in\mathbf{D}$ be fixed and let $|b|=1$,

$$\varphi(t)=\frac{t+z}{1+\bar{z}t},\qquad g(t)=b\frac{f\big(\varphi(t)\big)-f(z)}{1-\overline{f(z)}f\big(\varphi(t)\big)}\qquad(t\in\mathbf{D}). \tag{3.11}$$

Then g is again h-convex and moreover

$$g(0)=0,\qquad g'(0)=(1-|z|^2)|f'(z)|/\big(1-|f(z)|^2\big),$$

if $b\in\mathbf{T}$ is suitably chosen. Applying (1.8) to g, we obtain from (3.11) that

$$\frac{(1-|z|^2)|f'(z)|\big(1-|\varphi(t)|^2\big)\big|f'\big(\varphi(t)\big)\big|}{\big|f(z)-f\big(\varphi(t)\big)\big|^2}\lambda(t,0)^2=g'(0)\frac{(1-|t|^2)|g'(t)|}{|g(t)|^2}\lambda(t,0)^2\le c_1$$

which implies (1.7) because $\lambda(t,0)=\lambda\big(\varphi(t),\varphi(0)\big)=\lambda\big(\varphi(t),z\big)$. □

4. Proof of Theorem 2

First let $f(\varsigma) \in \mathbf{D}$. We prove that the limit $b(\varsigma)$ exists and is $=0$. We may assume that f has the form (1.3). It follows from [MP1, (2.6)] that

$$(1-r^2)|f'(r\varsigma)| \leq 4|f(\varsigma)-f(r\varsigma)| \leq 8(1-r)^{\alpha^2\delta/4}$$

for r close to 1, where $\delta = 1-|f(\varsigma)|^2 > 0$.

Now let $f(\varsigma) \in \mathbf{T}$. We may assume that $\varsigma = 1$ and $f(\varsigma) = 1$; otherwise we consider the h-convex function $\overline{f(\varsigma)} f(\varsigma z)$. For the proof of (1.9) we may furthermore assume that there exist $r_n \to 1$ such that

$$(1-r_n^2)|f'(r_n)|\lambda(r_n,0)^2 \to b \neq 0 \qquad \text{as } n \to \infty; \tag{4.1}$$

see (1.8). By Theorem 1, we have,

$$\frac{(1-|z|^2)|f'(z)|(1-r_n^2)|f'(r_n)|}{|f(r_n)-f(z)|^2}\lambda(r_n,z)^2 \leq c_1.$$

Since $\lambda(r_n,z) \sim \lambda(r_n,0)$ as $n \to \infty$, we obtain from (4.1) that

$$\frac{(1-|z|^2)|f'(z)|}{|1-f(z)|^2} \leq \frac{c_1}{b} \qquad \text{for } z \in \mathbf{D}. \tag{4.2}$$

Ma and Minda [MM1, Theorem 4] have shown that the function

$$p(z) = \frac{\bigl(1-f(z)\bigr)^2}{(1-z)^2 f'(z)} \qquad (z \in \mathbf{D}) \tag{4.3}$$

satisfies $\operatorname{Re} p > 0$. The Julia–Wolff lemma shows that

$$\frac{\bigl(1-f(z)\bigr)^2}{(1-z^2)f'(z)} = \frac{1-z}{1+z}p(z) \to a \qquad \text{as } z \to 1,\ z \in \Delta \tag{4.4}$$

for every Stolz angle Δ, where $0 \leq a < +\infty$. It follows from (4.2) that $a \geq b/c_1 > 0$.

We obtain from (4.4) that

$$\frac{f'(z)}{\bigl(1-f(z)\bigr)^2} = \left(\frac{1}{a}+o(1)\right)\frac{1}{1-z^2} \qquad \text{as } z \to 1,\ z \in \Delta,$$

and by integration we deduce that

$$\frac{1}{1-f(z)} = \left(\frac{1}{a}+o(1)\right)L(z) \qquad \text{as } z \to 1,\ z \in \Delta, \tag{4.5}$$

where

$$L(z) = \frac{1}{2}\log\frac{1+z}{1-z}.$$

Together with (4.4) this implies

$$(1-z^2)f'(z)L(z)^2 \to a \qquad \text{as } z \to 1,\ z \in \Delta.$$

Hence the limit (1.9) exists and $b(1) = a$. The assertions (1.10) and (1.11) now follow from (4.5) and (4.4). □

5. Proofs of Theorems 3 and 4

Let $c_8, c_9, \dots$ denote suitable positive absolute constants.

Proposition 5. *Let f be h-convex and $f(0)=0$, $|f'(0)|=\alpha$. Let $\varsigma\in\mathbf{T}$, $f(\varsigma)\in\mathbf{T}$, and $\omega\in\mathbf{T}\cap f(\mathbf{T})$ with $\omega\neq f(\varsigma)$. For given $r\in(0,1)$, there are only two possibilities*:

(i) $|f(\varsigma)-f(r\varsigma)|<c_8|f(\varsigma)-\omega|$,

(ii) $(1-r^2)|f'(r\varsigma)|>c_9\min(\alpha,|f(\varsigma)-\omega|)$.

Proof. We may assume that $\varsigma=1$ and $f(\varsigma)=1$. Let $G=f(\mathbf{D})$. We write

$$q=|f(\varsigma)-\omega|=|1-\omega|. \tag{5.1}$$

There exists a smooth crosscut $Q=Q(r)$ of G with $f(r)\in Q$ that separates 0 from 1 and satisfies

$$\operatorname{length} Q<c_{10}(1-r^2)|f'(r)|. \tag{5.2}$$

Let C be the circle orthogonal to $\mathbf{T}$ through 1 and ω. Since $0\in G$ and $1,\omega\in\partial G$ and since G is h-convex, the non-euclidean triangle T bounded by $[0,1],[0,\omega]$ and $\mathbf{D}\cap C$ lies in G.

The crosscut Q has to meet $[0,1]$ because Q separates 0 and 1. Let $A^{\pm}$ be the arcs of Q from the last points of intersection with $[0,1]$ to ∂G. Then A^+ and A^- go to different sides of $[0,1]$. Hence one of these arcs, say A^+, has to enter the triangle T. Since $T\subset G$ the endpoint of A^+ on ∂G cannot lie in T. Hence A^+ has to meet C or $[0,\omega]$.

First we consider the case that there exists $a\in A^+\cap C\subset Q\cap C$. Since $f(r)\in Q$ we obtain from (5.1) and (5.2) that

$$|1-f(r)|\le|1-a|+|a-f(r)|\le q+c_{10}(1-r^2)|f'(r)|.$$

If $(1-r^2)|f'(r)|<q$ then (i) holds; if $(1-r^2)|f'(r)|\ge q$ then (ii) holds trivially by (5.1).

Now we consider the case that there exists $a\in A^+\cap[0,\omega]$. If $0\le r<\frac12$ then

$$(1-r^2)|f'(r)|\ge c_{11}|f'(0)|=c_{11}\alpha$$

by the Koebe distortion theorem. Hence (ii) holds.

Hence we may assume that $\frac12\le r<1$. Then $|f(r)|\ge c_{12}\alpha$. Now $A^+\subset Q$. Thus Q intersects $[0,1]$ and $[0,\omega]$ and furthermore $f(r)\in Q$. It follows that

$$\operatorname{length} Q\ge\max\bigl[\operatorname{dist}\bigl(f(r),[0,1]\bigr),\operatorname{dist}\bigl(f(r),[0,\omega]\bigr)\bigr]\ge c_{13}\min(\alpha,|1-\omega|).$$

Hence (ii) holds by (5.2). □

Proof of Theorem 3. Let c_8 and c_9 be the constants of the proposition and put $c_2 = c_9/(2c_8)$, $c_3 = c_2c_8$. It follows from (i), (ii) and (2.1) that, for every fixed $r \in [r_0, 1)$ there are only the two cases

(i′) $|f(\varsigma) - f(r\varsigma)| < c_8 \min\bigl(\alpha, |f(\varsigma) - \omega|\bigr)$,

(ii′) $|f(\varsigma) - f(r\varsigma)| > 2c_8 \min\bigl(\alpha, |f(\varsigma) - \omega|\bigr)$.

Since $|f(\varsigma) - f(r\varsigma)|$ is continuous in $[r_0, 1)$, we conclude that either (i′) or (ii′) holds for all $r \in [r_0, 1)$. But (ii′) is impossible for r close to 1. Hence (i′) holds, which implies our assertion (2.2) with $c_4 = 1/c_8$. □

Proof of Theorem 4. Let $\varsigma, f(\varsigma) \in \mathbf{T}$ and $b(\varsigma) > 0$. It follows from (4.2) (where $f(\varsigma) = 1$) that

$$(1 - r^2)|f'(r\varsigma)| \le \frac{c_1}{b(\varsigma)}|f(\varsigma) - f(r\varsigma)|^2. \tag{5.3}$$

Let c_2, c_3, c_4 be the constants of Theorem 3 and let

$$s = s(\varsigma) \min\left(\frac{c_2}{c_1}b(\varsigma), \frac{c_3}{c_2}\alpha, \alpha\right). \tag{5.4}$$

Since $|f(\varsigma) - f(r\varsigma)| = 1 \ge \alpha$ for $r = 0$ and $= 0$ for $r = 1$, there exists $r_0 \in [0, 1)$ such that

$$|f(\varsigma) - f(r\varsigma)| \le s \qquad \text{for } r_0 \le r < 1, \qquad = s \qquad \text{for } r = r_0. \tag{5.5}$$

We obtain from (5.3), (5.4) and (5.5) that, for $r_0 \le r < 1$,

$$(1 - r^2)|f'(r\varsigma)| \le \frac{c_1 s}{b(\varsigma)}|f(\varsigma) - f(r\varsigma)| \le c_2|f(\varsigma) - f(r\varsigma)|.$$

Furthermore we see from (5.4) and (5.5) that

$$c_2|f(\varsigma) - f(r\varsigma)| \le c_2 s \le c_3\alpha.$$

Hence it follows from Theorem 3 and (5.5) that

$$|\omega - f(\varsigma)| \ge c_4|f(\varsigma) - f(r_0\varsigma)| = c_4 s$$

for $\omega \in \mathbf{T} \cap f(\mathbf{T})$, $\omega \ne f(\varsigma)$.

The open set $\mathbf{T}\backslash f(\mathbf{T})$ is the union of disjoint open arcs I_n. We have just shown that, for every ς with $b(\varsigma) > 0$, there exists $n = n(\varsigma)$ such that $f(\varsigma)$ is the right-hand endpoint of I_n and

$$\operatorname{diam} I_n \ge c_4 s(\varsigma). \tag{5.6}$$

Let X be the set of ς for which $s(\varsigma) = c_2 b(\varsigma)/c_1$ and let Y be the set for which $s(\varsigma) < c_2 b(\varsigma)/c_1$; see (5.4). Then, by (5.6),

$$\sum_{\varsigma \in X} b(\varsigma) \leq \frac{c_1}{c_2 c_4} \sum_n \operatorname{diam} I_n < \frac{2\pi c_1}{c_2 c_4} = c_{14},$$
$$\sum_{\varsigma \in Y} c_4 \min\left(\frac{c_3}{c_2}\alpha, \alpha\right) \leq \sum_n \operatorname{diam} I_n < 2\pi$$

and thus card $Y < c_{15}/\alpha$, furthermore $b(\varsigma) \leq c_1/\alpha$ by (1.8) and (1.9). Hence

$$\sum_{b(\varsigma)>0} b(\varsigma) < c_{14} + \frac{c_1 c_{15}}{\alpha^2}.$$

This proves (2.3). In particular we can have $b(\varsigma) > 0$ only for countably many ς. □

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