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NONLINEAR PERTURBATION OF BALAYAGE SPACES

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Abstract. For a large class of nonlinear perturbations of balayage spaces existence and uniqueness of solutions to the Dirichlet problem are shown. In the special case of a harmonic space given by a linear second order partial differential operator L the perturbed equation is $Lu - \varphi(\,\cdot\,,u)\mu = 0$ where $\varphi(x,t)$ is continuous in $t \in \mathbf{R}$, the functions $\varphi_c := \sup\{|\varphi(\,\cdot\,,t)| : -c \le t \le c\}$, $c > 0$, are locally μ-Kato, i.e., yield continuous real L-potentials ${}^U G_L^{\varphi_c \mu}$, and the functions $t \mapsto \varphi(x,t)$, $x \in X$, have a weak form of joint lower Lipschitz property, i.e., $\psi := \sup_{s<t}(\varphi(\,\cdot\,,t) - \varphi(\,\cdot\,,s))^- /(t-s)$ is locally μ-Kato and perturbation by $-\psi\mu$ still leads to a $\mathscr{P}$-harmonic structure.

1. Introduction and basic notions

Recently the Dirichlet problem for nonlinear perturbation of partial differential equations of the type

$$Lu - u\varphi(\,\cdot\,,u)\mu = 0$$

(L being a linear elliptic or parabolic operator of second order) has been studied in the potential-theoretic setting of a harmonic space ([BM], [BBM]).

We shall be able to weaken the assumptions, our model case being

$$Lu - \varphi(\,\cdot\,,u)\mu = 0$$

(cf. (2.2)), to get better results, and, nevertheless, have shorter proofs.

In fact, our method works even for balayage spaces (see [BH]) covering in addition nonlocal situations as e.g. given by Riesz potentials. So we shall study nonlinear perturbation of balayage spaces. The reader who is mainly interested in the PDE case leading to harmonic spaces (which can be viewed as balayage spaces where harmonic measures for open sets live on the boundaries) might consult [He], [CC], [BH], [K], [HH] and [Bo].

It will be convenient to assume that our balayage space has a base of regular sets. We adopt the same notations as in [Ha] and recall briefly the basic definitions:

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Let X be a locally compact space with countable base. For every open set U in X, let $\mathscr{B}(U)$ denote the set of all numerical Borel measurable functions on U. Further, $\mathscr{C}(U)$ will denote the space of all continuous real functions on U and $\mathscr{K}(U)$ the set of all functions in $\mathscr{C}(U)$ having compact support in U. Occasionally, functions on U will be identified with functions on X which are zero on U^c. Finally, given any set $\mathscr{A}$ of functions let $\mathscr{A}_b$ ($\mathscr{A}^+$ resp.) denote the set of all functions in $\mathscr{A}$ which are bounded (positive resp.).

Let $\mathscr{U}$ be a base of relatively compact open subsets of X and, for every $U \in \mathscr{U}$, let H_U be a kernel on X such that $H_U(x, \cdot\,) = \varepsilon_x$ for every $x \in U^c$ and $H_U 1_U = 0$. Let us suppose that $\mathscr{U}$ is stable with respect to finite intersections (by [BH, Remark VII.3.2.4] this is no restriction of generality). Define

$$(1.1) \qquad \mathscr{W} := \{v \mid v : X \to [0, \infty] \text{ l.s.c., } H_U v \le v \text{ for every } U \in \mathscr{U}\}$$

and, for every numerical function $f \ge 0$ on X, let

$$R_f := \inf\{v \in \mathscr{W} : v \ge f\}.$$

A function $s \in \mathscr{C}^+(X)$ is called *strongly* ($\mathscr{W}$)*-superharmonic* if, for every $U \in \mathscr{U}$, $H_U s < s$ on U.

Then $(H_U)_{U\in\mathscr{U}}$ is a family of (regular) harmonic kernels and $(X, \mathscr{W})$ is a balayage space provided the following holds (where $U, V \in \mathscr{U}$):

(H_1) Given $x \in X$, $\lim_{U \downarrow \{x\}} H_U f(x) = f(x)$ for all $f \in \mathscr{K}(X)$ or $R_{1_{\{x\}}}$ is l.s.c. at x.
(H_2') $H_V H_U = H_U$ if $V \subset U$.
(H_3) For every $f \in \mathscr{B}_b(X)$ with compact support, the function $H_U f$ is continuous on U.
(H_4') For every $f \in \mathscr{K}(X)$, the function $H_U f$ is continuous on $\overline{U}$.
(H_5') There exists a strongly superharmonic function $s \in \mathscr{C}^+(X)$.

In the following $(X, \mathscr{W})$ will always denote a balayage space associated with a family $(H_U)_{U\in\mathscr{U}}$ of regular harmonic kernels. For simplicity let us suppose that there exists a strictly positive bounded function in $\mathscr{W}$. This is no real loss of generality, since we may always replace $\mathscr{W}$ by $\{w/s : w \in \mathscr{W}\}$, s being an arbitrary strictly positive function in $\mathscr{W} \cap \mathscr{C}(X)$. (Note, however, that we would have to replace $\mathscr{B}_b(X)$ by the space of all s-bounded functions.)

For every open subset U of X let $\mathscr{H}(U)$ denote the set of all *harmonic functions* on U, i.e.,

$$\mathscr{H}(U) := \{h \in \mathscr{B}(X) : h|_U \text{ continuous, } H_V h(x) = h(x) \\ \qquad\qquad \text{for every } x \in V \in \mathscr{U},\ \overline{V} \subset U\}.$$

One way of defining the convex cone $\mathscr{P}(X)$ of all continuous real potentials is the following:

$$\mathscr{P}(X) := \{p \in \mathscr{W} \cap \mathscr{C}(X) : 0 \le g \le p,\ g \in \mathscr{H}^+(X) \Longrightarrow g = 0\}.$$

We recall from [BH] that for every open subset U of X, regular or not, bounded or not, there is a harmonic kernel H_U. It is characterized by

$$H_U p = \inf\{q \in \mathscr{P}(X) : q \geq p \text{ on } U^c\} \qquad (p \in \mathscr{P}(X)).$$

By definition, a sequence (x_n) in U converging to a point $z \in \partial U$ is called *regular* if $\lim_{n\to\infty} H_U f(x_n) = f(z)$ for every $f \in \mathscr{K}(X)$. If it is regular, then $\lim_{n\to\infty} H_U f(x_n) = f(z)$ even for every $f \in \mathscr{B}_b(X)$ such that the restriction of f on U^c is continuous at z. The set U is regular, if every sequence in U converging to a boundary point of U is regular. In particular, every $U \in \mathscr{U}$ is regular by (H_4').

Let us fix a potential kernel K_X for $(X, \mathscr{W})$, i.e., K_X is a kernel such that

(1.2) $\quad K_X f \in \mathscr{P}(X) \cap \mathscr{H}\big(X \setminus \operatorname{supp}(f)\big)$ for $f \in \mathscr{B}_b^+(X)$ with compact support.

For every open subset U of X, we define a kernel K_U by

$$K_U := K_X - H_U K_X. \tag{1.3}$$

Since obviously $K_U(x, \cdot\,) = 0$ for every $x \in U^c$, we may view K_U as being a kernel on X or a potential kernel on U (restricting $H_V(x, \cdot\,)$ on U for $x \in V \in \mathscr{U}$ with $\overline{V} \subset U$ we obtain a family of harmonic kernels on U). We recall that K_U is a compact operator on $\mathscr{B}_b(X)$ if U is relatively compact (this follows easily from [Ha, Lemma 10.1] and (1.3)). Moreover,

$$K_U = K_V + H_V K_U \tag{1.4}$$

for all open U, V with $V \subset U$.

A function $f \in \mathscr{B}(X)$ is called a *Kato function* (with respect to K_X) if $K_X M_f$ is a potential kernel (M_f denotes multiplication by f) or, equivalently, if

$$K_X(1_{U_n} f^{\pm}) \in \mathscr{C}(X)$$

for a sequence (U_n) of open sets covering X. Of course, every locally bounded function in $\mathscr{B}(X)$ is a Kato function. More generally, every $f \in \mathscr{B}(X)$ which is locally bounded by a Kato function is a Kato function.

Remark 1.1. If the balayage space $(X, \mathscr{W})$ is given by a second order differential operator L with Green function G_L (such that $LG_L(\,\cdot\,, y) = -\delta_y$), then potential kernels are associated with (Kato) measures μ by $K_X f = G_L^{f\mu} := \int G_L(\,\cdot\,, y) f(y)\, \mu(dy)$.

2. Main result

Let φ be a Borel measurable real function on $X \times \mathbf{R}$ such that the functions $t \mapsto \varphi(x,t)$, $x \in X$, are continuous on $\mathbf{R}$. For every $x \in X$ and real $c > 0$, we define

$$\varphi_c(x) := \max\bigl\{|\varphi(x,t)| : -c \le t \le c\bigr\},$$
$$\psi(x) := \sup\biggl\{\biggl(\frac{\varphi(x,t)-\varphi(x,s)}{t-s}\biggr)^{-} : -\infty < s < t < \infty\biggr\}.$$

We note that $\psi\colon X \to [0,\infty]$ is the smallest function such that the functions $t \mapsto \varphi(x,t) + \psi(x)t$, $x \in X$, are increasing. Moreover,

$$\psi(x) = \sup_{t\in\mathbf{R}}\biggl(\frac{\partial\varphi}{\partial t}(x,t)\biggr)^{-}$$

if $\partial\varphi/\partial t$ exists.

Let us fix a relatively compact open subset U of X and assume the following:

(i) For every $c > 0$, $K_X(1_U\varphi_c) \in \mathscr{C}(X)$, i.e., $1_U\varphi_c$ is a Kato function.
(ii) $K_X(1_U\psi) \in \mathscr{C}(X)$, i.e., $1_U\psi$ is a Kato function.
(iii) Perturbation of $(X,\mathscr{W})$ by $-1_U\psi$ (with respect to K_X) yields a balayage space $(X,\widetilde{\mathscr{W}})$ (see [Ha]).

Remarks 2.1. 1. If (i) and (ii) hold for U, then (i) and (ii) hold for any open $V \subset U$.

2. Of course (i) holds if φ is locally bounded. Further, (ii) holds if $(\partial\varphi/\partial t)(x,t)$ exists and the functions $\bigl((\partial\varphi/\partial t)(x,\cdot)\bigr)^{-}$, $x \in U$, are uniformly bounded.

3. Assuming that $1_U\psi$ is a Kato function, (iii) holds provided there exist $s \in \mathscr{W}$ and $u \in \mathscr{B}^{+}(X)$ such that

$$v := s + K_X u \in \mathscr{C}(X), \qquad \psi v \le u \text{ on } U$$

and, for every $V \in \mathscr{U}$, $\{H_V s < s\} \cup \{K_V(1_U\psi v) < K_V u\} = V$ [Ha, Theorem 6.4]). A special case would be $p := K_X 1 \in C(X)$ strongly superharmonic and $\psi < 1/p$ on U $(s := 0,\ u := 1)$.

4. If $(X,\mathscr{W})$ is parabolic, (iii) is already a consequence of (ii). Indeed, suppose that $1_U\psi$ is a Kato function. By [Ha, Lemma 10.1], $K_X M_{1_U\psi}$ is a compact operator on $\mathscr{B}_b(X)$ and therefore, by [Ha, Theorem 10.2, Lemma 10.3],

$$L := \sum_{m=0}^{\infty} (K_X M_{1_U\psi})^m$$

is a bounded operator on $\mathscr{B}_b(X)$. Choose a strongly superharmonic bounded $s \in \mathscr{W} \cap \mathscr{C}(X)$ and define $v := Ls$, $u := 1_U\psi v$. Then $v = s + K_X u$ and $\psi v = u$ on U. Thus (iii) holds by the preceding remark.

We define a (nonlinear) operator $K_U^\varphi\colon \mathscr{B}_b(X) \to \mathscr{B}_b(X)$ by

$$K_U^\varphi v := K_U\bigl(\varphi(\,\cdot\,, v)\bigr) \qquad (v \in \mathscr{B}_b(X)).$$

Of course, K_U^φ lives on $\mathscr{B}_b(U)$: $K_U^\varphi v$ depends only on the restriction of v on U and vanishes on U^c.

Given $f \in \mathscr{B}_b(X)$, we shall say that a function $H_U^\varphi f \in \mathscr{B}_b(X)$ is a (generalized) *solution to the perturbed Dirichlet problem* associated with U and f provided

$$H_U^\varphi f + K_U^\varphi H_U^\varphi f = H_U f. \tag{2.1}$$

In the situation of Remark 1.1 equation (2.1) implies that

$$L H_U^\varphi f - \varphi(\,\cdot\,, H_U^\varphi f)\mu = 0. \tag{2.2}$$

Moreover, $H_U^\varphi f$ has essentially the same boundary behavior as $H_U f$. If e.g. U is regular, then $K_U^\varphi H_U^\varphi f$ tends to zero at ∂U whence $\lim_{x\to z} H_U^\varphi f(x) = f(z)$ for any $z \in \partial U$ where f is continuous.

Our main result is the following:

Theorem 2.2. 1. *For every $f \in \mathscr{B}_b(X)$ there exist a unique generalized solution $H_U^\varphi f$ to the perturbed Dirichlet problem. Moreover,*

$$\begin{aligned}-(I - K_U M_\psi)^{-1}\bigl(H_U f^- + K_U\bigl(\varphi(\,\cdot\,,0)^+\bigr)\bigr) &\le H_U^\varphi f \\ &\le (I - K_U M_\psi)^{-1}\bigl(H_U f^+ + K_U\bigl(\varphi(\,\cdot\,,0)^-\bigr)\bigr),\end{aligned}$$

$H_V^\varphi H_U^\varphi f = H_U^\varphi f$ for every open V with $\overline{V} \subset U$, $H_U^\varphi f$ is continuous on U, and $\lim_{n\to\infty} H_U^\varphi f(x_n) = f(z)$ for every regular sequence (x_n) in U converging to a point $z \in \partial U$ where the restriction of f to the complement of U is continuous.

2. *If $f, g \in \mathscr{B}_b(X)$ such that $f \le g$, then $H_U^\varphi f \le H_U^\varphi g$.*

3. *If a bounded sequence (f_n) in $\mathscr{B}_b(X)$ converges pointwise to a function f, then $\lim_{n\to\infty} H_U^\varphi f_n = H_U^\varphi f$.*

Obviously, K_V^φ for $V \subset U$ is not changed if we replace φ by $1_{(U\cap\{\psi<\infty\})\times\mathbf{R}}\varphi$ (note that $K_X(1_{\{\psi=\infty\}}) = 0$ by (ii)). Therefore we may assume without loss of generality that $\varphi(x,\cdot) = 0$ for every $x \in U^c$ and that ψ is a real function. To prove Theorem 2.2 we may in addition suppose that $|\varphi| \le 1$ and all functions $t \mapsto \varphi(x,t)$, $x \in X$, are increasing. Indeed, fix $c > 0$ and define

$$\tilde\varphi(x,t) := \frac{\varphi(x,t_c) + t_c\psi(x)}{\varphi_c(x) + c\psi(x) + 1} \qquad (x \in X,\ t \in \mathbf{R})$$

where $t_c := \min\bigl(\max(-c,t), c\bigr)$.

Obviously, $|\tilde{\varphi}| \le 1$, $\tilde{\varphi} = 0$ on $U^c \times \mathbf{R}$ and every function $t \mapsto \tilde{\varphi}(x,t)$, $x \in X$, is increasing and continuous. By (iii), for every relatively compact open W in X, the operator $I - K_W M_\psi$ is invertible, $(I - K_W M_\psi)^{-1} = \sum_{m=0}^\infty (K_W M_\psi)^m$, and

$$\widetilde{H}_W = (I - K_W M_\psi)^{-1} H_W \tag{2.3}$$

is the harmonic kernel for W with respect to $(X, \widetilde{\mathscr{W}})$. By (i) and (ii),

$$K'_X := K_X M_{\varphi_c + c\psi + 1}$$

is a potential kernel with respect to $(X, \mathscr{W})$. Clearly, $K_X M_\psi = K'_X M_{\psi/(\varphi_c + c\psi + 1)}$. This implies that the balayage space $(X, \widetilde{\mathscr{W}})$ is obtained perturbing $(X, \mathscr{W})$ by $-\psi/(\varphi_c + c\psi + 1)$ with respect to K'_X. We finally note that there exists a unique potential kernel $\widetilde{K}_X$ for $(X, \widetilde{\mathscr{W}})$ such that, for every relatively compact open W in X,

$$\widetilde{K}_W = (I - K_W M_\psi)^{-1} K'_W$$

(this follows from (2.3) and [Ha, Proposition 10.5]).

Fix $a > 0$ and let $f \in \mathscr{B}_b(X)$ such that $|f| \le a$. Since the functions $H_U 1$ and $K_U|\varphi(\,\cdot\,, 0)|$ are bounded, we may choose $b > 0$ such that

$$(I - K_U M_\psi)^{-1}\big(a H_U 1 + K_U|\varphi(\,\cdot\,,0)|\big) \le b.$$

Then

$$(I - K_U M_\psi)^{-1}\big(H_U|f| + K_U|\varphi(\,\cdot\,,0)|\big) \le b.$$

Moreover, fix $g \in \mathscr{B}_b(X)$ and $c \ge b$ such that $|g| \le c$.

Suppose now that the statements of Theorem 2.2 hold for $(X, \widetilde{\mathscr{W}})$, $\widetilde{K}_X$, and $\tilde{\varphi}$ (defined using this constant c). Then

$$|\widetilde{H}_U^{\tilde{\varphi}} f| \le \widetilde{H}_U|f| + \widetilde{K}_U|\tilde{\varphi}(\,\cdot\,,0)| = (I - K_U M_\psi)^{-1}(H_U|f| + K_U|\varphi(\,\cdot\,,0)|) \le b \le c.$$

Further,

$$K'_U\big(\tilde{\varphi}(\,\cdot\,,g)\big) = K_U\big(\varphi(\,\cdot\,,g) + \psi g\big) = K_U\big(\varphi(\,\cdot\,,g)\big) + K_U M_\psi g$$

whence

$$g + K_U\big(\varphi(\,\cdot\,,g)\big) = (I - K_U M_\psi) g + K'_U\big(\tilde{\varphi}(\,\cdot\,,g)\big) = (I - K_U M_\psi)\big(g + \widetilde{K}_U\big(\tilde{\varphi}(\,\cdot\,,g)\big)\big).$$

Thus

$$g + K_U^\varphi g = H_U f \iff g + \widetilde{K}_U^{\tilde{\varphi}} g = \widetilde{H}_U f \iff g = \widetilde{H}_U^{\tilde{\varphi}} f.$$

This implies that $H_U^\varphi f = \widetilde{H}_U^{\tilde{\varphi}} f$ and that Theorem 2.2 holds for $(X, \mathscr{W})$, K_X, and φ.

3. Existence and uniqueness of the solution to the perturbed Dirichlet problem

As indicated in the previous section we shall assume from now on that $|\varphi| \le 1$ and all functions $t \mapsto \varphi(x,t)$, $x \in X$, are continuous and increasing. For every real function v on X we define a function $\Phi(v)$ by

$$\Phi(v)(x) := \varphi(x, v(x)) \qquad (x \in X).$$

Clearly, for every $v \in \mathscr{B}_b(X)$,

$$K_U^\varphi v = K_U\big(\Phi(v)\big).$$

Our assumption on φ implies that $|\Phi(v)| \le 1$ for every $v \in \mathscr{B}_b(X)$. Moreover, $\Phi(v) \le \Phi(w)$ on $\{v \le w\}$, and $\big(\Phi(v_n)\big)$ converges pointwise to $\Phi(v)$ if (v_n) converges pointwise to v.

Lemma 3.1. *Let (v_n) be a sequence in $\mathscr{B}_b(X)$. Then there exists a subsequence (w_n) of (v_n) such that the sequence $(K_U^\varphi w_n)$ in $\mathscr{B}_b(X)$ converges uniformly. Moreover, if (v_n) converges pointwise to a function v, then $(K_U^\varphi v_n)$ converges uniformly to $K_U^\varphi v$.*

Proof. $\big(\Phi(v_n)\big)$ is a bounded sequence in $\mathscr{B}_b(X)$ and K_U is a compact operator on $\mathscr{B}_b(X)$. This implies the first statement. The second statement now follows from the continuity of the functions $t \mapsto \varphi(x,t)$ and the fact that K_U is a kernel. □

Proposition 3.2. *The operator $I + K_U^\varphi \colon \mathscr{B}_b(X) \to \mathscr{B}_b(X)$ is surjective.*

Proof. We fix $g \in \mathscr{B}_b(X)$ and consider the mapping T from $\mathscr{B}_b(X)$ into $\mathscr{B}_b(X)$ defined by

$$Tu := g - K_U^\varphi u.$$

By Lemma 3.1, T is continuous and $T(\mathscr{B}_b(X))$ is relatively compact. By Schauder's fixed point theorem there exists $u \in \mathscr{B}_b(X)$ such that $Tu = u$, i.e., we have $u + K_U^\varphi u = g$. Thus $I + K_U^\varphi$ is surjective. □

Let ${}^*\mathscr{H}^+(U)$ denote the set of all functions $s \in \mathscr{B}^+(X)$ such that s is l.s.c. on U and $H_V s \le s$ for every $V \in \mathscr{U}$ with $\overline{V} \subset U$. If $s \in {}^*\mathscr{H}^+(U)$, then obviously $1_U s \in {}^*\mathscr{H}^+(U)$.

Lemma 3.3. *Let $v, w, g \in \mathscr{B}_b(X)$ and $s \in {}^*\mathscr{H}^+(U)$ such that $v + K_U w = g \le s$ and $\{w > 0\} \subset \{v \ge 0\}$. Then $v \ge g - s$. In particular, $v \ge 0$ if $g \in {}^*\mathscr{H}_b^+(U)$.*

Proof. Obviously, $K_U w \le g \le s$ on $\{v \ge 0\}$, hence on $\{w > 0\}$. Consequently $K_U w \le s$ and $v = g - K_U w \ge g - s$. □

Proposition 3.4. *Let $v, w, g \in \mathscr{B}_b(X)$ and $s \in {}^*\mathscr{H}^+(U)$ such that $|g| \le s$, $vw \ge 0$, and $v + K_U w = g$. Then $|v - g| \le s$.*

Proof. Apply Lemma 3.3 to v, w, g and $-v, -w, -g$. □

Corollary 3.5. *The operator $I + K_U^\varphi \colon \mathscr{B}_b(X) \to \mathscr{B}_b(X)$ is bijective.*

Proof. By Proposition 3.2, the operator $I + K_U^\varphi$ is surjective. To show that it is injective, we fix v_1, v_2 in $\mathscr{B}_b(U)$ such that $v_1 + K_U^\varphi v_1 = v_2 + K_U^\varphi v_2$. Taking $v := v_1 - v_2$ and $w := \Phi(v_1) - \Phi(v_2)$ we have $vw \ge 0$ and $v + K_U w = 0$, hence $|v| \le 0$ by Proposition 3.4 (taking $g = s = 0$). Thus $v_1 = v_2$. □

An immediate consequence is the following:

Theorem 3.6. *For every $f \in \mathscr{B}_b(X)$, there exists a unique solution $H_U^\varphi f$ to the perturbed Dirichlet problem.*

4. Properties of the solution to the perturbed Dirichlet problem

As before we suppose that U is a relatively compact open subset of X, $|\varphi| \le 1$, and the functions $t \mapsto \varphi(x, t)$, $x \in X$, are continuous and increasing.

Proposition 4.1. *Let $f \in \mathscr{B}_b(X)$. Then $H_V^\varphi H_U^\varphi f = H_U^\varphi f$ for every $V \in \mathscr{U}$ with $\overline{V} \subset U$. Moreover, $\lim_{n\to\infty} H_U^\varphi f(x_n) = f(z)$ for every regular sequence (x_n) in U converging to a point $z \in \partial U$ where the restriction of f to the complement of U is continuous.*

Proof. Define $h := H_U^\varphi f$. Then h is continuous on U, since $K_U^\varphi H_U f \in \mathscr{C}(U)$ and $H_U f$ is harmonic on U. Moreover, for every $V \in \mathscr{U}$ with $\overline{V} \subset U$,

$$h + K_V^\varphi h = h + K_U^\varphi h - H_V K_U^\varphi h = H_U f - H_V(H_U f - h) = H_V h.$$

Fix $z \in \partial U$ such that $f|_{U^c}$ is continuous at z and let (x_n) be a regular sequence in U such that $\lim_{n\to\infty} x_n = z$. Then $\lim_{n\to\infty} H_U f(x_n) = f(z)$ and we conclude from (1.3) that $\lim_{n\to\infty} K_U g(x_n) = 0$ for every $g \in \mathscr{B}_b(X)$. Thus $\lim_{n\to\infty} H_U^\varphi f(x_n) = f(z)$ by (2.1). □

Moreover, we easily obtain the following (note that a combination of (1) and (2) yields the first inequalities in Theorem 2.2):

Proposition 4.2. *Let $f, f_1, f_2, \ldots \in \mathscr{B}_b(X)$. Then the following holds:*

1. $-K_U\big(\varphi(\,\cdot\,, 0)^+\big) \le H_U^\varphi 0 \le K_U\big(\varphi(\,\cdot\,, 0)^-\big)$.
2. $-H_U\big((f_1 - f_2)^-\big) \le H_U^\varphi f_1 - H_U^\varphi f_2 \le H_U\big((f_1 - f_2)^+\big)$.
3. *If $f_2 \le f_1$ then $H_U^\varphi f_2 \le H_U^\varphi f_1$.*
4. *If the sequence (f_n) is bounded and converges pointwise to a function f, then $\lim_{n\to\infty} H_U^\varphi f_n = H_U^\varphi f$.*

Proof. 1. Let $v = H_U^\varphi 0$ and $s_\pm = K_U\big(\varphi(\,\cdot\,,0)^\pm)\big)$. Then $v \in \mathscr{B}_b(X)$, $s_\pm \in {}^*\mathscr{H}_b^+(U)$, and taking $w := \Phi(v) - \Phi(0)$ we obtain that

$$v + K_U w = -K_U\big(\varphi(\,\cdot\,,0)\big) = s_- - s_+ =: g.$$

Of course, $g \le s_-$ and $-g \le s_+$. Since $\{w > 0\} = \{\Phi(v) > \Phi(0)\} \subset \{v \ge 0\}$, Lemma 3.3 implies that $v \ge g - s_- = -s_+$. Since $\{-w > 0\} \subset \{-v \ge 0\}$, we obtain that $-v \ge -g - s_+ = -s_-$. Thus $-s_+ \le v \le s_-$.

2. Let $v_1 = H_U^\varphi f_1$, $v_2 = H_U^\varphi f_2$, $v = v_1 - v_2$, and $w = \Phi(v_1) - \Phi(v_2)$. Then $\{w > 0\} \subset \{v \ge 0\}$ and $v + K_U w = H_U(f_1 - f_2) =: g$. In particular, $g \le H_U\big((f_1 - f_2)^+\big) \in {}^*\mathscr{H}_b^+(U)$ and $-g \le H_U\big((f_1 - f_2)^-\big) \in {}^*\mathscr{H}_b^+(U)$. Thus, by Lemma 3.3,

$$\begin{aligned} v &\ge g - H_U\big((f_1 - f_2)^+\big) = -H_U\big((f_1 - f_2)^-\big), \\ -v &\ge -g - H_U\big((f_1 - f_2)^-\big) = -H_U\big((f_1 - f_2)^+\big). \end{aligned}$$

3. Immediate consequence of (2), since $(f_1 - f_2)^- = 0$ if $f_2 \le f_1$.

4. For every $n \in \mathbf{N}$,

$$H_U^\varphi f_n + K_U^\varphi H_U^\varphi f_n = H_U f_n. \tag{4.1}$$

Of course, $\lim_{n\to\infty} H_U f_n = H_U f$ and the sequence $(H_U^\varphi f_n)$ is bounded by (2). Moreover, by Lemma 3.1, there exists a subsequence (g_n) of (f_n) such that the sequence $(K_U^\varphi H_U^\varphi g_n)$ is convergent. So we conclude from (4.1) that the sequence $(H_U^\varphi g_n)$ converges to a function $G \in \mathscr{B}_b(X)$. Letting n tend to infinity we obtain from (4.1) that

$$G + K_U^\varphi G = H_U f.$$

Thus $H_U^\varphi f = G = \lim_{n\to\infty} H_U^\varphi g_n$. By a general argument on subsequences, this shows that in fact $\lim_{n\to\infty} H_U^\varphi f_n = H_U^\varphi f$. ▫

Proposition 4.3. *Let $h \in \mathscr{B}_b(X)$ such that, for every $z \in \partial U$, $\lim_{y\to z, y\notin U} h(y) = h(z)$ and $\lim_{n\to\infty} h(x_n) = h(z)$ for every regular sequence (x_n) converging to z. Moreover, suppose that U is covered by subsets $V \in \mathscr{U}$ satisfying $H_V^\varphi h = h$. Then $h = H_U^\varphi h$.*

Proof. Define $g := h + K_U^\varphi h$ and let $V \in \mathscr{U}$ be a subset of U such that $H_V^\varphi = h$. Then $h + K_V^\varphi h = H_V h$ is harmonic on V and $K_U^\varphi h - K_V^\varphi h$ is harmonic on $U \cap V$. Therefore g is harmonic on $U \cap V$ and we conclude that g is harmonic on U. So $g - H_U h$ is a function in $\mathscr{B}_b(X)$ which is harmonic on U, equal to zero on U^c, and tends to zero along every regular sequence converging to a boundary point of U. This implies that $g - H_U h = 0$ whence $h = H_U^\varphi h$. ▫

Corollary 4.4. *If the restriction of $f \in \mathscr{B}_b(X)$ to the complement of U is continuous at every $z \in \partial U$, then $H_U^\varphi f$ is the only function $g \in \mathscr{B}_b(X)$ such that $g = f$ on U^c, $H_V^\varphi g = g$ for every $V \in \mathscr{U}$ with $\overline{V} \subset U$, and $\lim_{n\to\infty} g(x_n) = f(z)$ for every regular sequence (x_n) convergent to a point $z \in \partial U$.*

Finally, let us suppose that our assumptions (i)–(iii) hold for every relatively compact open subset U in X. Then, for every open subset W of X, we may define ${}^\varphi\mathscr{H}(W)$ to be the set of all $h \in \mathscr{B}_b(X)$ such that h is continuous on W and $H_V^\varphi h = h$ for every $V \in \mathscr{U}$ with $\overline{V} \subset W$. The following sheaf property is an immediate consequence of Proposition 4.3.

Corollary 4.5. *The set $\{{}^\varphi\mathscr{H}(W) : W \text{ open } \subset X\}$ is a sheaf, i.e., for every family $(W_i)_{i\in I}$ of open subsets in X,*

$$
{}^\varphi\mathscr{H}\Big(\bigcup_{i\in I} W_i\Big) = \bigcap_{i\in I} {}^\varphi\mathscr{H}(W_i).
$$

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