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REVERSE MARKOV INEQUALITY

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Abstract. Let K be a compact convex set in $\mathbf{C}$. For each point $z_0 \in \partial K$ and each holomorphic polynomial $p = p(z)$ having all of its zeros in K, we prove that there exists a point $z \in K$ with $|z - z_0| \le 20 \operatorname{diam} K/\sqrt{\deg p}$ such that

$$|p'(z)| \ge \frac{(\deg p)^{1/2}}{20(\operatorname{diam} K)} |p(z_0)|;$$

i.e., we have a pointwise reverse Markov inequality. In particular,

$$\|p'\|_K \ge \frac{(\deg p)^{1/2}}{20(\operatorname{diam} K)} \|p\|_K.$$

1. Introduction

Let K be a compact set in the complex plane $\mathbf{C}$ and let $V_K(z)$ be the extremal function of K, i.e.,

$$V_K(z) = \max\left[0, \frac{1}{\deg p} \sup \log|p(z)|\right], \tag{1}$$

where the supremum is taken over all nonconstant holomorphic polynomials $p = p(z)$ with supremum norm $\|p\|_K = \sup_{z \in K} |p(z)| \le 1$. Suppose that the function V_K is Hölder continuous with exponent $0 < a \le 1$. Then from the Bernstein–Walsh inequality

$$|p(z)| \le \|p\|_K \exp\left[\deg p V_K(z)\right], \tag{2}$$

which follows from the definition of $V_K(z)$ in (1) (cf., [R]), and the Cauchy estimates, one obtains a *Markov inequality*: this estimates the size of the derivative

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of a polynomial p on K via the sup-norm $\|p\|_K$ of p, its degree, and the Hölder exponent a of V_K; namely,

$$\|p'\|_K \le C(K)(\deg p)^{1/a}\|p\|_K, \tag{3}$$

where $C(K)$ is a constant depending only on K. We outline a proof of (3); cf., [PP, Remark 3.2]. Given a polynomial p of degree n, take $z_0 \in K$ with $|p'(z_0)| = \|p'\|_K$. Apply the Cauchy estimates on the disk of radius $r = n^{-1/a}$ centered at z_0 to obtain

$$\|p'\|_K \le n^{1/a}\|p\|_{K_r}$$

where $K_r := \{z \in \mathbf{C} : \operatorname{dist}(z, K) \le r\}$. Since V_K is Hölder continuous with exponent a, (2) implies that

$$\|p\|_{K_r} \le \|p\|_K(1 + Mr^a)^n$$

for some constant $M = M(K)$. The choice of $r = n^{-1/a}$ then yields the result.

For example, if K is the unit disk, then (S.N. Bernstein)

$$\|p'\|_K \le (\deg p)\|p\|_K$$

and if K is the interval $[-1, 1]$, then (A.A. Markov)

$$\|p'\|_K \le (\deg p)^2\|p\|_K$$

(cf., [Lo]). Note that in these two examples we have precise knowledge of the constant: $C(K) = 1$, and this is sharp. In general, controlling $C(K)$ is difficult.

It is natural to ask whether one can improve the Markov inequality for some natural subclasses of polynomials. Indeed, P. Lax [L] proved a conjecture of P. Erdös that if all the zeros of p lie *outside* the open unit disk Δ, then

$$\|p'\|_{\bar\Delta} \le \tfrac{1}{2}\deg p\|p\|_{\bar\Delta}.$$

An example of $p(z) = z^n + 1$ shows that this is best possible.

However, in 1939, P. Turán [T] showed that if all the zeros of p lie in $\bar\Delta$, then a *reverse Markov inequality* holds:

$$\|p'\|_{\bar\Delta} \ge \tfrac{1}{2}\deg p\|p\|_{\bar\Delta}. \tag{4}$$

Again the same example of $p(z) = z^n + 1$ shows that it is sharp. Thus for this class the Markov inequality cannot be significantly improved.

Turán also proved in [T] that for the interval $I = [-1, 1]$ and polynomials with all their zeros in I, the reverse Markov inequality occurs in the following form:

$$\|p'\|_I \ge \tfrac{1}{6}(\deg p)^{1/2}\|p\|_I. \tag{5}$$

In this note, we are interested in a general form of the reverse Markov inequality for a polynomial p having all of its zeros in K:

$$\|p'\|_K \geq C(K)(\deg p)^b \|p\|_K. \tag{6}$$

Note that any polynomial $p(z) = cz^n + 1$, where $|c|$ is sufficiently small, provides a counterexample to any form of the reverse Markov inequality for polynomials with zeros *outside* K.

The inequality (6) contains two parameters: $C(K)$ and b. Since the holomorphic polynomials p with zeros in K are in one-to-one correspondence with the holomorphic polynomials $\tilde{p}$ with zeros in $aK := \{az \in \mathbf{C} : z \in K\}$ via $\tilde{p}(z) = p(z/a)$, $C(K)$ is inversely proportional to the diameter of K. We are interested in classes of compact sets K for which there is no other dependence on K for some value of b. Indeed, we show that for the class of R-*circular sets*, to be defined in the next section, which includes disks as well as circular arcs, the reverse Markov inequality holds with $b = 1$. For general *convex* sets we show that $b = \frac{1}{2}$. In both cases the value of b is best possible provided that $C(K)$ depends only on the diameter of K.

Moreover, we prove a *pointwise* version of (6):

$$|p'(z)| \geq C(K)(\deg p)^b |p(z)|,$$

where, in the case of R-circular sets, such an inequality is valid at *every* point $z \in \partial K$; while, in the case of general convex sets, given any point $z_0 \in \partial K$, we can find $z \in K$ lying within a distance $20 \operatorname{diam} K/\sqrt{\deg p}$ from z_0 where the above inequality holds.

Throughout the paper the following function will play a major role: given a polynomial $p(z) = c(z - z_1) \cdots (z - z_n)$, we define

$$\phi_p(z) = \phi(z) = \sum_{j=1}^{n} (z - z_j)^{-1}. \tag{7}$$

In the polynomial p (and thus in (7)), we allow repeated roots. Note that

$$p'(z) = p(z)\phi(z).$$

2. Reverse Markov for R-circular sets

We begin with the following result.

Proposition 2.1. *Let $K \subset \mathbf{C}$ be compact. Suppose that $z \in \partial K$ has the property that there exists a circle of radius $R = R(z, K)$ passing through z such* *that the closed disk it bounds contains K. Then for each complex polynomial p having all of its zeros in K,*

$$|p'(z)| \geq \frac{\deg p}{2R}|p(z)|.$$

Proof. Let p have degree n and zeros $z_1, \dots, z_n$ in K (with repeated zeros listed as often as they occur). Since $p'(z) = p(z)\phi(z)$, it suffices to show that $|\phi(z)| \geq n/(2R)$ for $z \in \partial K$. Rotating and translating the z-variable (which does not affect the absolute value of the derivative), we may assume that $z = R$ and that the closed disk Δ_R of radius R centered at zero contains K. Then

$$|\phi(z)| = \left|\sum_{j=1}^{n}(R - z_j)^{-1}\right|$$

and the conformal mapping $w = 1/(R - z)$ maps Δ_R onto the half-plane $\operatorname{Re} w \geq 1/(2R)$. Thus

$$|\phi(z)| \geq \sum_{j=1}^{n} \operatorname{Re} \frac{1}{R - z_j} \geq \frac{n}{2R}$$

and we have

$$|p'(z)| \geq \frac{n}{2R}|p(z)|$$

for $z \in \partial K$. □

Motivated by the proposition, we make the following definition. A compact set $K \subset \mathbf{C}$ will be called R*-circular* if for every point $z \in \partial K$ there is a circle of radius R passing through z such that the closed disk it bounds contains K.

Theorem 2.2. *If K is a compact R-circular set in $\mathbf{C}$, then for each complex polynomial p having all of its zeros in K, and for every point $z \in \partial K$*

$$|p'(z)| \geq \frac{\deg p}{2R}|p(z)|.$$

As a corollary we obtain the first result of Turán.

Corollary 2.3. *For a complex polynomial p having all of its zeros in $\bar{\Delta}$, and for every point $z \in \partial\Delta$,*

$$|p'(z)| \geq \tfrac{1}{2}\deg p|p(z)|.$$

In particular,

$$\|p'\|_{\bar{\Delta}} \geq \tfrac{1}{2}\deg p\|p\|_{\bar{\Delta}}.$$

Remark. This inequality is not valid when the zeros lie very close to $\partial\Delta$ but outside $\bar{\Delta}$. Indeed, take $p(z) = z^n - n^2$. The zeros of p are the n-th roots of n^2 and hence all have modulus $n^{2/n}$ which tends to 1 as $n \to \infty$. However, $\|p\|_{\bar{\Delta}} = n^2 + 1$ while $\|p'\|_{\bar{\Delta}} = n$.

Remark. There exist non-convex compact sets satisfying the hypothesis of the corollary. For example, let K be an arc of a circle. But as we will see right now this estimate fails on an interval.

3. Reverse Markov inequality for convex sets

What happens for general *convex* compact sets K? First of all, we simplify the proof of Turan's second result (5) concerning the real interval $I := [-1, 1]$.

Proposition 3.1. *Let $p = p(x)$ be a real polynomial of degree n with real zeros $x_1, \dots, x_n \in I$. Then*

$$\|p'\|_I \geq \frac{\sqrt{n}}{2\sqrt{e}} \|p\|_I. \tag{8}$$

Proof. Order the zeros $-1 \leq x_1 \leq \cdots \leq x_n \leq 1$. Fix $x_0 \in I$ with $|p(x_0)| = \|p\|_I$. We may assume that $p(x_0) > 0$. For some $m \in \{1, \dots, n-1\}$, $x_0 \in (x_m, x_{m+1})$ or else $x_0 \in [0, x_1) \cup (x_n, 1]$.

In this last case, Proposition 2.1 tells us that

$$|p'(x_0)| \geq \tfrac{1}{2} n p(x_0).$$

Thus we may assume that $x_0 \in (x_m, x_{m+1})$.

Since $p' = p\phi$, for $x \in (x_m, x_{m+1})$ we have

$$p(x) = p(x_0) \exp\biggl(\int_{x_0}^{x} \phi(t)\, dt\biggr).$$

Hence

$$p'(x) = p(x_0)\phi(x) \exp\biggl(\int_{x_0}^{x} \phi(t)\, dt\biggr)$$

or, integrating by parts,

$$p'(x) = p(x_0)\phi(x) \exp\biggl((x - x_0)\phi(x) - \int_{x_0}^{x} (t - x_0)\phi'(t)\, dt\biggr). \tag{9}$$

Now since $|x - x_j| \leq 2$,

$$\phi'(x) = -\sum_{j=1}^{n} \frac{1}{(x - x_j)^2} \leq -\frac{n}{4}. \tag{10}$$

Therefore, for $x_0 \leq x \leq x_{m+1}$ we have

$$\phi(x) \leq -\tfrac{1}{4} n (x - x_0) \tag{11}$$

because $\phi(x_0) = 0$.

Since $\phi(x)$ approaches $-\infty$ as x approaches x_{m+1} from the left, there exists some point $x \in (x_0, x_{m+1})$ at which $\phi(x) = -1/(x-x_0)$. From (11), we conclude that $1/(x-x_0) \geq \frac{1}{4}n(x-x_0)$ at this point; i.e.,

$$x - x_0 \leq 2/\sqrt{n}. \tag{12}$$

Thus, at this point x, since $\phi(x) = -1/(x-x_0)$, using (9) we obtain

$$p'(x) = -\frac{p(x_0)}{x-x_0}\exp\left(-1 - \int_{x_0}^{x}(t-x_0)\phi'(t)\,dt\right).$$

By (10) and (12) the integral

$$\int_{x_0}^{x}(t-x_0)\phi'(t)\,dt \leq -\frac{n}{4}\int_{x_0}^{x}(t-x_0)\,dt = -\frac{n(x-x_0)^2}{8}.$$

Thus

$$|p'(x)| \geq \frac{p(x_0)}{x-x_0}\exp\left(-1 + \frac{n(x-x_0)^2}{8}\right).$$

The function of x in the right side of the inequality above is decreasing when $0 < x - x_0 \leq 2/\sqrt{n}$ and attains its minimum when $x - x_0 = 2/\sqrt{n}$. Therefore

$$|p'(x)| \geq \frac{p(x_0)\sqrt{n}}{2\sqrt{e}}. \qquad \square$$

The polynomial $p(x) = (x^2-1)^{n/2}$ (n even) shows that the $\frac{1}{2}$ power of n is the correct exponent in this inequality. However, the constant is not sharp. Indeed, Erod [E] proved a sharp result for each degree.

We proceed to show that a reverse Markov inequality with exponent $\frac{1}{2}$ is valid for general convex compact sets.

Theorem 3.2. *For any convex compact set K in $\mathbf{C}$, any polynomial $p = p(z)$ of degree n with all of its zeros in K, and any point $z_0 \in \partial K$, there is a point $z \in K$ with $|z - z_0| \leq 20(\operatorname{diam} K)/\sqrt{n}$ such that*

$$|p'(z)| \geq \frac{\sqrt{n}}{20(\operatorname{diam} K)}|p(z_0)|.$$

In particular,

$$\|p'\|_K \geq \frac{\sqrt{n}}{20(\operatorname{diam} K)}\|p\|_K.$$

Proof. For simplicity, we assume the diameter of K is 1. Fix $p = p(z)$ of degree n with all of its zeros $z_1, \dots, z_n$ in K and let $z_0 \in \partial K$. We may assume that $z_0 = 0$. We want to prove that there is a constant $C > 0$—we will see that $C = 1/20$ will work—such that

$$|p'(z)| \geq C\sqrt{n}\,|p(0)|$$

for some point $z \in K$ with $|z| \leq r_n := 1/\big(C\sqrt{n}\big)$. We may assume that the set K lies in the upper half plane H. Then the points $w_k := -1/z_k$ lie in H as well. We fix the angle $\alpha = \pi/12$; note that

$$\tfrac{1}{4} < \sin\alpha < \tfrac{1}{3}.$$

We divide H into the sector $S_1 := \{w \in H : \alpha < \arg w < \pi - \alpha\}$ and $S_2 = H \setminus S_1$. As before, we set

$$\phi(z) := \sum_{k=1}^{n} \frac{1}{z - z_k} = \phi_1(z) + \phi_2(z),$$

where

$$\phi_1(z) = \sum{}' \frac{1}{z - z_k} \quad \text{and} \quad \phi_2(z) = \sum{}'' \frac{1}{z - z_k}.$$

Here $\sum'$ denotes the sum over k with $w_k \in S_1$ and $\sum''$ denotes the sum over k with $w_k \in S_2$.

Let n_1 be the number of of points w_k in S_1 and let n_2 be the number of points w_k in S_2. Since $\operatorname{Im} w_k \geq 0$, $\operatorname{Im} w_k \geq |w_k| \sin\alpha$ when $w_k \in S_1$, and since the diameter of K is 1 implies $|w_k| \geq 1$, we see that

$$|\phi(0)| \geq \sum_{k=1}^{n} \operatorname{Im} w_k \geq \sin\alpha \sum{}' |w_k| \geq \tfrac{1}{4} a \geq \tfrac{1}{4} n_1, \tag{13}$$

where $a = \sum' |w_k|$.

We let d denote the distance from $z_0 = 0$ to the nearest zero of p; i.e., $d = \min\{|z_1|, \dots, |z_n|\}$. Let $t = d\sin\alpha$. We want to estimate the values of

$$\phi'(z) = \phi_1'(z) + \phi_2'(z) = -\sum{}' \frac{1}{(z - z_k)^2} - \sum{}'' \frac{1}{(z - z_k)^2}$$

when $|z| \leq t$.

Note that for such z,

$$|z_k|(1 - \sin\alpha) \leq |z - z_k| \leq |z_k|(1 + \sin\alpha).$$

Hence

$$\begin{aligned} |\phi_1'(z)| &= \left| \sum{}' \frac{1}{(z - z_k)^2} \right| \leq \sum{}' \frac{1}{|z - z_k|^2} \\ &\leq \frac{1}{(1 - \sin\alpha)^2} \left(\sum{}' \frac{1}{|z_k|^2} \right) \leq \frac{1}{(1 - \sin\alpha)^2} \left(\sum{}' \frac{1}{|z_k|} \right)^2 \leq \frac{9a^2}{4}. \end{aligned} \tag{14}$$

Next we consider

$$-\operatorname{Re}\phi_2'(z) = \operatorname{Re}\sum{}'' \frac{1}{(z-z_k)^2}.$$

If $|z| \le t$ and $w_k \in S_2$, then $|\arg 1/z_k^2|$ does not exceed 2α. Since $|zw_k| \le \sin\alpha$, we see that $|\arg(1+zw_k)| \le \alpha$ and

$$\beta_k(z) := \left|\arg \frac{1}{(z-z_k)^2}\right| = \left|\arg \frac{1}{z_k^2(1+zw_k)^2}\right| \le 4\alpha.$$

Now

$$\operatorname{Re}\frac{1}{(z_k-z)^2} = \frac{1}{|z_k-z|^2}\cos\beta_k(z) \ge \frac{1}{|z_k|^2(1+\sin\alpha)^2}\cos 4\alpha.$$

Thus

$$\operatorname{Re}\sum{}'' \frac{1}{(z-z_k)^2} \ge \frac{\cos 4\alpha}{(1+\sin\alpha)^2}\sum{}'' \frac{1}{|z_k|^2} \ge \frac{9}{32}\sum{}'' \frac{1}{|z_k|^2}. \tag{15}$$

Recall again that the diameter of K is 1 implies $|w_k|^2 \ge 1$, so that

$$\sum{}'' \frac{1}{|z_k|^2} \ge n_2. \tag{16}$$

Plugging (16) into (15) we obtain

$$-\operatorname{Re}\phi_2'(z) = \operatorname{Re}\sum{}'' \frac{1}{(z-z_k)^2} \ge \frac{9}{32}n_2. \tag{17}$$

Fixing a constant $C > 0$ which we will specify later, recall that our goal is to prove that $|p'(z)| \ge C\sqrt{n}\,|p(0)|$ for some point $z \in K$ with $|z| \le r_n = 1/(C\sqrt{n})$ (the conclusion of the theorem is that we can take $C = 1/20$). First of all, we note that it suffices to consider the case when $d > r_n$. For if $d \le r_n$, then $|z_k| \le 1/(C\sqrt{n})$ for some k and $|p'(z)| \ge C\sqrt{n}\,|p(0)|$ for some point $z \in [0, z_k]$; this interval lies in K by convexity of K.

If $|\phi(0)| > C\sqrt{n}$, then $|p'(0)| = |p(0)\phi(0)| > C\sqrt{n}\,|p(0)|$ and the desired inequality is true. Thus we may assume that $|\phi(0)| \le C\sqrt{n}$, and, by (13), that both $\frac{1}{4}a$ and $\frac{1}{4}n_1$ are less than $C\sqrt{n}$.

Let $t_n = r_n \sin\alpha$. For the sake of obtaining a contradiction, we suppose that

$$|p'(z)| < C\sqrt{n}\,|p(0)| \quad \text{for all } z \in K \text{ satisfying } |z| < r_n. \tag{18}$$

Then for $|z| < t_n$ we find that

$$|p(z)| \ge |p(0)| - \left|\int_0^z p'(\zeta)\,d\zeta\right| > |p(0)|(1-\sin\alpha) > \tfrac{2}{3}|p(0)|.$$

We now take any point $z' \in K$ with $|z'| = \sin\alpha/\big(C\sqrt{n}\,\big)$. By the convexity of K and the assumption $d > r_n > t_n = |z'|$, such points exist. Since $n_2 = n - n_1 \geq n - 4C\sqrt{n}$, it follows from (17) and the inequality $\sin\alpha > \frac{1}{4}$ that

$$\left|\int_0^{z'} \phi_2'(z)\,dz\right| \geq \frac{9}{32}\big(n - 4C\sqrt{n}\,\big)\frac{\sin\alpha}{C\sqrt{n}} = \frac{9}{32}\left(\frac{\sqrt{n}}{4C} - 1\right). \tag{19}$$

Equation (14) and the inequalities $a < 4C\sqrt{n}$, $|z'| < 1/[3C\sqrt{n}\,]$ give

$$\left|\int_0^{z'} \phi_1'(z)\,dz\right| \leq \frac{9a^2}{4}|z'| < 12C\sqrt{n}\,. \tag{20}$$

Now

$$|\phi(z')| \geq \left|\int_0^{z'} \phi_2'(z)\,dz\right| - \left|\int_0^{z'} \phi_1'(z)\,dz\right| - |\phi(0)|.$$

Plugging (19) and (20) and the assumption $|\phi(0)| < C\sqrt{n}$ (from (18) into the latter inequality we obtain

$$|\phi(z')| > \frac{9}{32}\left(\frac{\sqrt{n}}{4C} - 1\right) - 12C\sqrt{n} - C\sqrt{n} = \sqrt{n}\left(\frac{9}{128C} - 13C\right) - \frac{9}{32}$$

and

$$|p'(z')| = |p(z')| \cdot |\phi(z')| > \frac{2}{3}\left[\sqrt{n}\left(\frac{9}{128C} - 13C\right) - \frac{9}{32}\right]|p(0)|.$$

If $C = 1/20$ then

$$|p'(z')| > .05\sqrt{n}\,|p(0)| = C\sqrt{n}\,|p(0)|$$

and this contradicts our assumption (18) since

$$|z'| = \sin\alpha/\big(C\sqrt{n}\,\big) < 1/\big[3C\sqrt{n}\,\big] < 1/\big[C\sqrt{n}\,\big] = r_n. \ \square$$

We end this section (and the paper) with a family of examples to show that for a star-shaped compact set the exponent b in a reverse Markov inequality can be arbitrarily small.

Proposition 3.3. *Given $\varepsilon > 0$, there exists a compact, star-shaped set $K \subset \mathbf{C}$ such that there is no constant $C > 0$ with*

$$\|p'\|_K \geq C(\deg p)^{\varepsilon}\|p\|_K$$

for all polynomials p.

Proof. Fix a positive integer m which will be chosen appropriately later and let $z_1,\dots,z_m$ be the m-th roots of unity. Consider the set consisting of m "spokes of a wheel"

$$K_m := \{tz_j : 0 \le t \le 1,\ j = 1,\dots,m\}.$$

For a positive integer n, let $p(z) = (z^m - 1)^n$. Then $\deg p = nm$; $\|p\|_{K_m} = 1$; and a calculation shows that

$$\|p'\|_{K_m} = mn\left(\frac{m-1}{mn-1}\right)^{(m-1)/m}\left[\frac{m-1}{mn-1} - 1\right]^n.$$

For n large,

$$\|p'\|_{K_m} \asymp m^{2-1/m} e^{-1+1/m} n^{1/m}.$$

Thus, taking $m > 1/\varepsilon$ proves the proposition. □

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