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# SCHWARZIAN DERIVATIVES OF TOPOLOGICALLY FINITE MEROMORPHIC FUNCTIONS

**Adam Lawrence Epstein**

University of Warwick, Mathematics Institute
Coventry CV4 7AL, United Kingdom; adame@maths.warwick.ac.uk

**Abstract.** Using results from dynamics, we prove sharp upper bounds on the local vanishing order of the nonlinearity and Schwarzian derivative for topologically finite meromorphic functions.

The entire functions $f\colon \mathbf{C} \to \mathbf{C}$ with rational *nonlinearity*

$$\mathfrak{N}_f = \frac{f''}{f'}$$

and more generally the meromorphic functions $f\colon \mathbf{C} \to \widehat{\mathbf{C}}$ with rational *Schwarzian derivative*

$$\mathfrak{S}_f = \mathfrak{N}_f' - \tfrac{1}{2}\mathfrak{N}_f^2$$

were discussed classically by the brothers Nevanlinna and a number of other researchers. These functions admit a purely topological characterization (for details, see [6, pp. 152–153], [7, pp. 391–393] and [11, pp. 298–303]). We show here that a *topological finiteness* property enjoyed by a considerably larger class of functions guarantees bounds on the local vanishing orders of $\mathfrak{N}_f$ and $\mathfrak{S}_f$. For meromorphic $f\colon \mathbf{C} \to \widehat{\mathbf{C}}$, let $S(f)$ denote the set of all *singular* (that is, *critical* or *asymptotic*) *values* in $\widehat{\mathbf{C}}$. It follows by elementary covering space theory that $\#S(f) \geq 2$, provided that $f$ is not a Möbius transformation; note further that $\infty \in S(f)$ for any entire $f\colon \mathbf{C} \to \mathbf{C}$, provided that $f$ is not affine. Our main result is the following:

**Theorem 1.** A. *If* $f\colon \mathbf{C} \to \widehat{\mathbf{C}}$ *is meromorphic but not a Möbius transformation then* $\operatorname{ord}_\zeta \mathfrak{S}_f \leq \#S(f) - 2$; *consequently* $\operatorname{ord}_\zeta \mathfrak{N}_f \leq \#S(f) - 1$, *for every* $\zeta \in \mathbf{C}$.

B. *If* $f\colon \mathbf{C} \to \mathbf{C}$ *is entire but not affine then* $\operatorname{ord}_\zeta \mathfrak{N}_f \leq \#S(f) - 2$ *for every* $\zeta \in \mathbf{C}$; *consequently, if* $\operatorname{ord}_\zeta \mathfrak{N}_f \geq 1$ *then* $\operatorname{ord}_\zeta \mathfrak{S}_f \leq \#S(f) - 3$.

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As to the converse, the mere existence of uniform bounds on $\operatorname{ord}_\zeta \mathfrak{N}_f$ or $\operatorname{ord}_\zeta \mathfrak{S}_f$ is not enough to guarantee the finiteness of $S(f)$. For example, the function $f(z) = z + 1 - e^z$ has infinitely many critical values $2\pi ki = f(2\pi ki)$, but $\mathfrak{N}_f(z) = e^z/(e^z - 1)$ has no zeros whatsoever and

$$\mathfrak{S}_f(z) = \frac{-e^z(e^z + 2)}{2(e^z - 1)^2}$$

has only simple zeros. On the other hand, the assertions in Theorem 1 are certainly sharp. Compare the following classical result discussed in the aforementioned references:

**Theorem 2.** A. *Let* $f\colon \mathbf{C} \to \widehat{\mathbf{C}}$ *be meromorphic; if* $\mathfrak{S}_f$ *is rational then* $\#S(f) - 2 \le \deg \mathfrak{S}_f$.
B. *Let* $f\colon \mathbf{C} \to \mathbf{C}$ *be entire; if* $\mathfrak{N}_f$ *is rational then* $\#S(f) - 2 \le \deg \mathfrak{N}_f$.

Combining Theorem 1 with Theorem 2 leads to the vacuous conclusion that $\operatorname{ord}_\zeta \mathfrak{R} \le \deg \mathfrak{R}$ at every $\zeta \in \mathbf{C}$ for any rational function $\mathfrak{R}$. In this sense, Theorem 1 may be regarded as a pointwise converse to Theorem 2; moreover, as $\operatorname{ord}_\zeta \mathfrak{R} = \deg \mathfrak{R}$ for $\mathfrak{R}(z) = (z - \zeta)^D$, the existence of meromorphic functions $f$ with $\mathfrak{S}_f = \mathfrak{R}$ and entire functions $f$ with $\mathfrak{N}_f = \mathfrak{R}$ shows that Theorem 1 is sharp. For a slightly less trivial example, the entire function $f(z) = \sin z$ has critical values at $\pm 1$, an asymptotic value at $\infty$ and no finite asymptotic values. As $\#S(f) = 3$, Theorem 1 guarantees that every zero of $\mathfrak{N}_f(z) = -\tan z$ is simple. Similarly, $S(g) = S(f)$ for the meromorphic function $g(z) = \csc(z) = 1/f(z)$, so Theorem 1 guarantees that every zero of $\mathfrak{S}_f(z) = -1 - \frac{3}{2}\tan^2 z = \mathfrak{S}_g(z)$ is simple, and that no such point is a zero of $\mathfrak{N}_f(z)$ or

$$\mathfrak{N}_g(z) = \frac{\sin^2 z - 2}{\sin z \cos z}.$$

The proof of Theorem 1 is more interesting than the actual statement. The strategy is to reduce the assertions to well-known results in holomorphic dynamics. This idea goes back at least as far as the work [2] of Bergweiler–Eremenko (see also [8]), and our application here is quite similar but perhaps even more basic. Let $F$ be meromorphic with an isolated fixed point at $\zeta \in \mathbf{C}$. The *multiplicity* of $F$ at $\zeta$ is the positive integer $\operatorname{mult}_\zeta F = \operatorname{ord}_\zeta\big(F(z) - z\big)$. Clearly, $\operatorname{mult}_\zeta F \ge 2$ precisely when $F'(\zeta) = 1$. Fatou showed that if $F$ is rational then the immediate basin of such a *multiple* fixed (or, more generally, periodic) point contains at least $\operatorname{mult}_\zeta(F) - 1$ critical values belonging to disjoint and infinite forward orbits. The argument adapts easily to entire and meromorphic functions, provided that one also counts forward orbits of asymptotic values; for details, see [10, p. 74]. The work [2] applied Fatou's theorem to help settle a long-standing conjecture in value-distribution theory. Part of the discussion involves showing that a particular auxiliary mapping, known by construction to have only finitely many singular values, necessarily has only finitely many multiple fixed points. In fact, as Bergweiler has observed, this particular conclusion may also be established without Fatou's theorem: indeed, if $S(f)$ is finite then $f'(\zeta_k) \to \infty$ for any sequence of distinct fixed points $\zeta_k$ (see [1] and also [5]). By contrast, our application exploits the actual bound on the multiplicity of a single fixed point:

**Theorem 3.** A. *If* $F\colon \mathbf{C} \to \widehat{\mathbf{C}}$ *is meromorphic but not a Möbius transformatiom then* $\mathrm{mult}_\zeta(F) \le \#S(F) + 1$ *for every fixed point* $\zeta \in \mathbf{C}$.

B. *If* $F\colon \mathbf{C} \to \mathbf{C}$ *is entire but not affine then* $\mathrm{mult}_\zeta(F) \le \#S(F)$ *for every fixed point* $\zeta \in \mathbf{C}$.

We begin by recalling some standard properties of the nonlinearity and Schwarzian derivative. If $f\colon U \to \widehat{\mathbf{C}}$ is meromorphic and nonconstant on a connected open set $U \subseteq \mathbf{C}$ then $\mathfrak{N}_f$ and $\mathfrak{S}_f$ are defined and meromorphic on $U$. It is well-known and easily verified that $\mathfrak{N}_f$ identically vanishes if and only if $f$ is the restriction of an affine transformation, while $\mathfrak{S}_f$ identically vanishes if and only if $f$ is the restriction of a Möbius transformation. Moreover, $\mathfrak{N}_{T\circ f} = \mathfrak{N}_f$ for any affine transformation $T$, while $\mathfrak{S}_{T\circ f} = \mathfrak{S}_f$ for any Möbius transformation $T$: these observations follow from the cocycle relations

$$\begin{aligned} \mathfrak{N}_{g\circ f}\, dz &= \mathfrak{N}_f\, dz + f^*(\mathfrak{N}_g\, dz), \\ \mathfrak{S}_{g\circ f}\, dz^2 &= \mathfrak{S}_f\, dz^2 + f^*(\mathfrak{S}_g\, dz^2). \end{aligned} \tag{1}$$

We further require two rather straightforward computations:

**Lemma 1.** *If* $f$ *is meromorphic near* $\zeta \in \mathbf{C}$ *then* $\mathrm{ord}_\zeta \mathfrak{N}_f \ge -1$ *and* $\mathrm{ord}_\zeta \mathfrak{S}_f \ge -2$.

A. *If* $\zeta$ *is a critical point of* $f$ *then* $\mathfrak{N}_f$ *has a simple pole at* $\zeta$ *and* $\mathfrak{S}_f$ *has a double pole at* $\zeta$.

B. *If* $\zeta$ *is not a critical point of* $f$ *then* $\mathfrak{S}_f$ *is holomorphic at* $\zeta$, *while* $\mathfrak{N}_f$ *is also holomorphic at* $\zeta$ *if and only if* $f$ *is holomorphic at* $\zeta$. *Moreover, if* $\mathrm{ord}_\zeta \mathfrak{N}_f \ge 1$ *then* $\mathrm{ord}_\zeta \mathfrak{S}_f = \mathrm{ord}_\zeta \mathfrak{N}_f - 1$.

*Proof.* Let $n = \deg_\zeta f$ be the local degree of $f$ at $\zeta$. If $f$ is holomorphic at $\zeta$ then we may write $f(z) = f(\zeta) + \alpha(z-\zeta)^n + O\big((z-\zeta)^{n+1}\big)$ for some $\alpha \ne 0$, whence

$$\begin{aligned} \mathfrak{N}_f(z) &= \frac{(n-1)na_n(z-\zeta)^{n-2} + O\big((z-\zeta)^{n-1}\big)}{na_n(z-\zeta)^{n-1} + O\big((z-\zeta)^n\big)} = \frac{n-1}{z-\zeta} + O(1), \\ \mathfrak{S}_f(z) &= \frac{n^2-1}{2(z-\zeta)^2} + O\bigg(\frac{1}{z-\zeta}\bigg). \end{aligned}$$

In particular, if $n = 1$ then $\mathfrak{N}_f$ is holomorphic at $\zeta$, so $\mathfrak{S}_f = \mathfrak{N}'_f - \frac{1}{2}\mathfrak{N}_f^2$ is also holomorphic at $\zeta$. Moreover, if $\mathrm{ord}_\zeta \mathfrak{N}_f = k \ge 1$ then

$$\mathrm{ord}_\zeta \mathfrak{N}'_f = k - 1 < 2k = \mathrm{ord}_\zeta \mathfrak{N}_f^2$$

and consequently $\mathrm{ord}_\zeta \mathfrak{S}_f = k-1 = \mathrm{ord}_\zeta \mathfrak{N}_f - 1$. On the other hand, if $f$ has a pole at $\zeta$ then we may apply the previous observations to the function $g = 1/f$ which is holomorphic at $\zeta$. It follows from (1) that $\mathfrak{N}_f = \mathfrak{N}_g - 2g'/g$, where $\mathfrak{N}_g(z) = (n-1)/(z-\zeta) + O(1)$ and $g'(z)/g(z) = n/(z-\zeta) + O(1)$ because $\mathrm{ord}_\zeta g = n = \deg_\zeta g$; consequently, $\mathfrak{N}_f(z) = -(n+1)/(z-\zeta) + O(1)$. Similarly, $\mathfrak{S}_f = \mathfrak{S}_g$ so

$$\mathfrak{S}_f(z) = \frac{n^2-1}{2(z-\zeta)^2} + O\left(\frac{1}{z-\zeta}\right). \ \square$$

**Lemma 2.** *Let $F$ be holomorphic with an isolated fixed point at $\zeta \in \mathbf{C}$.*
A. *If* $\mathrm{mult}_\zeta F \geq 2$ *then* $\mathrm{ord}_\zeta \mathfrak{N}_F = \mathrm{mult}_\zeta F - 2$.
B. *If* $\mathrm{mult}_\zeta F \geq 3$ *then* $\mathrm{ord}_\zeta \mathfrak{S}_F = \mathrm{mult}_\zeta F - 3$.

*Proof.* Writing $F(z) = z + \alpha(z-\zeta)^m + O\big((z-\zeta)^{m+1}\big)$ where $m = \mathrm{mult}_\zeta F \geq 2$ and $\alpha \neq 0$, we have

$$\begin{aligned}\mathfrak{N}_F(z) &= \frac{(m-1)m\alpha(z-\zeta)^{m-2} + O\big((z-\zeta)^{m-1}\big)}{1 + m\alpha(z-\zeta)^{m-1} + O\big((z-\zeta)^m\big)}\\ &= (m-1)m\alpha(z-\zeta)^{m-2} + O\big((z-\zeta)^{m-1}\big)\end{aligned}$$

whence $\mathrm{ord}_\zeta \mathfrak{N}_F = \mathrm{mult}_\zeta F - 2$. Consequently, if $\mathrm{mult}_\zeta F \geq 3$ then $\mathrm{ord}_\zeta \mathfrak{N}_f \geq 1$, so $\mathrm{ord}_\zeta \mathfrak{S}_F = \mathrm{mult}_\zeta F - 3$ by Part B of Lemma 1. $\square$

*Proof of Theorem* 1. It suffices to show that $\mathrm{ord}_\zeta \mathfrak{N}_f \leq \#S(f) - 2$ for any entire function $f\colon \mathbf{C} \to \mathbf{C}$, and that $\mathrm{ord}_\zeta \mathfrak{S}_f \leq \#S(f) - 2$ for any meromorphic function $f\colon \mathbf{C} \to \widehat{\mathbf{C}}$; the remaining claims then follow by Part B of Lemma 1.

Suppose first that $f$ is entire. If $\zeta$ is a critical point of $f$ then Part A of Lemma 1 implies $\mathrm{ord}_\zeta \mathfrak{N}_f = -1 < \#S(f) - 1$. If $\zeta$ is not a critical point of $f$ then the distinguished local inverse at $f(\zeta)$ is approximated to first-order by a unique affine transformation

$$T^{(1)}_{f,\zeta}(z) = \zeta + \frac{z - f(\zeta)}{f'(\zeta)}.$$

The composition $F = T^{(1)}_{f,\zeta} \circ f$ is an entire function which fixes $\zeta$; indeed, we have $F(z) = z + O\big((z-\zeta)^2\big)$ so $\zeta$ is a multiple fixed point. As $\mathfrak{N}_f = \mathfrak{N}_F$, it follows from Part A of Lemma 2 that $\mathrm{ord}_\zeta \mathfrak{N}_f = \mathrm{mult}_\zeta F - 2$; moreover, $S(F) = T^{(1)}_{f,\zeta}\big(S(f)\big)$ so

$$\mathrm{ord}_\zeta \mathfrak{N}_f = \mathrm{mult}_\zeta F - 2 \leq \#S(F) - 2 = \#S(f) - 2$$

by Part A of Theorem 3.

Suppose now that $f$ is meromorphic. If $\zeta$ is a critical point of $f$ then Part A of Lemma 1 implies $\mathrm{ord}_\zeta \mathfrak{S}_f = -2 < \#S(f) - 2$. If $\zeta$ is not a critical point of $f$

then the distinguished local inverse at $f(\zeta)$ is approximated to second-order by a unique Möbius transformation $T^{(2)}_{f,\zeta}$: in fact,

$$T^{(2)}_{f,\zeta}(z) = \zeta + \frac{2f'(\zeta)\big(z - f(\zeta)\big)}{f''(\zeta)\big(z - f(\zeta)\big) + 2f'(\zeta)^2}$$

provided that $\zeta$ is not a pole. The composition $F = T^{(2)}_{f,\zeta} \circ f$ is a meromorphic function fixing $\zeta$, and $\mathrm{mult}_\zeta F \geq 3$ because $F(z) = z + O\big((z-\zeta)^3\big)$. As we have $\mathfrak{S}_f = \mathfrak{S}_F$, it follows from Part B of Lemma 2 that $\mathrm{ord}_\zeta \mathfrak{S}_f = \mathrm{mult}_\zeta F - 3$; moreover, $S(F) = T^{(2)}_{f,\zeta}\big(S(f)\big)$ so

$$\mathrm{ord}_\zeta \mathfrak{S}_f = \mathrm{mult}_\zeta F - 3 \leq \#S(F) - 2 = \#S(f) - 2$$

by Part B of Theorem 3. □

For context, let us briefly sketch the proof of Theorem 2. The core of the argument is to show that every asymptotic value of $f$ corresponds to a *logarithmic end*: a simply connected region $U \subset \mathbf{C}$ such that the restriction $f|_U$ is an infinite cyclic cover of a punctured neigborhood of the asymptotic value. This assertion follows from asymptotic analysis of the solutions to the associated differential equations (see [7], [9] and [11]—for Part B one may appeal to the explicit formula $f = \int e^{\int \mathfrak{N}_f}$). Note that $\mathrm{ord}_\infty \mathfrak{S}_f \geq 2$ if and only the quadratic differential $\mathfrak{S}_f(z)\,dz^2$ is holomorphic at $\infty$; in this case, it follows that $f$ is meromorphic at $\infty$ and therefore rational, so that there are no logarithmic ends. On the other hand, if $\mathrm{ord}_\infty \mathfrak{S}_f \leq 2$ then further analysis shows that there are precisely $-\mathrm{ord}_\infty \mathfrak{S}_f(z)\,dz^2 = 2 - \mathrm{ord}_\infty \mathfrak{S}_f$ logarithmic ends. As $\mathrm{ord}_\zeta \mathfrak{S}_f = -2$ at each finite critical point $\zeta$ but $\mathfrak{S}_f$ is elsewhere holomorphic on $\mathbf{C}$, it follows that $\deg \mathfrak{S}_f = 2k + l - 2$, where $k$ is the number of critical points and $l$ is the number of logarithmic ends; consequently, $\#S(f) - 2 \leq k + l - 2 \leq \deg \mathfrak{S}_f$ as claimed in Part A. Furthermore, under the assumption of Part B we have $\mathrm{ord}_\infty \mathfrak{S}_f(z)\,dz^2 = 2\mathrm{ord}_\infty \mathfrak{N}_f(z)\,dz^2$, so that there are an even number of logarithmic ends, corresponding alternately to finite or infinite asymptotic values. As $\mathrm{ord}_\zeta \mathfrak{N}_f = -1$ at each critical point $\zeta$ but $\mathfrak{N}_f$ is elsewhere holomorphic on $\mathbf{C}$, it follows that $\#S(f) - 2 \leq k + \frac{1}{2}l - 1 = \deg \mathfrak{N}_f$.

Theorem 3 admits various extensions and refinements, each of which may be transformed as above into a version of Theorem 1. We conclude this note with a survey of such possible adaptations. In one direction, Fatou's theorem on the existence of singular orbits in attracting and parabolic immediate basins has been extended to the class of *finite type* complex analytic maps, namely those analytic maps $f\colon W \to X$ where $X$ is a finite union of compact Riemann surfaces, $W \subseteq X$ is open, and $S(f) \subset X$ is finite; details may be found in [4]. The argument proving Theorem 1 applies more generally to finite type maps $f\colon W \to \widehat{\mathbf{C}}$, with the conclusion that $\mathrm{ord}_\zeta \mathfrak{S}_f(z)\,dz^2 \leq \#S(f) - 2$ for every $\zeta \in W$. The case when $\widehat{\mathbf{C}} - W$

is finite is of particular interest, for then $\mathfrak{S}_f(z)\,dz^2$ is meromorphic except possibly at isolated essential singularities. The simplest examples of such maps would appear to be those for which $\mathfrak{S}_f$ is rational, and the above sketch suggests that Theorem 2 remains valid without the assumption that $f$ be meromorphic on all of $\mathbf{C}$. In fact, it is highly plausible that Nevanlinna's topological characterization extends, so that the following should be equivalent for analytic maps $f\colon W \to \widehat{\mathbf{C}}$ with $W \subseteq \widehat{\mathbf{C}}$:

(1) $f$ is a map of finite type with finitely many critical points and finitely many logarithmic ends,

(2) $\mathfrak{S}_f$ is a rational function and $\widehat{\mathbf{C}} - W$ is finite.

However, we have not seen such a claim demonstrated or even explicitly stated in the literature.

One might also seek to exploit the fact that Fatou's theorem actually bounds the parabolic multiplicity of any rationally indifferent periodic point, and sometimes furnishes a supplementary constraint on the associated holomorphic index (see [3] for precise statements). The obvious modification of the argument proving Theorem 1, namely consideration of the maps $\omega \cdot T^{(j)}_{f,\zeta} \circ f$ for other roots of unity $\omega$, thereby yields additional restrictions on various combinations of Taylor coefficients. However, it is not clear that the conclusions so obtained place illuminating restrictions on the nonlinearity and Schwarzian derivative.

### References

[1] Bergweiler, W.: Iteration of meromorphic functions. - Bull. Amer. Math. Soc. 29, 1993, 151–188.

[2] Bergweiler, W., and A. Eremenko: On the singularities of the inverse to a meromorphic function of finite order. - Rev. Math. Iberoamericana 11, 1995, 355–373.

[3] Epstein, A.: Infinitesimal Thurston rigidity and the Fatou–Shishikura inequality. - Stony Brook IMS Preprint 1999/1.

[4] Epstein, A.: Towers of finite type complex analytic maps. - Ph.D. Thesis, CUNY, 1993.

[5] Epstein, A.: Dynamics of finite type complex analytic maps. - Manuscript in preparation.

[6] Eremenko, A.: Entire and meromorphic solutions of ordinary differential equations. - In: Complex Analysis I, Chapter 6, Springer-Verlag, New York, 1997.

[7] Hille, E.: Ordinary Differential Equations in the Complex Domain. - Wiley-Interscience, New York, 1976.

[8] Hinkkanen, A.: Iteration and the zeros of the second derivative of a meromorphic function. - Proc. London Math. Soc. 65, 1992, 629–650.

[9] Laine, I.: Nevanlinna Theory and Complex Differential Equations. - Walter de Gruyter, Berlin, 1993.

[10] Morosawa, S., Y. Nishimura, M. Taniguchi, and T. Ueda: Holomorphic Dynamics. - Cambridge University Press, Cambridge, 2000.

[11] Nevanlinna, R.: Analytic Functions. - Springer-Verlag, New York, 1970.

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