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DIFFERENTIABILITY AND RIGIDITY THEOREMS

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Abstract. In this paper we establish two theorems in rigidity problems. In particular, suppose that G is a Möbius group of $\overline{\mathrm{R}}^n$ and $f\colon \overline{\mathrm{R}}^n \to \overline{\mathrm{R}}^n$ is a G-compatible map which has a non zero differential at a radial point of G. If all elements in G do not have a common fixed point, then f is a Möbius transformation. This improves a well-known result by P. Tukia.

1. Introduction

Mostow's rigidity theorem is a deep fundamental theorem in the theory of Möbius groups. Since the discovery of the theorem, there have been many discussions on this subject by S. Agard, D. Sullivan, P. Tukia, and some others.

In [5], P. Tukia extended Mostow's rigidity theorem to a very general situation. He proved ([5, Theorem A])

Theorem A. *Let G be a group of Möbius transformations of $\overline{\mathrm{R}}^n$ and let $f\colon \overline{\mathrm{R}}^n \to \overline{\mathrm{R}}^n$ be a G-compatible map which is differentiable with a non-vanishing Jacobian at a radial point of G. Then f is a Möbius transformation unless there is a point $z \in \overline{\mathrm{R}}^n$ fixed by every $g \in G$. If there is such a point z, then there are Möbius transformations h and h' such that $h(\infty)$ is fixed by every $g \in G$ and that $h'fh \mid R^n$ is an affine homeomorphism of R^n.*

The map f is G-*compatible*, if there is a homomorphism φ of G onto another Möbius group such that $fg(x) = \varphi(g)f(x)$.

Actually, a more general result, where f is defined on a G-invariant set A, was obtained in [5]. A striking fact is that the action of f at a radial point of G determines very much the behavior of f. It is natural to investigate the behavior of f without the assumption that the Jacobian of f at the radial point is non-vanishing. In [5], the assumption about the Jacobian is essential, because in the proof triples of distinct points are mapped by the derivative of f at the radial point to triples of distinct points (see Section 2 for the definitions). Hence the projections to H^{n+1} (defined in Section 2 or Section D in [5]) of the images of those triples by the derivative of f are defined. In this paper, we find a local way to avoid such a situation. Thus locally Tukia's technique can be applied and the local result can be extended globally.

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The main results of this paper are the following theorems.

Theorem 1. *Let G be a Möbius group of $\overline{\mathrm{R}}^n$ and let $A \subset \overline{\mathrm{R}}^n$ be a G-invariant set containing at least three points. Let $f\colon A \to \overline{\mathrm{R}}^n$ be G-compatible and be differentiable at a radial point of G such that the rank of the derivative is $k > 0$. Suppose that A is not contained in the union of two $(n-k)$-subspheres of $\overline{\mathrm{R}}^n$. Then there are Möbius transformations h' and h such that $h'fh$ is the restriction of an affine map on $h^{-1}(A) \cap R^n$. If $h(\infty) \in A$ and f is not continuous at $h(\infty)$, then $h(\infty)$ is fixed by every $g \in G$. Moreover, if A is $\overline{\mathrm{R}}^n$ or a k'-sphere $(0 < k' \le n)$ and $h(\infty)$ is not fixed by every $g \in G$, then f is a Möbius transformation.*

Since A can be a proper subset of $\overline{\mathrm{R}}^n$, the definition of differentiability and the rank of the differential needs careful definition. These definitions are given in Section 2.

Corollary 1. *Let G be a non-elementary Möbius group of $\overline{\mathrm{R}}^n$ and let $f\colon \overline{\mathrm{R}}^n \to \overline{\mathrm{R}}^n$ be G-compatible and differentiable at a radial point of G. Then either f is a Möbius transformation or the rank of the differential of f at this radial point is 0.*

More generally, let $c \in \overline{\mathrm{R}}^n$ and denote by S_c the minimal sphere containing $L(G) \cup \{c\}$. We have

Theorem 2. *Let A, f and G be as in Theorem 1. Assume that the point $h(\infty)$ in Theorem 1 is not fixed by every $g \in G$. Then $f \mid A \cap S_c$ is the restriction of a Möbius transformation.*

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2. Proofs of the theorems

We will now give the definition of differentiability of a map defined on an arbitrary subset of $\overline{\mathrm{R}}^n$, analogous and equivalent to the definition D1 in [5]. Assume that $f\colon A \to \overline{\mathrm{R}}^n$, $A \subset \overline{\mathrm{R}}^n$, is a map and that $x \in R^n \cap \bar{A}$. Denote by $V(A)$ the affine subspace generated by A and $D(A)$ the dimension of $V(A)$. Then we say that f is differentiable at x if there is an affine map α of $V(A)$ such that

$$\frac{|f(y) - \alpha(y)|}{|y - x|} \to 0$$

as $y \to x$ in A, where we require that f is continuous at x if $x \in A$. If β is a linear map such that $\alpha = \beta +$ (constant) on $V(A)$, then β is called the derivative or differential of f at x. If $A \neq \overline{\mathrm{R}}^n$, then α may not be unique and neither the rank of β is unique. What we call in this paper the rank of the differential of f at point x is the minimal rank of all possible β and we always choose α and β with the minimal rank. The affine part of f at point x is defined to be α. If β has a full rank on $V(A)$, then obviously α can be extended to an affine homeomorphism on R^n.

Notation. The hyperbolic metric of H^{n+1} is ϱ and $|\cdot|$ denotes the usual euclidean metric.

A map is an *affine* map if it is of the form $L+$ (constant), where L is a linear map.

Affine and linear maps of R^n are extended to $\overline{\mathrm{R}}^n$ so that $\infty \mapsto \infty$.

Assume that $u \in \overline{\mathrm{H}}^{n+1} = \overline{\mathrm{R}}^n \cup H^{n+1}$, $v \in \overline{\mathrm{R}}^n$, $u' \in R^{n+1}$ and $v' \in R^{n+1}$ such that $u \neq v$ and $u' \neq v'$. Then

$H(u,v)$ is the hyperbolic line or ray with end points u and v;

$L(u',v')$ is the euclidean line through u' and v'.

$x \in \overline{\mathrm{H}}^{n+1}$, y, $z \in R^n$ be distinct, and let Π be a plane. Then

$\operatorname{ang}(y,z,\Pi)$ denotes the angle between the euclidean straight line $L(y,z)$ and Π, taken between and Π, taken between 0 and $\frac{1}{2}\pi$.

$\operatorname{ang}(x,y,z)$ is the angle between the rays $H(x,y)$ and $H(x,z)$, taken between 0 and $\frac{1}{2}\pi$,

$P(x,y,z)$ is the orthogonal projection of z onto $H(x,y)$ in the hyperbolic geometry. In this case $x \in \overline{\mathrm{R}}^n$.

We set for $m \geq 0$:

$C_m = \bigl\{x \in H^{n+1} : \varrho\bigl(x, H(0,\infty)\bigr) \leq m\bigr\}$,

$C_{m,1} = \bigl\{(x,y,z) : P(x,y,z) \in C_m,\ |P(x,y,z)| = 1\bigr\}$.

$\mathrm{M\ddot{o}b}(n)$ is the group of Möbius transformations of $\overline{\mathrm{R}}^n$ and if $g \in \mathrm{M\ddot{o}b(n)}$ is loxodromic, then the *multiplier* of g is the number $\lambda > 1$ so that g can be conjugated to the map $z \mapsto \lambda\beta(z)$, where β is orthogonal.

Lemma 1. *Let $g \in \mathrm{M\ddot{o}b}(n)$ and $u \in R^n$. Then g is loxodromic with an attractive fixed point u if and only if for some $x \in R^n$ the following is true*:

$$|g^k(x) - u| = \lambda^{-k}|x-u|B(x,k), \tag{1}$$

for some $\lambda > 1$ and where the sequence $\{B(x,k) : k = 1,2,\dots\}$ is bounded. The number λ is the multiplier of g.

Proof. Suppose first that g is loxodromic with multiplier λ, attractive fixed point u and repelling fixed point v. If k is a positive integer and $u \neq x \neq v$, then the Möbius invariance of the cross-ratio implies

$$|g^k(x), x, u, v| = \lambda^{-k},$$

that is

$$|g^k(x) - u| = \lambda^{-k} \frac{|x-u|\,|g^k(x) - v|}{|x-v|}.$$

Here as usual in the definition of a cross ratio,

$$\frac{|g^k(x) - v|}{|x-v|} = 1$$

if $v = \infty$. Setting

$$B(x,k) = \frac{|g^k(x) - v|}{|x-v|}$$

our claim follows.

Conversely, suppose that (1) is true. It follows that g is either loxodromic or parabolic and that u must be the attractive fixed point of g. Since $\lambda > 1$, g must be loxodromic.

Obviously, the number λ in (1) is well defined and we have seen in the first part of the proof that it is the multiplier of g. □

Remark. If g is loxodromic with fixed points u and v, then (1) holds true for any $x \in R \backslash \{u, v\}$.

The following lemma is the extension of Lemma C1 in [5].

Lemma 2. *Let g, $g' \in \text{Möb}(n)$, where g is loxodromic, let $A \subset \overline{R}^n$ be a set which properly contains the fixed point set of g and is invariant under g (i.e. $gA = A$). Suppose that β is an affine map which is not constant on A. If for any point $x \in A$ and all $k \in Z$, we have that*

$$\beta g^k(x) = g'^k \beta(x),$$

then g' is loxodromic, both g and g' have the same multiplier, and β maps the attractive (repelling) fixed point of g to attractive (repelling) fixed point of g'. Moreover, $\beta \mid V$ is a similarity if $a \in A$ and V is the affine subspace of minimal dimension such that $V \cup \{\infty\}$ contains a and all fixed points of g.

Proof. Let u be the attractive fixed point of g. We can assume that $u \in R^n$, possibly by replacing g by g^{-1}. Our assumption implies that we can find $x \in A$ such that $g(x) \neq x$ and $\beta(x) \neq \beta(u)$; if necessary, we replace u by the other fixed point of g (and g by g^{-1}). Set

$$B_0(x,k) = \frac{|\beta g^k(x) - \beta(u)|}{|g^k(x) - u|}.$$

Since $g^k(x) \to u$ as $k \to \infty$, the numbers $B_0(x,k)$, $k = 1, 2, \ldots$, are bounded. Write

$$|g'^k \beta(x) - \beta(u)| = |\beta g^k(x) - \beta(u)| = |g^k(x) - u| B_0(x,k).$$

Using Lemma 1 we find that g' is loxodromic and $\beta(u)$ is the attractive fixed point of g'. Similarly we see that β maps the repelling fixed point of g to a repelling fixed point of g', and from the remark of Lemma 1 we deduce that both g and g' have the same multiplier. It is easy to see that the original proof of Lemma C1 in [5] works if we replace α in Lemma C1 by β in our lemma here. Actually we have proven that β maps distinct fixed points of g to distinct fixed points of g' and this is what we need for the modification. The rest of the claim of our lemma follows from [5, Lemma C1] immediately. We remark that the right-hand side formula (C2) of [5] should be $(x_1 + \lambda x_2, \mu x_2)$ but the proof of Lemma C1 in [5] is unaffected (also $S_\varepsilon(u)$ should be the union, rather than the family, of the circles S mentioned in the defining formula). □

Let G, A and f be as in Theorem 1. Then there is a homomorphism φ of G onto another Möbius group such that $\varphi(g)f = fg$ for all $g \in G$. By composing with Möbius transformations, we can assume that $0 \in A$ is a radial point, $f(0) = 0$ and f is differentiable at 0 and the rank of the differential is k. We also assume that in the following lemmas $0 < k < D(A)$, where $D(A)$ is the dimension of the affine subspace $V(A)$ generated by A. Denote by β the differential and α the affine part of f at the radial point. We extend α to R^n so that α maps an affine subspace orthogonal to $V(A)$ onto a constant. The assumption that 0 is a radial point guarantees that we can find a sequence $\{g_i\}$, where $g_i \in G$, such that for any given $x \in H^{n+1}$, there is $m > 0$ such that $g_i(x) \in C_m$ and $g_i(x) \to 0$. Without loss of generality, we assume that $\infty \in A$ and $g_i^{-1}(\infty)$ converges to a point $a \neq 0$. Then $g_i(z) \to 0$ if $z \neq a$ as follows, for instance, by an application of 4°, p. 568, of [5].

Lemma 3. *Let $b \in \overline{R}^n \setminus \{a\}$. Then we can pass to a subsequence so that the following is true. There are x_0, $y_0 \in A\setminus\{a\}$, a neighborhood U of x_0 and a neighborhood V of y_0 such that $\alpha g_i(x)$, $\alpha g_i(b)$, and $\alpha g_i(y)$ are distinct for each $x \in U$ and $y \in V$ beginning from some $i = i_0$.*

Proof. Since $b \neq a$, then, as we have seen, $g_i(b) \to 0$ as $i \to \infty$. In particular, $g_i(b) \neq \infty$ for large i. Let $V(0)$ be the subspace whose dimension is $n-k$ and $\alpha \mid V(0)$ is constant. For each i, there is a unique $(n-k)$-sphere or plane Π_i containing b and $g_i^{-1}(\infty)$ such that $g_i(\Pi_i)$ is an $(n-k)$-plane parallel to $V(0)$. Since $g_i^{-1}(\infty) \to a \neq b$, we can find an $(n-k)$-sphere or plane S containing b and a such that $\Pi_i \to S$ in the sense of Hausdorff metric. Here we may substitute a subsequence for $\{g_i\}$ if necessary. By the assumption about A there exists $x_0 \in A \setminus S$. Since $a \in S$ and $\lim_{i\to\infty} g_i^{-1}(\infty) = a$, we have $x_0 \neq a$ and hence $g_i(x_0) \neq \infty$ for large i. Furthermore, there exists a neighborhood U of x_0 such that $U \cap S = \emptyset$ and $U \cap \Pi_i = \emptyset$ for large i. Thus, by possibly shrinking U, we can assume that if L is a line or a circle which contains b, $g_i^{-1}(\infty)$ and a point $x \in U$, then L intersects S at b at an angle at not less than a certain $\delta > 0$ for large i. It follows that if $x \in U$, then $\operatorname{ang}\bigl(g_i(b), g_i(x), g_i(\Pi_i)\bigr) > \delta$ and hence $\alpha\bigl(g_i(b)\bigr)$ and $\alpha\bigl(g_i(x)\bigr)$ are distinct for large i. Now for each i, denote by $\Pi_i{}'$ the unique $(n-k)$-sphere or plane which contains x_0 and $g_i^{-1}(\infty)$ such that $g_i(\Pi_i{}')$ is parallel to $V(0)$. In the same way we can find an $(n-k)$-sphere or plane S' containing x_0 and a such that $\Pi_i{}' \to S'$ in the sense of Hausdorff metric. Take $y_0 \in A \backslash (S \cup S')$ and choose a neighborhood V of y_0 such that $V \cap (S \cup S') = V \cap U = \emptyset$. Then, possibly by shrinking V, we see that they satisfy the requirements of the lemma. □

An immediate consequence is the following lemma.

Lemma 4. *Let b, $\{g_i\}$, x_0 and y_0 be as in Lemma 3. Then there exists a neighborhood W of b such that $a \notin W$ and a number $\delta > 0$ such that the following inequalities are true for all $x \in W$ and large i:*

$$\begin{aligned}
|\alpha g_i(x) - \alpha g_i(x_0)| &\geq \delta |g_i(x) - g_i(x_0)|,\\
|\alpha g_i(x) - \alpha g_i(y_0)| &\geq \delta |g_i(x) - g_i(y_0)|,\\
|\alpha g_i(y_0) - \alpha g_i(x_0)| &\geq \delta |g_i(y_0) - g_i(x_0)|.
\end{aligned}$$

Proof. Since $b \neq a$, $x_0 \neq a$ and $y_0 \neq a$, we have $g_i(b) \neq \infty$, $g_i(x_0) \neq \infty$ and $g_i(y_0) \neq \infty$ for large i. It follows that the euclidean lines (cf. Section 2) $L\bigl(g_i(b), g_i(x_0)\bigr)$, $L\bigl(g_i(b), g_i(y_0)\bigr)$ and $L\bigl(g_i(x_0), g_i(y_0)\bigr)$ are well defined for large i. Let U and $\{\Pi_i\}$ be as in Lemma 3 and $C(b, x_0, i) = g_i^{-1}\Bigl(L\bigl(g_i(b), g_i(x_0)\bigr)\Bigr)$ be the unique circle or line containing $g_i^{-1}(\infty)$, b and x_0. Since Π_i do not meet U for large i, we obtain that the angles of $C(b, x_0, i)$ and Π_i at b are bounded away from 0. It follows that $\operatorname{ang}\bigl(g_i(b), g_i(x_0), g_i(\Pi_i)\bigr)$ are bounded away from 0 for large i. In the same way we see that $\operatorname{ang}\bigl(g_i(b), g_i(y_0), g_i(\Pi_i)\bigr)$ and $\operatorname{ang}\bigl(g_i(x_0), g_i(y_0), g_i(\Pi_i)\bigr)$ are bounded away from 0 for large i. This is also true if we replace b by nearby points. The conclusion follows immediately. □

Denote by $S_{\alpha,m,\delta}$ the set of all triples $(x,y,z) \in C_{m,1}$ which satisfy

$$\begin{aligned} &|\alpha(x)-\alpha(y)| \ge \delta|x-y|, \\ &|\alpha(x)-\alpha(z)| \ge \delta|x-z|, \qquad \text{and} \\ &|\alpha(z)-\alpha(y)| \ge \delta|z-y|. \end{aligned}$$

Consider the map

$$\begin{aligned} &P\alpha\colon S_{\alpha,m,\delta} \to (H^{n+1}, \varrho), \\ &P\alpha(x,y,z) = P\big(\alpha(x), \alpha(y), \alpha(z)\big). \end{aligned}$$

It is easy to see $S_{\alpha,m,\delta}$ is compact in the usual product topology. Consequently, $P\alpha(S_{\alpha,m,\delta})$ is also compact as the continuous image of a compact set. This implies the following lemma.

Lemma 5. *The image of $S_{\alpha,m,\delta}$ by $P\alpha$ is bounded in the hyperbolic metric.*

Combining Lemmas 4 and 5, and Tukia's methods, we can prove the following result.

Lemma 6. *Let $\{g_i\}$, W, x_0 and y_0 be as in Lemma 4. Then for each $x \in W \cap A$,*

$$f(x) = \lim_{i\to\infty} g_i' \alpha g_i(x),$$

where $g_i' = \varphi(g_i)^{-1}$.

Proof. Denote by $y_1 = x_0$, $y_2 = y_0$ and $y_3 = x \in W \cap A$. Let $z = P(y_1, y_2, y_3)$. Since $g_i^{-1}(\infty) \to a$ as $i \to \infty$, we can find $\varepsilon_0 > 0$ such that

$$\operatorname{ang}\big(g_i(z), \infty, g_i(y_j)\big) = \operatorname{ang}\big(z, g_i^{-1}(\infty), y_j\big) \ge \varepsilon_0 \tag{2}$$

for all j if i exceeds a certain i_0. Also since $z_i = g_i(z)$ approach radially 0, there exists $m \ge 0$ such that all z_i are in C_m. Choose $\lambda_i > 0$ such that $\big(\lambda_i g_i(y_1), \lambda_i g_i(y_2), \lambda_i g_i(y_3)\big)$ are in the compact set $C_{m,1}$ defined in Section 2. By Lemma 4, the triples $\big(\lambda_i g_i(y_1), \lambda_i g_i(y_2), \lambda_i g_i(y_3)\big)$ are in the set $S_{\alpha,m,\delta}$ which is also compact. By Lemma 5, $P\alpha\big(\lambda_i g_i(y_1), \lambda_i g_i(y_2), \lambda_i g_i(y_3)\big)$ are contained in a set of H^{n+1} which is bounded in the hyperbolic metric. Set

$$z_i'' = P\alpha\big(g_i(y_1), g_i(y_2), g_i(y_3)\big).$$

Then the hyperbolic distances $\varrho(z_i, z_i'') = \varrho(\lambda_i z_i, \lambda_i z_i'')$ are bounded. It follows from (2) and 4° in [5, p. 568] that

$$\operatorname{ang}\big(z_i'', f g_i(y_j), \alpha g_i(y_j)\big) \to 0, \qquad i \to \infty,\ j = 1,2,3.$$

This together with 6° in [5, p. 568] implies that $z_i' = P\big(fg_i(y_1), fg_i(y_2), fg_i(y_3)\big)$ is defined for large i and $\varrho(z_i', z_i'') \to 0$ as $i \to \infty$. It follows that $f(y_1)$, $f(y_2)$ and $f(y_3)$ are distinct and $\operatorname{ang}\big(z_i', fg_i(y_j), \alpha g_i(y_j)\big) \to 0$ as $i \to \infty$ for $j = 1, 2, 3$. Let

$$z' = g_i'(z_i') = P\big(g_i' fg_i(y_1), g_i' fg_i(y_2), g_i' fg_i(y_3)\big) = P\big(f(y_1), f(y_2), f(y_3)\big).$$

Since

$$\operatorname{ang}\big(z', f(y_3), g_i'\alpha g_i(y_3)\big) = \operatorname{ang}\big(z_i', fg_i(y_3), \alpha g_i(y_3)\big),$$

we deduce that

$$\operatorname{ang}\big(z', f(y_3), g_i'\alpha g_i(y_3)\big) \to 0,$$

implying that

$$f(y_3) = \lim_{i\to\infty} g_i'\alpha g_i(y_3). \ \square$$

Lemma 7. *There are Möbius transformations h and h' such that $h'fh$ is an affine map on $h^{-1}(A) \cap R^n$.*

Proof. Choose $b \in A\backslash\{a\}$ and a neighborhood W of b in $\overline{\mathrm{R}}^n$ as in Lemma 4. Pass to the subsequence of $\{g_i\}$ such that Lemma 6 is true. By Lemma 6, we have

$$f(x) = \lim_{i\to\infty} g_i'\alpha g_i(x), \qquad x \in W \cap A. \tag{3}$$

We can assume that the affine space $V(A)$ generated by A is generated already by $W \cap A$. If this is not the case, we pick points $b_1, \dots, b_m \in A \setminus \{a\}$ such that $\{b_p\}$ contain b and generate $V(A)$. We choose for each p a subsequence so that (3) is true for $x \in W_i \cap A$. We can choose the subsequence so that the subsequence for $p+1$ is a sub-subsequence of the subsequence for p. Thus the subsequence for p satisfies (3) for all $x \in W = W_1 \cup \dots \cup W_m$. So we can assume that W contains points $b_1, \dots, b_m$ so that the affine subspace generated by $\{b_p\}$ is $V(A)$.

Assume that $g_i'(\infty) \to c$ as $i \to \infty$. Let h_i, h_i', h and h' be Möbius transformations such that $h_i' g_i'(\infty) = \infty$, $g_i h_i(\infty) = \infty$, and $h_i' \to h'$ and $h_i \to h$ as $i \to \infty$ on $\overline{\mathrm{R}}^n$ in the chordal metric. Thus $h_i(\infty) = g_i^{-1}(\infty) \to a$ as $i \to \infty$. Hence $h_i^{-1}(b_p) \to h^{-1}(b_p) \in R^n$ as $i \to \infty$.

Now, for each i, $\alpha_i = h_i' g_i' \alpha g_i h_i$ is an affine map and since $\{h_i^{-1}(b_p)\}$ generate the affine space $h_i^{-1}\big(V(A)\big)$, it is the well-defined affine map of $h_i^{-1}(V(A))$ sending $c_{pi} = h_i^{-1}(b_p)$ to $d_{pi} = h_i' g_i' \alpha g_i(b_p)$. Since $c_{pi} \to h^{-1}(b_p)$ and $d_{pi} \to h'f(b_p)$, it follows that α_i have a limit on $h^{-1}\big(V(A)\big)$, which is the well-defined affine map such that $h^{-1}(b_p) \mapsto h'f(b_p)$. Obviously, we can assume that α_i have the limit β on the whole R^n. Thus

$$f(x) = h'^{-1}\beta h^{-1}(x), \qquad x \in W \cap A. \tag{4}$$

Actually, (4) is true for all $x \in A \setminus \{a\}$. To see this, pick a point $b' \in A \setminus \{a\}$. Choose a neighborhood W' of b' and a subsequence of g_i for which (3) is true with W replaced by W'. We can pick the subsequence from the sequence for which (3) was true (with W as above). Since limits do not change when passing to a subsequence, we see that (4) is true for $x \in W' \cap A$. We conclude that (4) is true for all $x \in A \setminus \{a\}$. □

Proof of Theorem 1. If $k = D(A)$, the dimension of the affine subspace generated by A, then α can be extended to an affine homeomorphism on R^n and the claim follows from Theorem D in [5] immediately.

Assume then that $0 < k < D(A)$. It follows from Lemma 7 that there are Möbius transformations h and h' such that $h'fh$ is an affine map on $h^{-1}(A) \cap R^n$. By (D14) in [5] we see that $a \in \mathrm{cl}\bigl(A \setminus \{a\}\bigr)$. If $a = h(\infty) \in A$ and f is not continuous at a, then a must be fixed by all g since f is compatible with G and is continuous on $A \setminus \{a\}$.

Now suppose that $A = \overline{\mathrm{R}}^n$. Let h and h' be as above and suppose that $h(\infty)$ is not fixed by every $g \in G$. Thus f is continuous at $h(\infty)$ and if we replace f by $h'fh$, we can assume that f is an affine map, extended to ∞ so that $\infty \mapsto \infty$.

By Lemma B2 in [5], there are loxodromic elements in G. Let g be such a loxodromic element. Let u be its attractive fixed point. We can assume that $u \in R^n$, otherwise replace g by g^{-1}. By Lemma 2, we see that $g' = \varphi(g)$ is also loxodromic. Now let u' and v' be the fixed points of g'. Suppose that $f^{-1}\{u'\}$ contains more than one point. It is easy to check that $f^{-1}\{u'\}$ is closed and invariant under both g and g^{-1}. It follows that $f^{-1}\{u'\}$ contains all fixed points of g. On the other hand, since the set $f^{-1}\{v'\}$ is also closed and invariant under g, it must contain a fixed point of g. This leads to a contradiction. Thus $f^{-1}\{u'\}$ contains exactly one point and we conclude that f is injective. An immediate consequence is that $k = n$ and our theorem follows from [5]. The case that A is a k'-sphere follows similarly. □

Obviously, Theorem 1 implies Corollary 1.

Proof of Theorem 2. We can assume by Theorem 1 that f is the restriction of an affine map (which is extended to ∞ if necessary). Thus we can assume that A is closed and hence $L(G) \subset A$. We see as in the proof of Theorem 1 that the pre-images of fixed points of loxodromic $g \in G$ elements are mapped onto the fixed points of $\varphi(g)$ which is loxodromic and the pre-image of a fixed point of $\varphi(g)$ is a point. It follows that if $a \in A$ and $g \in G$ are loxodromic, then f is a similarity on the affine subspace generated by a and the fixed points of g. After this fact, the proof of Theorem 2 follows as the proof of (c) of Theorem D of [5]; we only use Lemma 2 instead of Lemma C1 of [5]. □

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