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ASYMPTOTIC MAXIMUM PRINCIPLE

Boris Korenblum

University at Albany, Department of Mathematics and Statistics
Albany, NY 12222, U.S.A.

Dedicated to W. K. Hayman on his 75th birthday

Abstract. By definition the asymptotic maximum principle (AMP) is valid for a class $\mathscr{K}$ of analytic functions in the unit disk $\mathbf{D}$ if, whenever $f \in \mathscr{K}$

$$\sup_{0\le\theta<2\pi} \limsup_{r\to 1} |f(re^{i\theta})| = \sup_{z\in\mathbf{D}} |f(z)|.$$

Let $\mathscr{K}_t$ $(t \ge 0)$ be the class of all f for which

$$\limsup_{|z|\to 1} \{(1-|z|)^2 \log |f(z)|\} \le t.$$

Then AMP is valid for $\mathscr{K}_0$ and not valid for $\mathscr{K}_t$ with $t > 0$.

1. Historical background and the main result

It is well known (Lusin–Privalov theorem) [3] that there are non-zero analytic functions in $\mathbf{D}$ that have zero radial limits on a subset E of $\partial\mathbf{D}$ having full Lebesgue measure. (However, this cannot happen if $|E| > 0$ and at the same time E is of second Baire category.) More specifically, no matter how slowly $k(r)$ tends to $+\infty$ as $r \to 1$, there always is a non-zero analytic f with $\log|f(z)| \le k(|z|)$ and $\lim_{r\to 1} f(re^{i\theta}) = 0$ almost everywhere on $[0, 2\pi)$ (see [2] for this and other similar results). Recently [1] a connection was found between such radial null sets E and the notion of k-entropy of E.

The radial null sets represent only one kind of asymptotic behavior, namely convergence to 0. If the radial null sets are replaced by sets of "asymptotic boundedness", then we are led to the following two general problems whose purpose is to expand the scope of the classical maximum principle.

Problem 1. *Given a class $\mathscr{K}$, characterize those subsets E of $\partial\mathbf{D}$ such that $f \in \mathscr{K}$ together with*

$$\sup_{\zeta\in E} \limsup_{r\to 1} |f(r\zeta)| < \infty$$

imply $f \in H^\infty$.

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Problem 2 (Dual to the above). *Given a subset E of $\partial\mathbf{D}$, identify the class $\mathscr{K}$, in terms of maximal allowed radial growth, such that the above inequality together with $f \in \mathscr{K}$ imply $f \in H^\infty$.*

Definition. We say that the asymptotic maximum principle (AMP) is valid for the class $\mathscr{K}$ of analytic functions in the unit disk $\mathbf{D}$ if, whenever $f \in \mathscr{K}$,

$$\sup_{0\le\theta<2\pi} \limsup_{r\to 1} |f(re^{i\theta})| = \sup_{z\in\mathbf{D}} |f(z)|.$$

We give a solution to Problem 2 for the case $E = \partial\mathbf{D}$, i.e. for AMP. More specifically, we identify maximal radial growth of functions that implies validity of the AMP.

Let $\mathscr{K}_t$, $t \ge 0$, be the class of all f for which

$$\limsup_{|z|\to 1} \big\{(1-|z|)^2 \log|f(z)|\big\} \le t.$$

(Note that for $t = 0$ and a non-zero f the above inequality becomes an equality.)

We are now in a position to state our main result.

Theorem. *AMP is valid for $\mathscr{K}_0$ and not valid for $\mathscr{K}_t$ with $t > 0$.*

Remark. A similar result holds for general subharmonic functions instead of $\log|f(z)|$. We emphasize here the analytic case for the sake of comparison with Lusin–Privalov's theorem and other results.

2. Use of a Baire-category argument

We proceed now to proving the above theorem. If $t > 0$ then the AMP fails for $\mathscr{K}_t$ as the example

$$f(z) = \exp\left\{ci\left(\frac{1+z}{1-z}\right)^2\right\}$$

shows, where c is a suitable positive constant depending on t. We therefore assume $t = 0$, which means for $f \in \mathscr{K}_0$ that

$$\limsup_{|z|\to 1} \big\{(1-|z|)^2 \log|f(z)|\big\} = 0,$$

and proceed to prove that

$$C = \sup_{\zeta\in\partial\mathbf{D}} \limsup_{r\to 1} |f(r\zeta)| = \sup_{z\in\mathbf{D}} |f(z)|.$$

If the first supremum above is infinite then there is nothing to prove. We therefore suppose that $C < \infty$.

If now $A > C$, we define the following sets:

$$E_r = E_{r,f} = \big\{\zeta \in \partial\mathbf{D} : |f(\varrho\zeta)| \le A,\ r \le \varrho < 1\big\}, \qquad \tfrac{1}{2} \le r < 1.$$

It is easily seen that $\{E_r : \frac{1}{2} \le r < 1\}$ is a nested family of closed sets whose union equals $\partial\mathbf{D}$. Let G_r denote the interior of E_r and write $G = \bigcup_{r \ge 1/2} G_r$; $\zeta \in G$ means that there is some $\delta = \delta(\zeta)$ such that $|z - \zeta| < \delta$, $z \in \mathbf{D}$ imply $|f(z)| \le A$.

We now claim that the open set G is dense in $\partial\mathbf{D}$, and therefore the closed set $F = \partial\mathbf{D} \setminus G$ is nowhere dense. In fact, picking an arbitrary monotone sequence $\frac{1}{2} \le r_1 < r_2 < \cdots \to 1$ we see that every ζ in G ultimately is in G_{r_n} and thus not in $E_{r_n} \setminus G_{r_n}$; on the other hand, every ζ in $\partial\mathbf{D}$ is either in some G_{r_n} or in some $E_{r_n} \setminus G_{r_n}$. Therefore $G \supset \partial\mathbf{D} \setminus \bigcup_n (E_{r_n} \setminus G_{r_n})$. By the Baire category theorem, the set $\bigcup_n (E_{r_n} \setminus G_{r_n})$ being a countable union of nowhere dense subsets of $\partial\mathbf{D}$ must have a dense complement, which proves our claim.

By the Heine–Borel theorem, in order to prove our theorem it is enough to show that $F = \emptyset$. We proceed now to show that the assumption $F \ne \emptyset$ leads to a contradiction.

Recall that F consists of those points $\zeta \in \partial\mathbf{D}$ such that for every $\delta > 0$ there is some z in $\mathbf{D}$ with $|z - \zeta| < \delta$ and $|f(z)| > A$.

Pick now some $\tilde{A}$, $C < \tilde{A} < A$, and construct the sets $\widetilde{E}_r$, $\frac{1}{2} \le r < 1$, that relate to F essentially as E_r relate to $\partial\mathbf{D}$:

$$\widetilde{E}_r : \big\{\zeta \in F : |f(\varrho\zeta)| \le \tilde{A},\ r \le \varrho < 1\big\}.$$

As before, $\big\{\widetilde{E}_r : \frac{1}{2} \le r < 1\big\}$ is a nested family of closed sets whose union equals F. Since F, as well as all its relatively open subsets, are of second Baire category in themselves, we can use again the Baire category argument to find that there is an open arc $I \subset \partial\mathbf{D}$ and some r_0, $\frac{1}{2} \le r_0 < 1$, such that $\emptyset \ne I \cap F \subset \widetilde{E}_{r_0}$. Let

$$K = (\bar{I} \setminus I) \cup (I \cap F).$$

This set is a closed and nowhere dense subset of $\bar{I}$ such that

$$|f(\varrho\zeta)| \le \tilde{A} \qquad \text{for all } \zeta \in K \text{ and } r_1 \le \varrho < 1,$$

where r_1 is some number between r_0 and 1. It then follows that there is a constant $B > \tilde{A}$ such that

$$|f(\varrho\zeta)| \le B \qquad \text{for all } \zeta \in K,\ 0 \le \varrho < 1.$$

The critical step now is to extend this estimate to the whole sector $S = \{z \in \mathbf{D} : z/|z| \in I\}$. Let $\bar{I} \setminus K = \bigcup_n J_n$ where J_n are disjoint open arcs with end points $e^{i\alpha_n}$ and $e^{i\beta_n}$. Consider the open sector S_n abutting on J_n. We need to prove that for all n, $|f(z)| \le B$, $z \in S_n$. Recall that $J_n \subset G$, which implies that for every closed subarc $\overline{\gamma} \subset J_n$ there is some number $a = a(\overline{\gamma}) < 1$ so that $|f(\varrho\zeta)| \le A$ if $\zeta \in \gamma$ and $a \le \varrho < 1$. Construct an increasing sequence of closed arcs $\bar{J}_{n1} \subset \bar{J}_{n2} \subset \bar{J}_{n3} \subset \cdots$ having a common midpoint with J_n and such that $\bigcup_k \bar{J}_{nk} = J_n$. For the corresponding numerical sequence $a_{nk} = a(\bar{J}_{nk})$, $k = 1, 2, \dots$, we assume that $a_{n1} < a_{n2} < \cdots \to 1$. This construction yields a piecewise constant function $\psi_n(\theta)$ equal to a_{nk} if $e^{i\theta} \in \bar{J}_{nk} \setminus \bar{J}_{n,k-1}$. By the construction we have $|f(\varrho e^{i\theta})| \le A$ if $\psi_n(\theta) \le \varrho < 1$, $\alpha_n < \theta < \beta_n$. We would like to replace ψ by a continuous function $\varphi_n(\theta) \ge \psi_n(\theta)$ having similar properties; this can be achieved by "cutting the corners" of the graph of ψ_n, i.e., linearly interpolating between discontinuity points of ψ_n. We thus obtain a continuous curve $\varrho = \varphi_n(\theta)$, $\alpha_n \le \theta \le \beta_n$, joining the end points $e^{i\alpha_n}$ and $e^{i\beta_n}$ of $\bar{J}_n$ and satisfying

$$|f(\varrho e^{i\theta})| \le A \qquad \text{if } \varphi_n(\theta) \le \varrho < 1,\ \alpha_n < \theta < \beta_n.$$

The required estimate

$$|f(z)| \le B, \qquad z \in S_n,$$

is then a consequence of the following lemma that is in fact a sharper form of a classical Phragmén–Lindelöf theorem (see [4], v.2, p. 214).

3. A Phragmén–Lindelöf type lemma

Lemma. *Let S be an open sector bounded by an arc $J = \{z : |z| = 1,\ -\alpha \le \operatorname{Arg} z \le \alpha\}$ and two radii $R^{\pm} = \{z : 0 \le |z| < 1,\ \operatorname{Arg} z = \pm\alpha\}$. Let Γ be the subdomain of S bounded by J and a curve γ whose polar equation is $\varrho = \varphi(\theta)$ with φ continuous on $[-\alpha, \alpha]$, $0 < \varphi(\theta) < 1$, $-\alpha < \theta < \alpha$, $\varphi(\pm\alpha) = 1$.*

If f is analytic in S, continuous on $R^+ \cup R^- \cup S$, and satisfies

$$(*) \qquad \limsup_{|z|\to 1,\, z\in S} \{(1 - |z|)^2 \log |f(z)|\} = 0,$$

then

$$\sup\bigl\{|f(z)| : z \in R^+ \cup R^- \cup \Gamma\bigr\} = \sup\bigl\{|f(z)| : z \in S\bigr\}.$$

Proof. It is convenient to change the setting from the sector S to that of the half-strip

$$\Omega = \{x + iy : -1 < x < 1,\ y > 0\},$$

which is done by an elementary conformal mapping. We have

$$\partial\Omega = [-1, 1] \cup l^- \cup l^+ \cup \{\infty\},$$

where $l^{\pm} = \{\pm 1 + iy : y > 0\}$. We are going to prove a somewhat stronger version of the lemma (cf. Remark at the end of Section 1), in which $\log|f(z)|$ is replaced by an arbitrary subharmonic v: $\Omega \to [-\infty, \infty)$ which extends to be continuous on $l^- \cup l^+ \cup \Omega \cup \{\infty\}$. (Note that continuity here is equivalent to the usual continuity of e^v.) Our assumptions are:

(i) there is a region Γ between $[-1, 1]$ and a continuous curve $\gamma : y = \varphi(x)$, with $\varphi(-1) = \varphi(1) = 0$, $\varphi(x) > 0$, $-1 < x < 1$, such that

$$\sup\bigl\{v(z) : z \in l^- \cup l^+ \cup \Gamma\bigr\} = M < \infty;$$

(ii)

$$(**) \qquad \limsup_{y \to 0+,\ -1<x<1} \{y^2 v(x + iy)\} = 0.$$

Our aim is to show that

$$\sup\{v(z) : z \in \Omega\} = M.$$

First we modify v by subtracting a harmonic function that is positive in Ω. Let

$$\Psi(x, y) = -1/2 \operatorname{Im} z^{-2} = \frac{xy}{(x^2 + y^2)^2}.$$

Clearly, Ψ is harmonic and positive for $x > 0$, $y > 0$, $\Psi = 0$ if $x = 0$ or $y = 0$, and $\Psi(x, mx) = m/(1 + m^2)^2 x^2$.

Consider now

$$v_a(z) = v(z) - a\bigl(\Psi(1 + x, y) + \Psi(1 - x, y)\bigr), \qquad a > 0.$$

Let $\Omega_1 = \{z = x + iy \in \Omega : y > \min(1 - x, 1 + x)\}$. By $(**)$ $v_a(z)$ is bounded above on $\partial\Omega_1$. By applying the classical Phragmén–Lindelöf theorem referred to earlier, we obtain that $v_a(z)$ is bounded above in Ω_1; it is also continuous on $\overline{\Omega}_1$ except, perhaps, at ± 1. To make it continuous on $\overline{\Omega}_1$ we adjust it as follows with $\varepsilon_1 > 0$:

$$v_{a,\varepsilon_1}(z) = v_a(z) + \log|(z + 1)/(z + 1 + \varepsilon_1)| + \log|(1 - z)/(1 + \varepsilon_1 - z)|.$$

We then have $v_{a,\varepsilon_1}(z) < v_a(z) < v(z)$, $z \in \Omega_1$; also, v_{a,ε_1} is continuous on $\overline{\Omega}_1$, with $v_{a,\varepsilon_1}(\pm 1) = -\infty$.

The next step requires the curve $\gamma : y = \varphi(x)$, $-1 \le x \le 1$, to satisfy the following additional condition: φ is differentiable, with $|\varphi'(x)| \le \frac{1}{2}$; this does not result in any loss of generality, as $\varphi(x)$ can be replaced by any smaller positive function. With this assumption made, we see that $\Gamma \cap \Omega_1 = \emptyset$; also, every straight line with slope m, $|m| > 1$, intersects the boundary of the region Ω_2 lying between Ω_1 and Γ at most at two points.

Consider now:

$$v_{a,\varepsilon_1,\varepsilon_2}(z) = v_{a,\varepsilon_1}(z) - a\big(\Psi(x+1-\varepsilon_2, y) + \Psi(1-\varepsilon_2-x, y)\big), \quad 0 < \varepsilon_2 < 1.$$

This function is subharmonic in the region $\Omega^{(\varepsilon_2)} \subset \Omega$ lying between the two lines $l^{\pm}_{\varepsilon_2} = \{z = x+iy : x = \pm(1-\varepsilon_2)\}$; also,

$$v_{a,\varepsilon_1,\varepsilon_2}(z) < v_{a,\varepsilon_1}(z) < v_a(z) < v(z), \qquad z \in \Omega^{(\varepsilon_2)}.$$

We apply now the classical maximum principle to $v_{a,\varepsilon_1,\varepsilon_2}(z)$ in the region $\Omega_3^{(\varepsilon_2)} \subset \Omega^{(\varepsilon_2)} \setminus \Gamma$ whose boundary is made up of the following parts:

B_1: the segment of γ between the lines $\lambda^{+}_{\varepsilon_2} : y = -2(x-1+\varepsilon_2)$ and $\lambda^{-}_{\varepsilon_2} : y = 2(x+1-\varepsilon_2)$;

B_2: the two segments of the above lines lying in Ω_2;

B_3: the two segments of $\partial\Omega_1$ lying between $l^{-}_{\varepsilon_2}$ and $\lambda^{-}_{\varepsilon_2}$, and between $\lambda^{+}_{\varepsilon_2}$ and $l^{+}_{\varepsilon_2}$, respectively;

B_4: parts of $l^{\pm}_{\varepsilon_2}$ lying in Ω_1.

We thus obtain for $z_0 \in \Omega_3^{(\varepsilon_2)}$ that $v_{a,\varepsilon_1,\varepsilon_2}(z_0)$ is less than or equal to

$$\max\big(\max\{v(z) : z \in B_1\}, \max\{v_{a,\varepsilon_1,\varepsilon_2}(z) : z \in B_2 \cup B_3\}, \max\{v_{a,\varepsilon_1}(z) : z \in B_4\}\big).$$

Now, we let $\varepsilon_2 \to 0$, holding ε_1 and a fixed. The second maximum in parentheses then tends to 0 due to $(**)$, and the third maximum tends to $\max\{v_{a,\varepsilon_1}(z) : z \in l^+ \cup l^-\}$ due to continuity of v_{a,ε_1} on $\overline{\Omega}_1$; at the same time $v_{a,\varepsilon_1,\varepsilon_2}(z_0) \to v_{2a,\varepsilon_1}(z_0)$ and $\Omega_3^{(\varepsilon_2)}$ eventually includes all $z_0 \in \Omega_3 = \Omega \setminus \Gamma$. Thus for all $z_0 \in \Omega \setminus \Gamma$

$$v_{2a,\varepsilon_1}(z_0) \le \max\big(\sup\{v(z) : z \in \gamma\}, \sup\{v(z) : z \in l^+ \cup l^-\}\big).$$

The right-hand side depends neither on a nor on ε_1. Letting $a \to 0$, $\varepsilon_1 \to 0$ we get

$$\sup\{v(z_0) : z_0 \in \Omega \setminus \Gamma\} \le \max\big(\sup\{v(z) : z \in \gamma\}, \sup\{v(z) : z \in l^+ \cup l^-\}\big),$$

which obviously implies

$$\sup\{v(z) : z \in \Omega\} = \sup\{v(z) : z \in l^- \cup l^+ \cup \Gamma\}. \ \square$$

4. Completion of the proof of the theorem

Applying the above lemma to each S_n we obtain

$$\sup\{|f(z)| : z \in S_n\} \le B.$$

Since the closure of $\bigcup_n S_n$ is the whole sector $\bar{S}$ we deduce that $|f(z)|$ is bounded by B in S. We also have $\limsup_{\varrho \to 1} |f(\varrho\zeta)| \le C < \tilde{A}$ on I. On the two boundary radii R_1 and R_2 $|f(\varrho\zeta)| \le \tilde{A} < A$, $r_1 \le \varrho < 1$, $\zeta \in \bar{I} \setminus I$. This implies that $\log|f(z)|$ is dominated in S by a harmonic function $U(z)$ whose boundary values on R_1 and R_2 are $\log B$, $0 \le |z| < r_1$, $\log \tilde{A}$, $r_1 \le |z| < 1$, and on I, $U(\zeta) = \log C$. To make $U(z)$ continuous we increase its piecewise constant boundary values by linearly interpolating them between the discontinuity points. The resulting harmonic function $U_1(z) \ge U(z)$ is continuous in the closed sector $\bar{S}$, and on I $U_1(\zeta) = \log \tilde{A}$. This implies that there is some r_2, $r_1 < r_2 < 1$, such that

$$|f(z)| \le A, \qquad z \in S,\ r_2 \le |z| < 1,$$

which means that $I \cap F = \emptyset$, contrary to the assumption. This contradiction proves our theorem.

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