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MAXIMAL INEQUALITY IN (s,m)-UNIFORM DOMAINS

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Abstract. We define a class of bounded domains $\Omega \subset \mathbf{R}^n$ which we call (s,m)-uniform, $s \geq 1$ and $0 < m \leq 1$. In this class we show that every Sobolev function $u \in W^{1,p}(\Omega)$, $1 \leq p \leq \infty$, satisfies

$$|u(x) - u(y)| \leq C|x-y|^{\alpha}(\mathscr{M}\nabla u(x) + \mathscr{M}\nabla u(y))$$

for almost every $x, y \in \Omega$ with

$$\alpha = \frac{m}{s}(n - s(n-1)).$$

Our result extends the previous result for Sobolev extension domains by P. Hajłasz. Classical bounded uniform domains or equivalently bounded (ε,∞) domains form a proper subclass of the (s,m)-uniform domains, when $s > 1$ or $0 < m < 1$, but our class of domains allows more irregular behavior for the boundary than in the classical case.

1. Introduction

P. Hajłasz showed that if $\Omega \subset \mathbf{R}^n$ is a Sobolev extension domain or $\Omega = \mathbf{R}^n$, then every $u \in W^{1,p}(\Omega)$, $1 \leq p \leq \infty$, satisfies

$$|u(x) - u(y)| \leq C|x-y|^{\alpha}\big(\mathscr{M}\nabla u(x) + \mathscr{M}\nabla u(y)\big) \tag{1.1}$$

for almost every $x, y \in \Omega$ with $\alpha = 1$, [H2]. Here $\mathscr{M}\nabla u$ is the Hardy–Littlewood maximal operator of a weak gradient of a function u. Hajłasz and O. Martio proved that under a weak geometric condition the inequality (1.1) with $\alpha = 1$ implies that the domain Ω is a Sobolev extension domain for $1 < p \leq \infty$, [HM]. A variant of the inequality (1.1) in the domain whose boundary is locally a graph of a Lipschitz continuous function, and also the case $\Omega = \mathbf{R}^n$, has been studied in [DS], [H1] and [HM].

We define a new class of bounded domains which we call (s,m)-uniform, $s \geq 1$ and $0 < m \leq 1$. The special case $s = m = 1$ is the class of bounded uniform domains defined by Martio and J. Sarvas, [MS] or equivalently the class of bounded (ε,∞) domains defined by P.W. Jones, [J]. An example of $(s,1)$-uniform domains in the plane is an s-cusp, $\{(x,y) \in \mathbf{R}^2 : 0 < x < 1,\ 0 < y < x^s\}$, with $s \geq 1$. The class of (s,m)-uniform domains is a proper subclass of the class of s-John

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domains. We prove that if Ω is bounded and its boundary is locally a graph of a λ-Hölder continuous function, $0 < \lambda \le 1$, then Ω is $(1/\lambda, \lambda)$-uniform. In the case $\lambda = 1$ this result seems to be well known, although we have not been able to find a reference. The converse does not hold. There exists a bounded domain which is even a $(1,1)$-uniform domain, but whose boundary fails to be a graph of a continuous function.

Our main theorem shows that if $\Omega \subset \mathbf{R}^n$ is a bounded (s,m)-uniform domain, $1 \le s < n/(n-1)$ and $0 < m \le 1$, then every $u \in W^{1,p}(\Omega)$, $1 \le p \le \infty$, satisfies the inequality (1.1) for almost every $x, y \in \Omega$ with $\alpha = m\bigl(n - s(n-1)\bigr)/s$. Hajłasz and Martio proved the case $s = 1$, [HM, Lemma 14, p. 243]. Our proof is based on their proof. We calculate an upper bound for the exponent α of the inequality (1.1) in the class of (s,m)-uniform domains: if $1 < s < n/(n-1)$ then

$$0 < \alpha \le \frac{s(n-1)+1}{n}\bigl(n - s(n-1)\bigr) < 1$$

and if $s \ge n/(n-1)$ then the inequality does not hold with any $\alpha > 0$ for every $1 < p < \infty$.

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2. Notation

Throughout this paper C will denote a constant which may change even in a single string of an estimate. We write $C(M)$ to denote that the constant C depends on M. We let Ω and D be bounded domains in the Euclidean n-space $\mathbf{R}^n$, $n \ge 2$. We denote the boundary of a domain Ω by $\partial\Omega$. By an open ball centered at x and with a radius $r > 0$ we mean the set $B^n(x,r) = \{y \in \mathbf{R}^n : |y-x| < r\}$. We write kB for the ball with the same center as B and dilated by a factor $k > 0$. We let $\bar{A}$ denote the closure of a set A in $\mathbf{R}^n$. The Lebesgue n-measure of a set $A \subset \mathbf{R}^n$ is denoted by $|A|$.

Following J. Väisälä [V] we say that γ is a curve if it is either a path or an arc. A path is a continuous mapping from a closed interval to $\Omega \subset \mathbf{R}^n$. A set in Ω is an arc if it is homeomorphic to a closed interval. We assume that every curve is rectifiable. A length of a curve γ is denoted by $|\gamma|$. If γ_1 is a curve from a point x to a point z and γ_2 is a curve from a point z to y then by $\gamma_1 \cup \gamma_2$ we denote a curve from x to y via γ_1 and γ_2.

The set of p-integrable functions in D is denoted by $L^p(D)$, $1 \le p \le \infty$. We denote by $W^{1,p}(D)$, $1 \le p \le \infty$, the class of all functions in $L^p(D)$ whose first weak derivatives are in $L^p(D)$. We equip the Sobolev space $W^{1,p}(D)$ with the norm $\|u\|_{W^{1,p}(D)} = \|u\|_{L^p(D)} + \|\nabla u\|_{L^p(D)}$, where ∇u is the weak gradient.

The class of λ-Hölder continuous functions, $0 < \lambda \le 1$, in a domain D is denoted by $C^{0,\lambda}(D)$: $u \in C^{0,\lambda}(D)$ if there exists a constant $C > 0$ such that

$$|u(x) - u(y)| \le C|x-y|^\lambda$$

for every $x, y \in D$. If $\lambda = 1$ we say that the function u is a Lipschitz-continuous function.

For a measurable function defined in a set A, $|A| > 0$, we write

$$\fint_A u(x)\,dx = \frac{1}{|A|}\int_A u(x)\,dx.$$

Let $v \in L^1(D)$ and $x \in D$. We put $v = 0$ in the complement of the domain D. For every $0 < R \le \infty$ we define

$$\mathscr{M}_R v(x) = \sup_{0<r<R} \fint_{B^n(x,r)} |v(z)|\,dz.$$

We let $\mathscr{M} u$ denote $\mathscr{M}_\infty u$. The operator $\mathscr{M}$ is the classical Hardy–Littlewood maximal operator. Recall that for $1 < p \le \infty$ we have $\|\mathscr{M} u\|_{L^p(D)} \le A\|u\|_{L^p(D)}$, where the constant A depends only on the dimension n and p, [St, Theorem 1, p. 6].

3. (s,m)-uniform domains

We define a new class of domains. The definition was suggested to the author by P. Hajłasz.

3.1. Definition. Let $s \ge 1$ and $0 < m \le 1$. A bounded domain $\Omega \subset \mathbf{R}^n$ is an (s,m)-uniform domain if there exists a constant $M \ge 1$ such that each pair x, y of points in Ω can be joined by a rectifiable curve $\gamma\colon [0,l] \to \Omega$ parametrized by arclength, such that $\gamma(0) = x$, $\gamma(l) = y$,

$$l \le M|x-y|^m \tag{3.2}$$

and

$$\min(t, l-t)^s \le M \operatorname{dist}\bigl(\gamma(t), \partial\Omega\bigr). \tag{3.3}$$

The idea of (s,m)-uniform domains is that every two points in Ω can be joined by a twisted double cusp inside the domain Ω. The exponent s describes which kind of outer peaks are allowed and the exponent m which kind of inner peaks. The special case $s = m = 1$ is the class of bounded uniform domains defined by Martio and J. Sarvas, [MS]. The class of bounded uniform domains, and thus the class of $(1,1)$-uniform domains, coincides with the class of bounded (ε, ∞) domains defined by P.W. Jones, [J]. It is easy to see that the class of (s,m)-uniform domains is a proper subset of the class of (s',m')-uniform domains if $s < s'$ and $m' \le m$ or if $s \le s'$ and $m' < m$. The standard examples in the plane are an s-cusp, $\{(x,y) \in \mathbf{R}^2 : 0 < x < 1,\ 0 < y < x^s\}$, with $s \ge 1$ which is

$(s,1)$-uniform, and the interior of its complement with respect to the ball $B^2(0,1)$, which is $(1,1/s)$-uniform.

We say that $\partial\Omega$ is λ-Hölder, $0<\lambda\le 1$, if for every point $x\in\partial\Omega$ there exists $r(x)=(r_1(x),\ldots,r_n(x))$, $r_i(x)>0$ for every i, and a λ-Hölder continuous function $\phi\colon \mathbf{R}^{n-1}\to\mathbf{R}$ such that, upon rotating and relabeling the coordinate axes such that x is at the origin, we have

$$\Omega\cap U\big(x,r(x)\big)=\big\{y\in\mathbf{R}^n:\phi(y_1,\ldots,y_{n-1})>y_n\big\}\cap U\big(x,r(x)\big)$$

and

$$\tfrac12 r_n(x)>\phi>-\tfrac12 r_n(x)$$

where $U\big(x,r(x)\big)=\big\{y\in\mathbf{R}^n:|y_i-x_i|<r_i(x),\ i=1,\ldots,n\big\}$ is an open rectangle. If $\lambda=1$ we say that $\partial\Omega$ is Lipschitz.

In the case $\lambda=1$ the following lemma seems to be well known, although we have not been able to find a reference.

3.4. Lemma. *Let $0<\lambda\le 1$ and let $\Omega\subset\mathbf{R}^n$ be a bounded domain. If $\partial\Omega$ is λ-Hölder then the domain Ω is $(1/\lambda,\lambda)$-uniform.*

The converse does not hold. There exists even a $(1,1)$-uniform domain, whose boundary is not locally a graph of a continuous function at any point. An example is the Koch snowflake domain. In Example 5.2 we construct for every $s\ge 1$ an $(s,1)$-uniform domain whose boundary fails to be a graph of a continuous function.

Proof. Since $\partial\Omega$ is bounded we may choose a finite covering of open rectangles $\big\{U\big(z_i,r(z_i)\big)\big\}_{i=1}^k$. Let ϕ_i be a λ-Hölder continuous function with a constant L_i related to $U\big(z_i,r(z_i)\big)$. We write $L=\max_{1\le i\le k}\{L_i\}$. For technical reasons we assume that $\operatorname{diam}(\Omega)=1$.

First we prove that every pair of points inside each $U\big(z_i,r(z_i)\big)\cap\Omega$ can be joined by a curve satisfying the conditions (3.2) and (3.3). Let $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$ be in $U\big(z_i,r(z_i)\big)\cap\Omega$. We fix a two-coordinate axis in $\mathbf{R}^n$ so that x is the point $(0,x_n)$ and y is the point (l,y_n),

$$l=\sqrt{(x_1-y_1)^2+\ldots+(x_{n-1}-y_{n-1})^2}\,.$$

We may assume that $x_n\ge y_n$. Let I_1 be a curve

$$\big\{(\xi_1,\xi_2):0\le\xi_1\le l,\ \xi_2=-L\xi_1^\lambda+x_n\big\}$$

and I_2 a curve

$$\big\{(\xi_1,\xi_2):0\le\xi_1\le l,\ \xi_2=-L|\xi_1-l|^\lambda+y_n\big\},$$

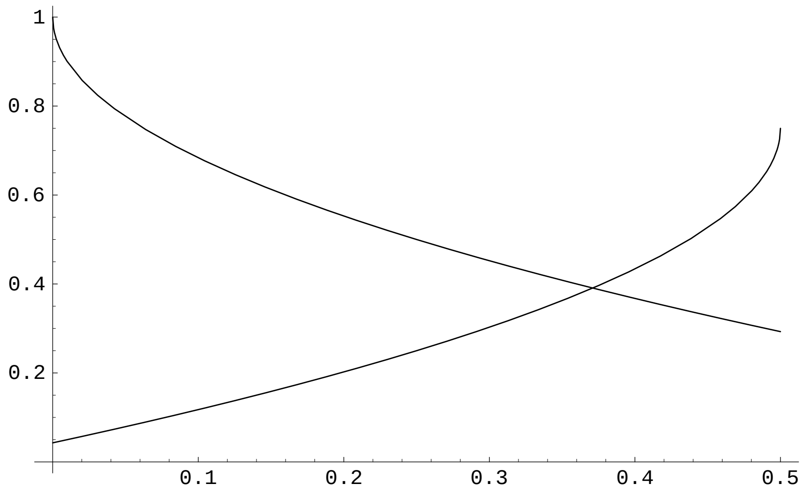

Figure 1. The curves I_1 and I_2.

The curves I_1 and I_2 are presented in Figure 1 with $L=1$, $\lambda=0.5$, $x_n=1$, $y_n=0.75$ and $l=0.5$.

If the curve I_1 intersects the curve I_2, as in Figure 1, we let J be a curve connecting x and y via I_1 and I_2. Let ξ be a point in I_1 with $\operatorname{dist}(I_1,y)=\operatorname{dist}(\xi,y)$. Otherwise we let J be a curve connecting x to y via I_1 and a line segment from ξ to y. It is easy to see that $l(J)\le C|x-y|^{\lambda}$ here C is a constant, depending on s, L and $\operatorname{diam}(\Omega)$, and $l(J)$ is the length of the curve J. Let J^* be a curve from x to y via the curves $J_1^*=\bigl\{(\xi_1,\xi_2):\xi_1=0,\ \xi_2\le x_n\bigr\}$, J' and $J_2^*=\{(\xi_1,\xi_2):\xi_1=|x-y|,\ \xi_2\le y_n\}$. Here J' is defined as follows: if $(\xi_1,\xi_2)\in J$ then $(\xi_1,\xi_2-\frac{1}{10}|x-y|)\in J'$. If necessary we replace a part of J^* by a line segment in the hyperplane

$$\bigl\{(\xi_1,\xi_2)\in U\bigl(z_i,r(z_i)\bigr)\cap\Omega:\xi_2=-\tfrac{3}{4}r_n(z_i)\bigr\}.$$

This yields

$$\operatorname{dist}(\xi,\partial\Omega)\ge C(L)|x_n-\xi|^{1/\lambda}$$

for every $\xi\in J_1^*$,

$$\operatorname{dist}(\xi,\partial\Omega)\ge C(L)|y_n-\xi|^{1/\lambda}$$

for every $\xi\in J_2^*$ and

$$\operatorname{dist}(\xi,\partial\Omega)\ge\min\bigl\{\tfrac{1}{10}|x-y|,\tfrac{1}{4}r_n(z_i)\bigr\}$$

for every $\xi\in J'$. It is easy to see that J^* satisfies the conditions (3.2) and (3.3) with $s=1/\lambda$, $m=\lambda$ and a constant M depending on L, $\operatorname{diam}\bigl(U\bigl(z_i,r(z_i)\bigr)\bigr)$ and $r_n(z_i)$.

Let $\mathscr{W}_0$ be a Whitney composition of Ω, [St, Theorem 1, p. 167]. Let $\mathscr{W}$ be a collection of cubes Q_i from $\mathscr{W}_0$ dilated by a factor $\frac{9}{8}$ with $Q_i\not\subset\bigcup_{i=1}^{k}U\bigl(x_i,r(z_i)\bigr)$.

There exists $\varepsilon > 0$ depending on the collection $\{U(z_i, r(z_i))\}_{i=1}^{k}$ such that for every $w \in \Omega$ we have $B^n(w,\varepsilon) \subset \frac{9}{8}Q_j$ for some $\frac{9}{8}Q_j \in \mathscr{W}$ or $B^n(w,\varepsilon) \subset U(z_i, r(z_i))$ for some $i = 1,\dots,k$. Since every cube is a $(1/\lambda,\lambda)$-uniform domain we see that each pair of points $x,y \in \Omega$ with $|x-y| < \varepsilon$ can be joined by a curve satisfying the conditions (3.2) and (3.3) with the constant M.

To complete the proof we use the same method as in [HK1, Theorems 2.4 and 3.3, pp. 175 and 178].

Let $x,y \in \Omega$ with $|x-y| \geq \varepsilon$. An elementary covering argument shows that there exists a positive integer N, depending on $\operatorname{diam}(\Omega)$, ε and n, such that Ω can be covered by balls B_i, $i = 1,\dots,N$, with radius $\frac{1}{4}\varepsilon$. Now there exists a chain of balls B_i, $i \in \{1,\dots,K\}$ and $K \leq N$, such that $x \in B_1$, $y \in B_K$ and $B_i \cap B_{i+1} \cap \Omega \neq \emptyset$ for each $j = 1,\dots,K-1$. We set $x = z_1$, $y = z_K$ and choose $z_i \in B_i \cap \Omega$. Since $|z_i - z_{i+1}| < \varepsilon$, there exists a curve γ_i joining z_i to z_{i+1} in Ω with $l(\gamma_i) \leq M|z_i - z_{i+1}|^\lambda < M\varepsilon^\lambda$. Thus we obtain

$$l(\gamma) = l\biggl(\bigcup_{i=1}^{K} \gamma_i\biggr) \leq KM\varepsilon^\lambda \leq KM|x-y|^\lambda.$$

We choose points $w_1 = x, w_2, \dots, w_l = y$ on the curve γ satisfying

$$\left(\frac{\varepsilon}{2M}\right)^{1/\lambda} \leq |w_i - w_{i+1}| < \left(\frac{\varepsilon}{M}\right)^{1/\lambda}$$

for $i = 1,2,\dots,l-1$. Let β_i be a curve joining w_i to w_{i+1} as in the definition of (s,m)-uniform domains, hence $l(\beta_i) \leq M|w_i - w_{i+1}|^\lambda < \varepsilon$ and

$$l\biggl(\bigcup_{i=1}^{l-1} \beta_i\biggr) \leq \frac{KM|x-y|^\lambda}{\left(\dfrac{\varepsilon}{2M}\right)^{1/\lambda}}\varepsilon \leq 2^{1/\lambda}KM^{1+1/\lambda}\varepsilon^{1-1/\lambda}|x-y|^\lambda.$$

By the definition of (s,m)-uniform domains every curve β_i has arclength as its parameter. We choose b_i to be the arclength midpoint of β_i. Since $|b_i - b_{i+1}| < \varepsilon$ there exists a curve α_i joining b_i to b_{i+1} as in the definition of (s,m)-uniform domains. We denote by $\beta_i(\xi_1,\xi_2)$ that part of the curve β_i from the point ξ_1 to the point ξ_2. We write

$$\alpha = \beta_1(x,b_1) \cup \alpha_1 \cup \ldots \cup \alpha_{l-2} \cup \beta_{l-1}(b_l,y).$$

This yields

$$l(\alpha) \leq C|x-y|^\lambda,$$

where the constant C depends on M, ε, λ, L, $\operatorname{diam}\bigl(U(z_i, r(z_i))\bigr)$ and $r_n(z_i)$ for each $i = 1,\dots,k$. Since $|\beta_i| \geq \frac{1}{2}\varepsilon$ and since the point b_i is the arclength midpoint of β_i we obtain

$$\operatorname{dist}(b_i,\partial\Omega) \geq \frac{1}{M}\left(\tfrac{1}{4}\varepsilon\right)^{1/\lambda}.$$

Hence it is easy to see that the curve α satisfies the conditions (3.2) and (3.3). This completes the proof of Lemma 3.4. □

Let $s \geq 1$. A domain $\Omega \subset \mathbf{R}^n$ is an s-John domain if there exists a distinguished point $x_0 \in \Omega$ and a constant $C \geq 1$ such that each point $x \in \Omega$ can be joined to x_0 by a rectifiable curve γ: $[0,l] \to \Omega$ parametrized by arclength, such that $\gamma(0) = x$, $\gamma(l) = x_0$,

$$l \leq C$$

and

$$t^s \leq C \operatorname{dist}\bigl(\gamma(t), \partial\Omega\bigr).$$

The definition implies that every s-John domain is bounded. When $s = 1$ these domains coincide with the class of John domains defined by Martio and Sarvas [MS]. The s-John domains for $s > 1$ are much wider than John domains. If a domain $\Omega \subset \mathbf{R}^n$ is an s-John domain with a distinguished point $x_0 \in \Omega$ then it is an s-John also with any other point $x \in \Omega$. This means that the distinguished point can be changed. Note that the constant C depends on the distance between the distinguished point and the boundary of Ω. For more information about s-John domains we refer to [SS], [HK2] and [KM].

3.5. Lemma. *Let $s \geq 1$ and $0 < m \leq 1$. A bounded (s,m)-uniform domain is an s-John domain.*

The case $s = 1$ of Lemma 3.5 is proved by F.W. Gehring and Martio, [GM, Lemma 2.18, p. 209]. The case $s > 1$ is similar.

4. Main theorem

First we prove a chain condition for (s,m)-uniform domains. This is a modification of the standard chaining argument for uniform domains and John domains, see [HM] and [HK2].

4.1. Lemma. *Let $\Omega \subset \mathbf{R}^n$ be a bounded (s,m)-uniform domain. Let $x,y \in \Omega$. Then there exists a sequence of balls $\{B_i\}_{i=-\infty}^{\infty}$, where $B_i = B^n(x_i, r_i)$, and constants $C, d \geq 1$ with the following properties*:

(1) $|B_i \cup B_{i+1}| \leq C|B_i \cap B_{i+1}|$,

(2) $\operatorname{dist}(x, B_i) \leq d r_i^{1/s}, B_i \subset B^n(x, C|x-y|^{m/s})$ *if $i \leq 0$ and $r_i \to 0$ as $i \to -\infty$*,

(3) $\operatorname{dist}(y, B_i) \leq d r_i^{1/s}$, $B_i \subset B^n(y, C|x-y|^{m/s})$ *if $i \geq 0$ and $r_i \to 0$ as $i \to \infty$*,

(4) *no point of the domain Ω belongs to more than C balls B_i.*

The constants depend only on s, m, the dimension n and the uniform constant M of the domain Ω.

Proof. We may assume that $\operatorname{diam}(\Omega) \leq 1$. Fix $x,y \in \Omega$ and let γ be a curve joining x and y as in the definition of (s,m)-uniform domains, $\gamma(0) = x$ and $\gamma(l) = y$. Fix $x_0 = \gamma(\frac{1}{2}l)$. Let $B_0' = B^n\bigl(x_0, \frac{1}{4}\operatorname{dist}\bigl(x_0, \partial\Omega \cup \{x\}\bigr)\bigr)$. We let γ' be the subcurve of γ from x to x_0. We cover $\gamma' \setminus \{x\}$ with balls as follows. Consider the collection of balls $\overline{B^n}\bigl(\gamma(t), \frac{1}{4}\operatorname{dist}(\gamma(t), \partial\Omega \cup \{x\})\bigr)$, $t \in$

$(0, \frac{1}{2}l)$, and $\overline{B'_0}$. By Besicovitch covering theorem [M, Theorem 2.7, p. 30] we find a sequence of closed balls $\overline{B'_0}, \overline{B'_1}, \overline{B'_2}, \dots$ that cover $\gamma' \setminus \{x\}$ and have uniformly bounded overlap depending only on n.

We define open balls $B_i = 2B'_i$, $i = 0, 1, 2, \dots$. Here $2B'_i$ is the ball with same center as B'_i but twice the radius of the ball B'_i. We write $x_i = \gamma(t_i)$ and $r_i = \frac{1}{2} \operatorname{dist}(x_i, \partial\Omega \cup \{x\})$.

If $r_i = \frac{1}{2}|x_i - x|$ then $\operatorname{dist}(x, B_i) = 2r_i \le 2r_i^{1/s}$. If $r_i = \frac{1}{2}\operatorname{dist}(x_i, \partial\Omega)$ then the definition of an (s,m)-uniform domain yields

$$\operatorname{dist}(x, B_i) \le \operatorname{dist}(x, x_i) \le t_i \le M^{1/s} \operatorname{dist}(x_i, \partial\Omega)^{1/s} \le 2M^{1/s} r_i^{1/s}.$$

We choose $d = \max\{2, 2M^{1/s}\}$. Since $r_i \le t_i$ properties of (s,m)-uniform domains imply

$$\begin{aligned}\operatorname{dist}(x, B_i) + 2r_i &\le d r_i^{1/s} + 2r_i \le (d+2) r_i^{1/s} \\ &\le \tfrac{1}{2}(d+2) t_i^{1/s} \le \tfrac{1}{2} M^{1/s}(d+2)|x-y|^{m/s}.\end{aligned}$$

Hence, we obtain $B_i \subset B^n(x, C|x-y|^{m/s})$ for every i, $i = 0, 1, \dots$, where $C = \frac{1}{2}M^{1/s}(d+2)$.

We renumber the balls. Let B_0 be as above. If we have chosen balls B_i, $i = 0, 1, \dots, m$, then we choose a ball B_{m+1} that is the ball for which $x_j \in B_m$ and $t_j < t_m$. We recall that $\gamma'(t_j) = x_j$ and $\gamma'(t_m) = x_m$. Hence $r_i \to 0$ and $x_i \to x$, as $i \to \infty$.

Next we prove that every point in the domain Ω belongs to a finite number of balls B_i only. The point x does not belong to any ball. Let x' be an arbitrary point in the domain Ω. Let $r = |x' - x|$. The point x' cannot belong to those balls B_i for which $r_i \le \frac{1}{2}|x_i - x| < \frac{1}{2}r$. If $x' \in B_i$ then $\operatorname{dist}(x, B_i) < r$ and furthermore $|x - x_i| \le 2r$. Thus we obtain that if $x' \in B_i$ then $\frac{1}{2}r \le r_i \le r$. The construction of the Besicovitch covering theorem [M, Theorem 2.7, p. 30] implies that balls with radius of $\frac{1}{4}$ of original balls are disjoint. Thus x' belongs to less than or equal to

$$C\frac{|B^n(x', 2r)|}{|B^n(0, \frac{1}{8}r)|} = 16^n C$$

balls B_i. The constant C is from the Besicovitch covering theorem.

Finally we prove the property (1). Assume that $r_i = \frac{1}{2}\operatorname{dist}(x_i, \partial\Omega)$ and $r_{i+1} = \frac{1}{2}\operatorname{dist}(x_{i+1}, \partial\Omega)$. Since $x_{i+1} \in B(x_i, r_i)$ we obtain $\operatorname{dist}(x_{i+1}, \partial\Omega) \ge r_i$. This yields

$$\frac{|B_i|}{|B_{i+1}|} \le \left(\frac{r_i}{\frac{1}{2}r_i}\right)^n = 2^n.$$

If $r_i = \frac{1}{2}|x_i - x|$ and $r_{i+1} = \frac{1}{2}|x_{i+1} - x|$ then

$$\frac{|B_i|}{|B_{i+1}|} \le \left(\frac{r_i}{\frac{1}{2}r_i}\right)^n = 2^n.$$

If $r_i = \frac{1}{2}\operatorname{dist}(x_i, \partial\Omega)$ and $r_{i+1} = \frac{1}{2}|x_{i+1} - x|$ we obtain

$$\frac{|B_i|}{|B_{i+1}|} = \left(\frac{r_i}{r_{i+1}}\right)^n \le \left(\frac{\frac{1}{2}|x_i - x|}{r_{i+1}}\right)^n = 2^n.$$

Similarly if $r_i = \frac{1}{2}|x_i - x|$ and $r_{i+1} = \frac{1}{2}\operatorname{dist}(x_{i+1}, \partial\Omega)$ then

$$\frac{|B_i|}{|B_{i+1}|} \le 2^n.$$

We have proved that $|B_i| \le 2^n|B_{i+1}|$. Similar arguments imply that $|B_i| \ge 3^{-n}|B_{i+1}|$. This yields $|B_i \cup B_{i+1}| \le C|B_i \cap B_{i+1}|$; here the constant C depends only on the dimension n.

Using again the same arguments for the point y imply Lemma 4.1. □

Next we prove our main theorem. In the proof we need only the chain of balls constructed in Lemma 4.1, the Lebesgue differentiation theorem, the Poincaré inequality in a ball and properties of the Riesz potential.

4.2. Theorem. *Let $1 \le s < n/(n-1)$, $0 < m \le 1$ and $1 \le p \le \infty$. If $\Omega \subset \mathbf{R}^n$ is a bounded (s,m)-uniform domain then there exists a constant $C > 0$ such that every $u \in W^{1,p}(\Omega)$ satisfies the inequality*

$$|u(x) - u(y)| \le C|x - y|^\alpha \big(\mathscr{M}\nabla u(x) + \mathscr{M}\nabla u(y)\big), \tag{4.3}$$

for almost every $x, y \in \Omega$ with $\alpha = m\big(n - s(n-1)\big)/s$. Here $\mathscr{M}\nabla u$ is the Hardy–Littlewood maximal operator of the function ∇u. The constant C depends only on n, s, m and the uniform constant of Ω.

Hajłasz and Martio proved that if $\Omega \subset \mathbf{R}^n$ is a bounded uniform domain then every $u \in W^{1,p}(\Omega)$ satisfies the inequality (4.3) for every $1 \le p \le \infty$, with $\alpha = 1$, [HM, Lemma 14, p. 243]. Our proof is a modification of the proof of Hajłasz and Martio.

Proof. We may assume that $\operatorname{diam}(\Omega) \le 1$. Let $\{B_i\}_{i=-\infty}^{\infty}$ be a chain of balls from the point $x \in \Omega$ to the point $y \in \Omega$ as in Lemma 4.1. Then by the Lebesgue differentiation theorem [St, Chapter 1, Section 1.8] we have $u_{B_i} \to u(x)$, whenever

$i \to -\infty$, and $u_{B_i} \to u(y)$, whenever $i \to \infty$, for almost every $x, y \in \Omega$. Thus we have

$$\begin{aligned}|u(x)-u(y)| &\le \sum_{i=-\infty}^{\infty} |u_{B_i} - u_{B_{i+1}}| \\ &\le \sum_{i=-\infty}^{\infty} \big(|u_{B_i} - u_{B_i\cap B_{i+1}}| + |u_{B_{i+1}} - u_{B_i\cap B_{i+1}}|\big) \\ &\le \sum_{i=-\infty}^{\infty} \bigg(\fint_{B_i\cap B_{i+1}} |u-u_{B_i}| + \fint_{B_i\cap B_{i+1}} |u-u_{B_{i+1}}|\bigg)\end{aligned}$$

and furthermore by Lemma 4.1

$$\begin{aligned}|u(x)-u(y)| &\le \sum_{i=-\infty}^{\infty} \bigg(\frac{1}{|B_i\cap B_{i+1}|}\int_{B_i\cap B_{i+1}} |u-u_{B_i}| \\ &\qquad + \frac{1}{|B_i\cap B_{i+1}|}\int_{B_i\cap B_{i+1}} |u-u_{B_{i+1}}|\bigg) \\ &\le \sum_{i=-\infty}^{\infty} \bigg(\frac{1}{|B_i\cap B_{i+1}|}\int_{B_i} |u-u_{B_i}| \\ &\qquad + \frac{1}{|B_i\cap B_{i+1}|}\int_{B_{i+1}} |u-u_{B_{i+1}}|\bigg) \\ &\le \sum_{i=-\infty}^{\infty} \bigg(\frac{C}{|B_i|}\int_{B_i} |u-u_{B_i}| + \frac{C}{|B_{i+1}|}\int_{B_{i+1}} |u-u_{B_{i+1}}|\bigg) \\ &\le 2\cdot C \sum_{i=-\infty}^{\infty} \fint_{B_i} |u-u_{B_i}|.\end{aligned}$$

The Poincaré inequality in a ball with a radius r_i, [GT, 7.45, p. 157], yields

$$|u(x)-u(y)| \le C \sum_{i=-\infty}^{\infty} r_i \fint_{B_i} |\nabla u| \le C \sum_{i=-\infty}^{\infty} \int_{B_i} \frac{|\nabla u|}{r_i^{n-1}}.$$

Lemma 4.1 implies that for each $z \in B_i$, $|x-z| \le (d+2)r_i^{1/s}$ and $B_i \subset B^n(x, C|x-y|^{m/s})$, when $i \le 0$ and $|y-z| \le (d+2)r_i^{1/s}$ and, when $i \ge 0$,

$B_i \subset B^n(y, C|x-y|^{m/s})$. We obtain

$$\begin{aligned}|u(x)-u(y)| &\le C \sum_{i=-\infty}^{0} \int_{B_i} \frac{|\nabla u(z)|}{|x-z|^{s(n-1)}}\,dz + C\sum_{i=0}^{\infty}\int_{B_i}\frac{|\nabla u(z)|}{|y-z|^{s(n-1)}}\,dz \\ &\le C \int_{B^n(x,C|x-y|^{m/s})} \frac{|\nabla u(z)|}{|x-z|^{s(n-1)}}\,dz \\ &\quad + C\int_{B^n(y,C|x-y|^{m/s})} \frac{|\nabla u(z)|}{|y-z|^{s(n-1)}}\,dz.\end{aligned}$$

We put $|\nabla u| = 0$ in the complement of the domain Ω. Since $s(n-1) < n$ we obtain by [Z, Lemma 2.8.3, p. 85] that

$$\begin{aligned}|u(x)-u(y)| &\le C\big(|x-y|^{m(n-s(n-1))/s}\mathscr{M}_{C|x-y|^{m/s}}\nabla u(x) \\ &\quad + |x-y|^{m(n-s(n-1))/s}\mathscr{M}_{C|x-y|^{m/s}}\nabla u(y)\big) \\ &= C|x-y|^{m(n-s(n-1))/s}\big(\mathscr{M}_{C|x-y|^{m/s}}\nabla u(x) + \mathscr{M}_{C|x-y|^{m/s}}\nabla u(y)\big).\end{aligned}$$

This completes the proof of Theorem 4.2. □

5. Sharpness of Theorem 4.2

Assume that a bounded domain $\Omega \subset \mathbf{R}^n$ satisfies the inequality (4.3) for all $1 < p < \infty$ with some exponent $\alpha > 0$. We obtain by the inequality (4.3) that

$$\begin{aligned}\left|u(x) - \fint_\Omega u(y)\,dy\right| &\le \fint_\Omega |u(x)-u(y)|\,dy \\ &\le C\operatorname{diam}(\Omega)^\alpha\left(M\nabla u(x) + \fint_\Omega M\nabla u(y)\,dy\right) \qquad (5.1)\\ &\le C\operatorname{diam}(\Omega)^\alpha\left(M\nabla u(x) + \left(\fint_\Omega \big(M\nabla u(y)\big)^p\,dy\right)^{1/p}\right)\end{aligned}$$

and the boundedness of the Hardy–Littlewood maximal operator, [St, Theorem 1, p. 6], yields

$$\|u-u_\Omega\|_{L^p(\Omega)} \le C\operatorname{diam}(\Omega)^\alpha\|M\nabla u\|_{L^p(\Omega)} \le C\operatorname{diam}(\Omega)^\alpha\|\nabla u\|_{L^p(\Omega)}$$

as in [H2, Lemma 2, p. 407]. Thus Theorem 4.2 implies that a bounded (s,m)-uniform domain $\Omega\subset\mathbf{R}^n$, $1\le s<n/(n-1)$ and $0<m\le 1$, is a p-Poincaré domain for every $1<p<\infty$. W. Smith and D. Stegenga showed that an s-John domain is a p-Poincaré domain for every $1<p<\infty$, if $1\le s\le n/(n-1)$, [SS, Theorem 10, p. 86]. Hajłasz and Koskela proved with a "mushroom" example that the limit is sharp in the sense that s cannot be greater than $n/(n-1)$, [HK2, Corollary 6].

We show that if $s>n/(n-1)$ then an $(s,1)$-uniform domain is not necessarily a p-Poincaré domain for every $1<p<\infty$. The following rooms and passages example is by R. Hurri [Hu, Chapter 5, p. 17].

5.2. Example. Let $\Omega = \bigcup_{i=1}^{\infty}(R_{2i-1} \cup P_{2i})$, where the sets R_{2i-1} and P_{2i} are defined as follows. Let $a \geq 1$. Let $h_i = 2^{-i}$, $\delta_{2i} = 2\cdot 2^{-2ai}$ and $d_i = \sum_{j=1}^{i} 2^{-j}$ for every $i = 1, 2, \ldots$. We define

$$R_{2i-1} = (d_{2i-1} - h_{2i-1}, d_{2i-1}) \times \bigl(-\tfrac{1}{2}h_{2i-1}, \tfrac{1}{2}h_{2i-1}\bigr)^{n-1},$$
$$P_{2i} = \bigl[d_{2i-1}, d_{2i-1} + h_{2i}\bigr] \times \bigl(-\tfrac{1}{2}\delta_{2i}, \tfrac{1}{2}\delta_{2i}\bigr)^{n-1}.$$

By Hurri [Hu, Remark 5.9, p. 19] the domain Ω is a p-Poincaré domain if and only if $p \geq (n-1)(a-1)$.

Since there exists a constant $C > 0$ so that $\frac{1}{2}\delta_{2i} \geq C(1 - d_{2i-1})^a$ for every $i = 1, 2, \ldots$, the domain Ω is an $(a,1)$-uniform domain. Let $\varepsilon > 0$ be arbitrary. If $a = \bigl(n/(n-1)\bigr) + \varepsilon$, then the domain Ω is not a p-Poincaré domain for any $1 \leq p < 1 + \varepsilon(n-1)$.

5.3. Corollary. *Let $s > n/(n-1)$ and $0 < m \leq 1$. There exists a bounded (s,m)-uniform domain where the inequality* (4.3) *does not hold for all $1 < p < (s-1)(n-1)$ with any $\alpha > 0$.*

Proof. Let $\varepsilon > 0$. Let $\Omega \subset \mathbf{R}^n$ be the bounded (s,m)-uniform domain, $s = \bigl(n/(n-1)\bigr) + \varepsilon$ and $m = 1$, constructed in Example 5.2. Assume that there exist constants $C, \alpha > 0$ such that for every $u \in W^{1,p}(\Omega)$, $1 < p < \infty$, we have

$$|u(x) - u(y)| \leq C|x-y|^{\alpha}\bigl(\mathscr{M}\nabla u(x) + \mathscr{M}\nabla u(y)\bigr), \tag{5.4}$$

for almost every $x, y \in \Omega$. As in (5.1) this implies that the domain Ω is a p-Poincaré domain for all $1 < p < \infty$.

In Example 5.2 we showed that the domain Ω is not a p-Poincaré domain for any $1 < p < 1 + \varepsilon(n-1)$. Thus the inequality (5.4) cannot hold for all $1 < p < 1 + \varepsilon(n-1)$ with any $\alpha > 0$ in the domain Ω. □

Following Hajłasz, [H2], we say that a domain D is δ-regular, $\delta > 0$, if there exists a constant $b > 0$ such that

$$|B^n(x,r) \cap D| \geq br^{\delta} \tag{5.5}$$

for every $x \in D$ and for every $0 < r \leq \operatorname{diam}(D)$. It is easy to see that every bounded (s,m)-uniform domain is $\bigl(s(n-1)+1\bigr)$-regular.

Using the method of Hajłasz, [H2, Theorem 6, p. 410], it is easy to prove the following Sobolev–Poincaré inequality. In the proof we need only the inequality (4.3) and the property (5.5).

5.6. Lemma. *Assume that $\Omega \subset \mathbf{R}^n$ is a bounded δ-regular domain, $\delta > 1$, which satisfies the inequality* (4.3) *with an exponent $0 < \alpha \leq 1$. If $1 < p < \delta/\alpha$, then for every $u \in W^{1,p}(\Omega)$ we have*

$$\|u - u_{\Omega}\|_{L^{p^*}(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \tag{5.7}$$

with $p^* = \delta p/(\delta - \alpha p)$.

Proof. We may assume that $\operatorname{diam}(\Omega) \le 1$. Let

$$E_k = \{x \in \Omega : M\nabla u(x) \le 2^k\}, \qquad k \in \mathbf{Z}.$$

There exists a constant $C > 0$ such that

$$C^{-1} \sum_{i=-\infty}^{\infty} 2^{kp} |E_k \setminus E_{k-1}| \le \int_\Omega M|\nabla u|^p \, dx \le C \sum_{i=-\infty}^{\infty} 2^{kp} |E_k \setminus E_{k-1}|. \tag{5.8}$$

Let $a_k = \operatorname{ess\,sup}_{x \in E_k} |u(x)|$. We will estimate a_k in terms of a_{k-1}. Let $x \in E_k$. Let $B^n(x,r)$ be a ball with a radius $r = 2b^{-1/\delta} |\Omega \setminus E_{k-1}|^{1/\delta}$. We obtain by the δ-regularity property (5.5)

$$|B^n(x,r) \cap \Omega| \ge br^\delta > |\Omega \setminus E_{k-1}|.$$

Hence there exists $y \in B^n(x,r) \cap E_{k-1}$. By the inequality (4.3) the function $u|_{E_k}$ is α-Hölder continuous with a constant $C2^{k+1}$. We obtain

$$|u(x)| \le |u(x) - u(y)| + |u(y)| \le C|x-y|^\alpha 2^{k+1} + a_{k-1} \le C|\Omega \setminus E_{k-1}|^{\alpha/\delta} 2^{k+1} + a_{k-1}.$$

The definition of E_k yields

$$|\Omega \setminus E_{k-1}| 2^{kp} \le C \|\mathscr{M}\nabla u\|^p_{L^p(\Omega)}; \tag{5.9}$$

hence we obtain that

$$\begin{aligned} a_k &\le C 2^{-kp\alpha/\delta} \|\mathscr{M}\nabla u\|^{p\alpha/\delta}_{L^p(\Omega)} 2^{k+1} + a_{k-1} \\ &\le C 2^{k(1-(p\alpha/\delta))} \|\mathscr{M}\nabla u\|^{p\alpha/\delta}_{L^p(\Omega)} + a_{k-1}. \end{aligned} \tag{5.10}$$

We may assume that $\mathscr{M}\nabla u(x) > 0$ for every $x \in \Omega$ since otherwise $|\nabla u| = 0$ which implies that u is a constant function almost everywhere in Ω. Let $b_k = \operatorname{ess\,inf}_{x \in E_k} |u(x)|$. It is clear that $b_k \le \|u\|_{L^p(\Omega)} |E_k|^{-1/p}$. Since $\mathscr{M}\nabla u > 0$ everywhere then there exists k_0 such that $|E_{k_0-1}| < \frac{1}{2}|\Omega|$ and $|E_{k_0}| \ge \frac{1}{2}|\Omega|$. We obtain by the inequality (5.9) that

$$2^{k_0} \le C \|\mathscr{M}\nabla u\|_{L^p(\Omega)} |\Omega \setminus E_{k_0-1}|^{-1/p}.$$

Since the function $u|_{E_k}$ is α-Hölder continuous with a constant $C2^{k+1}$ we obtain $a_k \le b_k + 2^{k+1} \operatorname{diam}(\Omega)^\alpha$. This yields

$$\begin{aligned} a_{k_0} &\le \|u\|_{L^p(\Omega)} |E_{k_0}|^{-1/p} + C \operatorname{diam}(\Omega)^\alpha \|\mathscr{M}\nabla u\|_{L^p(\Omega)} |\Omega|^{-1/p} \\ &\le C|\Omega|^{-1/p} \big(\|u\|_{L^p(\Omega)} + \operatorname{diam}(\Omega)^\alpha \|\mathscr{M}\nabla u\|_{L^p(\Omega)} \big). \end{aligned} \tag{5.11}$$

Since $p < \delta/\alpha$, it follows, for $k > k_0$, by the inequality (5.10) and the monotonicity of a_k that

$$
\begin{aligned}
a_k &\le C\|\mathscr{M}\nabla u\|_{L^p(\Omega)}^{p\alpha/\delta}\Big(\sum_{i=k_0}^{k} 2^{i(1-(p\alpha/\delta))}\Big) + a_{k_0} \\
&\le C\|\mathscr{M}\nabla u\|_{L^p(\Omega)}^{p\alpha/\delta}\Big(\sum_{i=-\infty}^{k} 2^{i(1-(p\alpha/\delta))}\Big) + a_{k_0} \qquad (5.12)\\
&\le C\|\mathscr{M}\nabla u\|_{L^p(\Omega)}^{p\alpha/\delta} 2^{k(1-(p\alpha/\delta))} + a_{k_0}.
\end{aligned}
$$

Since $p^* = p\delta/(\delta-\alpha p)$ the inequalities (5.8), (5.11), (5.12) and the regularity property (5.5) yield that

$$
\begin{aligned}
\Big(\int_\Omega |u|^{p^*}\Big)^{1/p^*} &\le \Big(\sum_{k=k_0+1}^{\infty} a_k^{p^*}|E_k \setminus E_{k-1}| + a_{k_0}^{p^*}|E_{k_0}|\Big)^{1/p^*} \\
&\le C\Big(\|\mathscr{M}\nabla u\|_{L^p(\Omega)}^{p\alpha p^*/\delta} \sum_{k=-\infty}^{\infty} 2^{k(1-(p\alpha/\delta))p^*}|E_k \setminus E_{k-1}| + a_{k_0}^{p^*}|\Omega|\Big)^{1/p^*} \\
&\le C\Big(\|\mathscr{M}\nabla u\|_{L^p(\Omega)}^{p\alpha p^*/\delta}\|\mathscr{M}\nabla u\|_{L^p(\Omega)}^{p} \\
&\quad + \big(C|\Omega|^{-1/p}\big(\|u\|_{L^p(\Omega)} + \operatorname{diam}(\Omega)^\alpha\|\mathscr{M}\nabla u\|_{L^p(\Omega)}\big)^{p^*}|\Omega|\Big)^{1/p^*} \\
&\le C(\|u\|_{L^p(\Omega)} + \|\mathscr{M}\nabla u\|_{L^p(\Omega)}).
\end{aligned}
$$

Since $u - u_\Omega \in W^{1,p}(\Omega)$, Ω is a p-Poincaré domain and the Hardy–Littlewood maximal operator is bounded, [St, Theorem 1, p. 5], we obtain

$$
\|u-u_\Omega\|_{L^{p^*}(\Omega)} \le C(\|u-u_\Omega\|_{L^p(\Omega)} + \|\mathscr{M}\nabla(u-u_\Omega)\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)}. \ \square
$$

We write $\delta = s(n-1)+1$. Hajłasz and P. Koskela have proved the inequality (5.7) for s-John domains with a better exponent. Let $\Omega \subset \mathbf{R}^n$ be an s-John domain, $s \ge 1$, then the inequality (5.7) holds with $1 \le p \le p^* \le np/(\delta - p)$, [HK2, Corollary 6, p. 20]. The limiting case $p^* = np/(\delta-p)$ is by T. Kilpeläinen and J. Malý [KM]. The exponent is the best possible in the class of s-John domains, [HK2]. It is also the best possible in the class of (s,m)-uniform domains. Let $s > 1$. Using the $(s,1)$-uniform domain constructed by Hurri, see Example 5.2, we obtain as in [Hu, Remark 5.8, p. 19], by replacing the exponent $-n/p$ by the exponent $-n/p^*$, that the exponent $np/(\delta-p)$ is the best possible.

5.13. Corollary. *Let $\Omega \subset \mathbf{R}^n$ be a bounded (s,m)-uniform domain, with $1 < s < n/(n-1)$ and $0 < m \le 1$. If there exists an $\alpha > 0$ such that the inequality* (4.3) *holds for all $1 < p < \infty$ then*

$$
\alpha \le \frac{s(n-1)+1}{n}\big(n - s(n-1)\big) < 1.
$$

If $s = n/(n-1)$ then Ω does not satisfy the inequality (4.3) *for all* $1 < p < \infty$ *with any* $\alpha > 0$.

Proof. Let $1 \le s < n/(n-1)$. Lemma 5.6 shows that the inequality (4.3) with an exponent $\alpha > 0$ and the δ-regular property (5.5), $\delta = s(n-1)+1$, implies the Sobolev–Poincaré inequality with $p^* = \delta p/(\delta - \alpha p)$.

The exponent $\delta p/(\delta - \alpha p)$ has to be less than or equal to the best possible exponent $np/(\delta - p)$ for every $1 < p < \infty$. This gives

$$\alpha \le \frac{\delta}{np}(n - \delta + p)$$

for every $1 < p < \infty$. As $p \to 1$ we see that

$$\alpha \le \frac{\delta}{n}(n - \delta + 1).$$

Let $s = n/(n-1)$. Assume that Ω is a bounded (s,m)-uniform domain which satisfies the inequality (4.3) with some $\alpha > 0$ for every $1 < p < \infty$. By Lemma 5.6 we obtain that Ω satisfies the Sobolev–Poincaré inequality with $(n+1)p/(n+1-\alpha p)$. Thus we obtain

$$\alpha \le \left(1 + \frac{1}{n}\right)\left(1 - \frac{1}{p}\right)$$

for every $1 < p < \infty$. As $p \to 1$ we see that $\alpha \le 0$. Hence the domain Ω cannot satisfy the inequality (4.3) with any $\alpha > 0$ for small $p > 1$. This completes the proof of Corollary 5.13. □

References

[DS] DeVore, R.A., and R.C. Sharpley: Maximal Functions Measuring Smoothness. - Mem. Amer. Math. Soc. 47 (293), 1984.

[GM] Gehring, F.W., and O. Martio: Lipschitz classes and quasiconformal mappings. - Ann. Acad. Sci. Fenn. Ser. A I Math. 10, 1985, 203–219.

[GT] Gilbarg, D., and N.S. Trudinger: Elliptic Partial Differential Equations of Second Order. - Grundlehren Math. Wiss. 224, Springer-Verlag, Berlin, 1977.

[H1] Hajłasz, P.: Boundary behaviour of Sobolev mappings. - Proc. Amer. Math. Soc. 123, 1995, 1145–1148.

[H2] Hajłasz, P.: Sobolev spaces on an arbitrary metric space. - Potential Anal. 5, 1996, 403–415.

[Hak] Hajłasz, P., and P. Koskela: Isoperimetric inequalities and imbedding theorems in irregular domains. - J. London Math. Soc. (2) 58, 1998, 425–450.

[HeK] Herron, D.A., and P. Koskela: Uniform, Sobolev extension and quasiconformal circle domains. - J. Anal. Math. 57, 1991, 172–202.

[Hu] Hurri, R.: Poincaré domains in $\mathbf{R}^n$. - Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 71, 1988, 1–42.

[J] Jones, P.W.: Quasiconformal mappings and extendability of functions in Sobolev spaces. - Acta Math. 147, 1981, 71–88.

[KM] Kilpeläinen, T., and J. Malý: Sobolev inequalities on sets with irregular boundaries. - Z. Anal. Anwendungen 19, 2000, 369–380.

[MS] Martio, O., and J. Sarvas: Injectivity theorems in plane and space. - Ann. Acad. Sci. Fenn. Ser. A I Math. 4, 1979, 383–401.

[M] Mattila, P.: Geometry of Sets and Measures in Euclidean Spaces. - Cambridge University Press, Cambridge, 1995.

[SS] Smith, W., and D.A. Stegenga: Hölder domains and Poincaré domains. - Trans. Amer. Math. Soc. 319, 1990, 67–100.

[St] Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. - Princeton Math. Ser. 30, Princeton University Press, Princeton, N.J., 1970.

[V] Väisälä, J.: Lectures on n-dimensional Quasiconformal Mappings. - Lecture Notes in Math. 229, Springer-Verlag, Berlin, 1971.

[Z] Ziemer, W.P.: Weakly Differentiable Functions. - Springer-Verlag, New York, 1989.

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