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VARIABILITY SETS ON RIEMANN SURFACES AND FORGETFUL MAPS BETWEEN TEICHMÜLLER SPACES

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Abstract. We extend Strebel's theory of variability sets to the setting of arbitrary hyperbolic Riemann surfaces. Our extended theory depends on the behavior of the Teichmüller metric on the fibers of forgetful maps between Teichmüller spaces. We obtain new results about the metric properties of these fibers.

Introduction

Let f be a quasiconformal mapping of the open unit disk Δ onto itself, and let $K_0(f)$ be the smallest number K such that there is a K-quasiconformal self-mapping of Δ with the same boundary values as f. A quasiconformal mapping of Δ onto itself is called f-extremal if it has the same boundary values as f and it is $K_0(f)$-quasiconformal. There is always at least one f-extremal mapping, and Strebel showed in [19] that there can be more than one.

Given any a in Δ, Strebel [20] defined the variability set $V[a]$ to be the set of all points b in Δ such that $b = g(a)$ for some f-extremal map g. In the groundbreaking paper [22] he proved that $V[a]$ is compact and that both $V[a]$ and its complement in Δ are connected. In this paper we shall extend that result to the case when Δ is replaced by any Riemann surface whose universal covering surface is conformally equivalent to Δ. We call such Riemann surfaces hyperbolic.

Let x_0 be a point on the hyperbolic Riemann surface X, and let f be a quasiconformal map of X onto a (necessarily hyperbolic) Riemann surface Y. To define the variability set of x_0 with respect to f we begin by giving X and Y the basepoints x_0 and $y_0 = f(x_0)$ and forming the universal covering surfaces $(\widetilde{X}, \widetilde{x_0})$ and $(\widetilde{Y}, \widetilde{y_0})$. This means we are given basepoints $\widetilde{x_0}$ in $\widetilde{X}$ and $\widetilde{y_0}$ in $\widetilde{Y}$ and basepoint preserving holomorphic universal covering maps $\varpi_X \colon \widetilde{X} \to X$ and $\varpi_Y \colon \widetilde{Y} \to Y$. Since $\widetilde{X}$ and $\widetilde{Y}$ are conformally equivalent to Δ they have Poincaré metrics $\rho_{\widetilde{X}}$ and $\rho_{\widetilde{Y}}$.

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Let g be any quasiconformal map of X onto Y. By definition a lift of g is a quasiconformal map $\tilde{g}$ of $\widetilde{X}$ onto $\widetilde{Y}$ that satisfies $g \circ \varpi_X = \varpi_Y \circ \tilde{g}$. The given map f has a unique lift $\tilde{f}$ such that $\tilde{f}(\widetilde{x_0}) = \widetilde{y_0}$. If g has a lift $\tilde{g}$ such that the distance $\rho_{\widetilde{Y}}\big(\tilde{f}(\tilde{x}), \tilde{g}(\tilde{x})\big)$ is bounded by a number M independent of $\tilde{x}$ in $\widetilde{X}$, we say that g is Teichmüller equivalent to f and we use the symbol $\tilde{g}$ to denote the unique lift of g with this bounded distance property. (This property means that $\tilde{f}$ and $\tilde{g}$ have the same boundary values if we identify $\widetilde{X}$ and $\widetilde{Y}$ with Δ.)

Let $K_0(f)$ be the smallest number K such that there is a K-quasiconformal map $g\colon X \to Y$ that is Teichmüller equivalent to f. We call g an f-extremal map if it is both Teichmüller equivalent to f and $K_0(f)$-quasiconformal.

By definition the variability set $\widetilde{V}_f[x_0]$ of x_0 with respect to f is the subset of $\widetilde{Y}$ consisting of the points $\tilde{y}$ such that $\tilde{y} = \tilde{g}(\widetilde{x_0})$ for some f-extremal map g. This definition reduces to Strebel's when X and Y are both the unit disk, so the following theorem generalizes Theorem 7 of [22].

Theorem 1. *The variability set $\widetilde{V}_f[x_0]$ is compact, and both $\widetilde{V}_f[x_0]$ and its complement in $\widetilde{Y}$ are connected.*

Like Strebel's, our proof depends on a study of the level curves of an appropriate dilatation function, but we base our study on the implicit function theorem, as applied to a certain map from one Teichmüller space to the product of another Teichmüller space and the real numbers. Theorem 3 in Section 5, which establishes the crucial properties of that map, is one of our main results. We shall derive Theorem 1 from it in Section 6.

Since our approach depends on embedding variability sets in appropriate Teichmüller spaces we review some Teichmüller theory in Sections 1–4. Bers fiber spaces and forgetful maps between Teichmüller spaces play a central role. So does Strebel's frame mapping criterion, as it has in all previous work on variability sets.

Studying variability sets by embedding them in Teichmüller spaces is a promising technique. Li Zhong used it in the classical unit disk setting to prove Theorem 5 of his interesting paper [13].

Our study of Bers fiber spaces and forgetful maps led to new results about their fibers, which we prove in Sections 7–11. The most striking of them is the following result, which shows conclusively that the restriction of Teichmüller's metric to the fibers of the forgetful map is not an arc length metric.

Theorem 2. *Let X be a hyperbolic Riemann surface that is not conformally equivalent to $\mathbf{C} \setminus \{0,1\}$, let x_0 be a point of X, and let $d_{X'}$ be Teichmüller's metric on the Teichmüller space $T(X')$ of $X' = X \setminus \{x_0\}$. If a and b are distinct points on some fiber of the forgetful map from $T(X')$ to the Teichmüller space of X, then that fiber contains only finitely many points c such that*

$$d_{X'}(a,b) = d_{X'}(a,c) + d_{X'}(c,b).$$

Unlike some previous studies of the metric properties of these fibers (see [9], [14], and [16]) ours relies primarily on elementary construction of quasiconformal mappings and makes no use of the mapping class group.

We thank Kurt Strebel both for writing his inspirational paper [22] and for encouraging our work on this paper.

1. Teichmüller spaces and forgetful maps

In this section and the next we review some classical Teichmüller theory and introduce notation that will be used throughout this paper. More information about Teichmüller spaces can be found in the books [8] and [17]. Section 3 of the paper [5] offers a more detailed explanation of our description of the tangent spaces to Teichmüller space.

Let X be a hyperbolic Riemann surface. A measurable Beltrami form μ on X is called a *Beltrami coefficient* if its L^∞ norm is less than one. The set $M(X)$ of all Beltrami coefficients on X has a natural complex structure, as it is the open unit ball in the complex Banach space of bounded measurable Beltrami forms.

Let $\mathscr{H}$ be the upper half plane, and let $\varpi\colon \mathscr{H} \to X$ be a holomorphic universal covering map. Every quasiconformal self-mapping g of X lifts to a quasiconformal self-mapping $\tilde{g}$ of $\mathscr{H}$. We call g *Teichmüller trivial* if it has a lift $\tilde{g}$ whose continuous extension to the closed half plane equals the identity on the extended real axis. The Teichmüller trivial quasiconformal self-mappings of X form a group that we denote by $\mathscr{QC}_0(X)$.

Every μ in $M(X)$ is the Beltrami coefficient of a quasiconformal map of X onto some Riemann surface. We say that μ and ν in $M(X)$ are *Teichmüller equivalent* on X if there exist g in $\mathscr{QC}_0(X)$ and a quasiconformal map f with domain X such that μ is the Beltrami coefficient of f and ν is the Beltrami coefficient of $f \circ g$. We denote the Teichmüller equivalence class of μ on X by $[\mu]_X$.

The *Teichmüller space* $T(X)$ is the space of Teichmüller equivalence classes $[\mu]_X$ of Beltrami coefficients on X. By a fundamental theorem of Bers (see [8] and [17]), $T(X)$ has a unique complex manifold structure such that the quotient map $\Phi_X(\mu) = [\mu]_X$ from $M(X)$ to $T(X)$ is a holomorphic split submersion.

The derivative of Φ_X at $\mu = 0$ has the following useful description. Let $Q(X)$ be the complex Banach space of L^1 holomorphic quadratic differentials on X, and let $Q(X)^*$ be its dual space. The tangent space to $T(X)$ at its *basepoint* $[0]_X$ can be uniquely identified with $Q(X)^*$ in such a way (see Section 3 of [5]) that $\Phi'_X(0)$ is the map that takes a bounded measurable Beltrami form μ on X to the linear functional l_μ on $Q(X)$ defined by

$$l_\mu(\varphi) = \iint\limits_X \mu\varphi, \quad \varphi \in Q(X).$$

If x_0 is a point of X, the Riemann surface $X' = X \setminus \{x_0\}$ is also hyperbolic. Quadratic differentials in $Q(X')$ are holomorphic in X except for a possible simple pole at x_0, and $Q(X)$ is a codimension one subspace of $Q(X')$.

Since the set $\{x_0\}$ has measure zero, the spaces $M(X)$ and $M(X')$ are the same. Further, every quasiconformal map with domain X' is the restriction of a quasiconformal map with domain X, and every map in $\mathscr{QC}_0(X')$ is the restriction of a map in $\mathscr{QC}_0(X)$. Therefore $[\mu]_{X'} \subset [\mu]_X$ for all μ in $M(X)$. We define the *forgetful map* $P_{x_0}\colon T(X') \to T(X)$ by the formula $P_{x_0}([\mu]_{X'}) = [\mu]_X$, μ in $M(X)$.

Since both $\Phi_{X'}$ and Φ_X are holomorphic split submersions, P_{x_0} is also a holomorphic split submersion. By definition, P_{x_0} maps the basepoint of $T(X')$ to the basepoint of $T(X)$. A trivial chain rule calculation shows that the derivative of P_{x_0} at $[0]_{X'}$ is the restriction map $l \mapsto l \mid Q(X)$ from $Q(X')^*$ to $Q(X)^*$.

2. The Bers fiber space and isomorphism theorem

The Bers fiber space and its projection onto Teichmüller space provide an alternative model for the forgetful map defined in Section 1. We shall review the relevant facts here. See [2] or [17] for more details.

As in Section 1, we consider a hyperbolic Riemann surface X with a basepoint x_0, and we write $X' = X \setminus \{x_0\}$. We give $\mathscr{H}$ the basepoint i, and we require the holomorphic universal covering $\varpi\colon \mathscr{H} \to X$ of X by the upper half plane to preserve basepoints.

Let Γ be the group of covering transformations of ϖ. By definition the set $M(\Gamma)$ of Beltrami coefficients for Γ consists of the measurable functions μ on $\mathscr{H}$ that have L^∞ norm less than one and satisfy the Γ-invariance condition $(\mu \circ \gamma)\overline{\gamma'}/\gamma'$ almost everywhere for all γ in Γ. For each μ in $M(\Gamma)$ let w^μ be the quasiconformal mapping of the Riemann sphere $\widehat{\mathbf{C}}$ onto itself that fixes the points 0, 1, and ∞, is conformal in the lower half plane, and satisfies the Beltrami equation $w_{\bar z} = \mu w_z$ in $\mathscr{H}$.

Each μ in $M(\Gamma)$ projects to a well-defined Beltrami coefficient on X. The resulting map from $M(\Gamma)$ to $M(X)$ is a norm-preserving bijection, and we use it to identify $M(X)$ with $M(\Gamma)$. In particular, from now on we shall regard the Teichmüller equivalence classes $[\mu]_X$ and $[\mu]_{X'}$ as subsets of $M(\Gamma)$. It is not hard to verify that for each μ in $M(\Gamma)$ the class $[\mu]_X$ consists of the ν in $M(\Gamma)$ such that $w^\nu = w^\mu$ on the extended real axis. Thus $w^\nu(\mathscr{H}) = w^\mu(\mathscr{H})$ for all ν in $[\mu]_X$.

The *Bers fiber space*

$$F(\Gamma) = \bigl\{([\mu]_X, \zeta) : \mu \in M(\Gamma) \text{ and } \zeta \in w^\mu(\mathscr{H})\bigr\}$$

is defined in [2] (see also [17]) and is shown to be an open subset of $T(X) \times \mathbf{C}$. According to the Bers isomorphism theorem (see [2] or [17]), the formula

$$\mathscr{B}([\mu]_{X'}) = \bigl([\mu]_X, w^\mu(i)\bigr), \qquad \mu \in M(\Gamma), \tag{1}$$

produces a well defined biholomorphic map $\mathscr{B}$ of $T(X')$ onto $F(\Gamma)$. In particular, for each μ in $M(\Gamma)$, $[\mu]_{X'}$ consists of the ν in $M(\Gamma)$ such that $w^\nu = w^\mu$ on the union of the extended real axis and the set $\{i\}$.

Let $P\colon F(\Gamma) \to T(X)$ be the projection map that sends $([\mu]_X, \zeta)$ to $[\mu]_X$ for each μ in $M(\Gamma)$ and ζ in $w^\mu(\mathscr{H})$. The Bers isomorphism $\mathscr{B}$ maps each fiber $P_{x_0}^{-1}([\mu]_X)$ of the forgetful map biholomorphically to the fiber $P^{-1}([\mu]_X) = \{[\mu]_X\} \times w^\mu(\mathscr{H})$ of P. Thus $P_{x_0}^{-1}([\mu]_X)$ is biholomorphically equivalent to the region $w^\mu(\mathscr{H})$ for each μ in $M(\Gamma)$. In particular, $P_{x_0}^{-1}([0]_X)$ is biholomorphically equivalent to $\mathscr{H}$.

Remark 1. In equation (1) and the subsequent discussion the point i can be replaced by any other point a in $\mathscr{H}$, provided that $X' = X \setminus \{\varpi(i)\}$ is replaced by the Riemann surface $X \setminus \{\varpi(a)\}$.

Example. We can take X to be Δ, x_0 to be 0, ϖ to be the map $z \mapsto (z-i)/(z+i)$, and Γ to be the trivial group $\{I\}$. Then $M(\{I\})$ is the open unit ball of $L^\infty(\mathscr{H})$ and the Bers isomorphism identifies $F(\{I\})$ with the Teichmüller space of the punctured disk $\Delta' = \Delta \setminus \{0\}$.

3. Extremal Beltrami coefficients and variability sets

Both $T(X)$ and $T(X')$ carry Teichmüller metrics, which we denote by d_X and $d_{X'}$ respectively. We are particularly interested in distances from the basepoints $[0]_X$ and $[0]_{X'}$ of $T(X)$ and $T(X')$. These are defined as follows.

For each μ in $M(\Gamma)$ let $\|\mu\|$ be the L^∞ norm of μ and let

$$K(\mu) = (1 + \|\mu\|)/(1 - \|\mu\|)$$

be the maximal dilatation of the quasiconformal mapping w^μ. Then

$$\begin{aligned} &d_X([0]_X, [\mu]_X) = \min\{\tfrac{1}{2}\log K(\nu) : \nu \in [\mu]_X\} \qquad \text{and} \\ &d_{X'}([0]_{X'}, [\mu]_{X'}) = \min\{\tfrac{1}{2}\log K(\nu) : \nu \in [\mu]_{X'}\}, \quad \mu \in M(\Gamma). \end{aligned} \tag{2}$$

The Beltrami coefficients μ that attain these minima are called extremal. More precisely, we call μ in $M(\Gamma)$ X-*extremal* if $\frac{1}{2}\log K(\mu) = d_X([0]_X, [\mu]_X)$ and X'-*extremal* if $\frac{1}{2}\log K(\mu) = d_{X'}([0]_{X'}, [\mu]_{X'})$. Since $[\mu]_{X'} \subset [\mu]_X$ for all μ in $M(\Gamma)$, every X-extremal μ is X'-extremal and

$$d_X([0]_X, [\mu]_X) \le d_{X'}([0]_{X'}, [\mu]_{X'}) \qquad \text{for all } \mu \text{ in } M(\Gamma). \tag{3}$$

Now let f be a quasiconformal map of X onto a Riemann surface Y. In the introduction we defined the variability set $\widetilde{V}_f[x_0]$ as a subset of the universal covering surface $\widetilde{Y}$ of Y. The following lemma allows us to identify $\widetilde{V}_f[x_0]$ with a subset of $T(X')$ that we can study more conveniently.

Lemma 1. *Let $f\colon X\to Y$ be a quasiconformal map and let μ be the element of $M(\Gamma)$ that determines its Beltrami coefficient. There is a biholomorphic map of $\widetilde{Y}$ onto the fiber $P_{x_0}^{-1}([\mu]_X)$ of the forgetful map such that the image of $\widetilde{V}_f[x_0]$ is the set*

$$V^\mu(x_0)=\big\{[\nu]_{X'}\in P_{x_0}^{-1}([\mu]_X): d_X([0]_X,[\nu]_X)=d_{X'}([0]_{X'},[\nu]_{X'})\big\}. \tag{4}$$

Proof. Our assumptions on f and μ make the map $\varpi^\mu=f\circ\varpi\circ(w^\mu)^{-1}$ of $w^\mu(\mathscr{H})$ onto Y holomorphic. With i and $w^\mu(i)$ as basepoints for $\mathscr{H}$ and $w^\mu(\mathscr{H})$ respectively, the maps $\varpi\colon\mathscr{H}\to X$ and $\varpi^\mu\colon w^\mu(\mathscr{H})\to Y$ are models for the holomorphic universal covering maps $\varpi_X\colon\widetilde{X}\to X$ and $\varpi_Y\colon\widetilde{Y}\to Y$ of the introduction. In these models the basepoint preserving lift $\tilde f$ of f is w^μ, and the variability set is

$$\widetilde{V}_f[x_0]=\big\{w^\nu(i):\nu\in[\mu]_X \text{ and } \nu \text{ is } X\text{-extremal}\big\}.$$

That set is mapped to $V=\big\{([\nu]_{X'}\in P_{x_0}^{-1}([\mu]_X):\nu \text{ is } X\text{-extremal}\big\}$ by the biholomorphic map $\zeta\mapsto\mathscr{B}^{-1}([\mu]_X,\zeta)$ of $w^\mu(\mathscr{H})$ $(=\widetilde{Y})$ onto $P_{x_0}^{-1}([\mu]_X)$.

It remains to show that V equals the set $V^\mu(x_0)$ defined by (4). This is easy. If ν in $M(\Gamma)$ is X-extremal then it is also X'-extremal, so

$$d_{X'}([0]_{X'},[\nu]_{X'})=\tfrac12\log K(\nu)=d_X([0]_X,[\nu]_X)$$

and $V\subset V^\mu(x_0)$. Conversely, if $[\nu]_{X'}$ is in $V^\mu(x_0)$ and σ in $[\nu]_{X'}$ is X'-extremal, then

$$\tfrac12\log K(\sigma)=d_{X'}([0]_{X'},[\nu]_{X'})=d_X([0]_X,[\nu]_X),$$

so σ is X-extremal. Since $[\nu]_{X'}=[\sigma]_{X'}$ it belongs to V, so $V^\mu(x_0)\subset V$. □

4. Boundary dilatations

Like Strebel's, our study of variability sets requires the notion of boundary dilatation, which we shall review in this section. We denote the characteristic function of a set S by χ_S.

For any μ in $M(\Gamma)$ set

$$\begin{aligned}
H_X^*(\mu)&=\inf\big\{K\big(\mu\chi_{\varpi^{-1}(X\setminus E)}\big): E \text{ is a compact subset of } X\big\},\\
H_{X'}^*(\mu)&=\inf\big\{K\big(\mu\chi_{\varpi^{-1}(X'\setminus E)}\big): E \text{ is a compact subset of } X'\big\},\\
H_X(\mu)&=\inf\big\{H_X^*(\nu):\nu\in[\mu]_X\big\},\qquad\text{and}\\
H_{X'}(\mu)&=\inf\big\{H_{X'}^*(\nu):\nu\in[\mu]_{X'}\big\}.
\end{aligned}\tag{5}$$

The numbers $H_X(\mu)$ and $H_{X'}(\mu)$ are called the *boundary dilatations* of μ (with respect to X and X').

We need to prove the intuitively obvious fact that $H_X(\mu)=H_{X'}(\mu)$ for all μ in $M(\Gamma)$. The inequalities $H_X^*(\mu)\le H_{X'}^*(\mu)\le K(\mu)$ and $H_X(\mu)\le H_{X'}(\mu)\le K(\mu)$ are clear from (5). Our proof that $H_{X'}(\mu)\le H_X(\mu)$ uses the following useful lemma, which is closely related to a result obtained by Aleksander Bulatovic in the course of the proof of Theorem 11 in his dissertation [3].

Lemma 2. *For any a and b in $\mathscr{H}$ and μ in $M(\Gamma)$ there exists ν in $[0]_X$ such that $H_X^*(\nu) = 1$, $w^\nu(a) = b$, and $\nu = \mu$ in some neighborhood of a.*

Proof. The set of restrictions to $\mathscr{H}$ of the maps w^ν with ν in $[0]_X$ and $H_X^*(\nu) = 1$ is a group G of quasiconformal self-mappings of $\mathscr{H}$. To show that for each a and b in $\mathscr{H}$ there is g in G with $g(a) = b$ it suffices to prove that the G-orbit of each a in $\mathscr{H}$ contains a neighborhood of a. For that purpose choose any a in $\mathscr{H}$ and choose $r > 0$ so small that the closed disk $\overline{D}(a; r)$ with center a and radius r is contained in $\mathscr{H}$ and the map $\varpi\colon \mathscr{H} \to X$ is injective on $\overline{D}(a; r)$.

For any t in Δ we define $g_t\colon \mathscr{H} \to \mathscr{H}$ as follows. For z in $\overline{D}(a; r)$ and γ in Γ set

$$g_t\bigl(\gamma(z)\bigr) = \gamma\bigl(z + t(r - |z - a|)\bigr),$$

and for z in $\mathscr{H} \setminus \bigcup_{\gamma\in\Gamma} \gamma\bigl(\overline{D}(a; r)\bigr)$ set $g_t(z) = z$. It is easy to see that $g_t \in G$. Since $g_t(a) = a + rt$, the G-orbit of a contains the interior of $\overline{D}(a; r)$, which is the required neighborhood of a.

Now let a and b in $\mathscr{H}$ and μ in $M(\Gamma)$ be given. We have already proved that there is g in G with $g(a) = b$. Let σ be the Beltrami coefficient of the quasiconformal map $w^\mu \circ g^{-1}$. Suppose for the moment that h in G fixes b and that its Beltrami coefficient equals σ in a neighborhood of b. Then the Beltrami coefficient ν of $h \circ g$ equals μ in a neighborhood of a. In addition $w^\nu = h \circ g$ in $\mathscr{H}$, so $\nu \in [0]_X$, $H_X^*(\nu) = 1$, and $w^\nu(a) = b$ as required.

It remains to produce h. Choose $r > 0$ so that the closed disk $\overline{D}(b; r)$ is contained in $\mathscr{H}$ and ϖ is injective on $\overline{D}(b; r)$. Set $\hat\sigma = \sigma\chi_{\overline{D}(b; r/3)}$. Let h_0 be the quasiconformal self-mapping of $\overline{D}\left(b; \frac{1}{2}r\right)$ that fixes b and $b + \frac{1}{2}r$ and has the Beltrami coefficient $\hat\sigma$ in $\overline{D}\left(b; \frac{1}{2}r\right)$. It is easy to extend h_0 to a quasiconformal self-mapping h of $\overline{D}(b; r)$ that equals the identity on the boundary. As above, we set $h\bigl(\gamma(z)\bigr) = \gamma\bigl(h(z)\bigr)$ for z in $\overline{D}(b; r)$ and γ in Γ, and we set $h(z) = z$ in $\mathscr{H} \setminus \bigcup_{\gamma\in\Gamma} \gamma\bigl(\overline{D}(b; r)\bigr)$. Clearly $h \in G$, $h(b) = b$, and the Beltrami coefficient of h equals σ in a neighborhood of b. □

Corollary 1. *We have $H_X(\mu) = H_{X'}(\mu)$ for all μ in $M(\Gamma)$.*

Proof. Let μ in $M(\Gamma)$ be given. Suppose $\sigma \in [\mu]_X$. Let a be the point in $\mathscr{H}$ where $w^\sigma(a) = w^\mu(i)$. By Lemma 2, there is ν in $[0]_X$ such that $H_X^*(\nu) = 1$, $w^\nu(a) = i$, and $\nu = \sigma$ in a neighborhood of a.

Define ϱ in $M(\Gamma)$ by the equation $w^\varrho \circ w^\nu = w^\sigma$. Then $w^\varrho(i) = w^\sigma(a) = w^\mu(i)$ and $\varrho \in [\sigma]_X = [\mu]_X$, so $\varrho \in [\mu]_{X'}$. Since $\varrho = 0$ in a neighborhood of i and $H_X^*(\nu) = 1$, $H_{X'}^*(\varrho) = H_X^*(\varrho) = H_X^*(\sigma)$. Thus for each σ in $[\mu]_X$ there is ϱ in $[\mu]_{X'}$ with $H_{X'}^*(\varrho) = H_X^*(\sigma)$, so $H_{X'}(\mu) \le H_X(\mu)$. We already know that $H_X(\mu) \le H_{X'}(\mu)$. □

5. Strebel points and Theorem 3

Following the terminology of [7] and [10] we call the point $[\mu]_{X'}$ in $T(X')$ a *Strebel point* if every X'-extremal ν in $[\mu]_{X'}$ satisfies the strict inequality $K(\nu) > H_{X'}(\mu)$. The following lemma generalizes an important observation that was made by Strebel when X is the open unit disk (see [20] and [22]).

Lemma 3. *Every point in the open set*

$$W = \{[\mu]_{X'} \in T(X') : d_X([0]_X, [\mu]_X) < d_{X'}([0]_{X'}, [\mu]_{X'})\} \tag{6}$$

is a Strebel point.

Proof. Let ν in $[\mu]_{X'}$ be X'-extremal and let σ in $[\mu]_X$ be X-extremal. If $[\mu]_{X'} \in W$, then $K(\nu) > K(\sigma) \geq H^*_X(\sigma) \geq H_X(\mu) = H_{X'}(\mu)$ and $[\mu]_X$ is a Strebel point. Clearly (6) defines an open set. □

By (3), (4), and (6), $T(X') \setminus W$ is the union of the variability sets $V^\mu(x_0)$, μ in $M(\Gamma)$. That fact will allow us in Section 6 to deduce Theorem 1 from the following theorem.

Theorem 3. *The map* $\Psi\colon T(X') \to T(X) \times \mathbf{R}$ *defined by*

$$\Psi([\mu]_{X'}) = \big(P_{x_0}([\mu]_{X'}), d_{X'}([0]_{X'}, [\mu]_{X'})\big), \qquad \mu \in M(\Gamma), \tag{7}$$

is continuous and proper. Its image is the closed subset

$$\Psi\big(T(X')\big) = \{([\mu]_X, r) \in T(X) \times \mathbf{R} : d_X([0]_X, [\mu]_X) \leq r\} \tag{8}$$

of $T(X) \times \mathbf{R}$. *The inverse image of the interior of* $\Psi\big(T(X')\big)$ *is the set* W *defined by* (6). *The restriction of* Ψ *to* W *is a proper* C^1 *split submersion, and the inverse image under* Ψ *of each point in* $\Psi(W)$ *is a simple closed curve.*

Proof. The forgetful map P_{x_0} is holomorphic and the function $d_{X'}([0]_{X'}, \cdot\,)$ is continuous, so Ψ is continuous. To see that Ψ is proper, let $\{[\mu_n]_{X'}\}$ be a sequence in $T(X')$ such that the sequence $\{\Psi([\mu_n]_{X'})\}$ converges in $T(X) \times \mathbf{R}$. We must show that $\{[\mu_n]_{X'}\}$ has a convergent subsequence. We may assume that each μ_n is X'-extremal.

Since the sequence $\{d_{X'}([0]_{X'}, [\mu_n]_{X'})\}$ converges in $\mathbf{R}$, the numbers $K(\mu_n)$ are uniformly bounded. We can therefore pass to a subsequence $\{\mu_{n_j}\}$ such that $w^{\mu_{n_j}}$ converges uniformly on compact subsets of $\mathbf{C}$ to a quasiconformal map w^μ. In particular $w^{\mu_{n_j}}(i) \to w^\mu(i)$ as $j \to \infty$.

Since each μ_n belongs to $M(\Gamma)$ and the sequence $\{[\mu_n]_X\}$ converges in $T(X)$, it is easy to verify that μ belongs to $M(\Gamma)$ and $[\mu_{n_j}]_X \to [\mu]_X$ as $j \to \infty$. Therefore the sequence $\big\{\big([\mu_{n_j}]_X, w^{\mu_{n_j}}(i)\big)\big\}$ in $F(\Gamma)$ converges to $\big([\mu]_X, w^\mu(i)\big)$ as $j \to \infty$. Since the Bers isomorphism $\mathscr{B}$ is a homeomorphism, $\{[\mu_{n_j}]_{X'}\}$ is a convergent subsequence of $\{[\mu_n]_{X'}\}$, and Ψ is proper.

Now choose μ in $M(\Gamma)$ and consider the set

$$J = \{r \in \mathbf{R} : ([\mu]_X, r) \in \Psi(T(X'))\} = \{d_{X'}([0]_{X'}, [\nu]_{X'}) : [\nu]_{X'} \in P_{x_0}^{-1}([\mu]_X)\}.$$

Since P_{x_0} has connected fibers, J is an interval. It is unbounded because Ψ is proper and $\Psi^{-1}(\{[\mu]_X\} \times S)$ is the non-compact set $P_{x_0}^{-1}([\mu]_X)$ if $J \subset S \subset \mathbf{R}$. By (3), J is a subinterval of $[r_\mu, \infty)$, where $r_\mu = d_X([0]_X, [\mu]_X)$. If ν in $[\mu]_X$ is X-extremal, then $\Psi([\nu]_{X'}) = ([\mu]_X, r_\mu)$, so $r_\mu \in J$. That proves (8).

By (8), the open set $\Omega = \{([\mu]_X, r) \in T(X) \times \mathbf{R} : d_X([0]_X, [\mu]_X) < r\}$ is the interior of $\Psi(T(X'))$. By the definitions of Ψ and W, $\Psi^{-1}(\Omega) = W$. It follows that Ψ is a proper map of W onto Ω.

By Lemma 3, each point of W is a Strebel point, so by Corollary 2 in Section 6 of [10] the function $[\mu]_{X'} \mapsto d_{X'}([0]_{X'}, [\mu]_{X'})$ is C^1 on W. Therefore Ψ is a C^1 map of W onto its image. We must study its derivative in W.

For that purpose choose μ in $M(\Gamma)$ so that $[\mu]_{X'}$ belongs to W. Choose a Riemann surface Y and a quasiconformal map f of X onto Y whose Beltrami coefficient is the element of $M(X)$ determined by μ. Then f maps X' onto $Y' = Y \setminus \{f(x_0)\}$. The map f induces biholomorphic maps of $T(X)$ and $T(X')$ onto $T(Y)$ and $T(Y')$ respectively (see [5]), allowing us to identify the tangent spaces to $T(X')$ and $T(X)$ at $[\mu]_{X'}$ and $[\mu]_X$ with $Q(Y')^*$ and $Q(Y)^*$ respectively. Under that identification the derivative of P_{x_0} at $[\mu]_{X'}$ becomes the restriction map $l \mapsto l \mid Q(Y)$ from $Q(Y')^*$ to $Q(Y)^*$, as in Section 1 above.

Since $[\mu]_{X'}$ is a Strebel point, Strebel's frame mapping theorem (see [8]) implies that there is a unique X'-extremal Beltrami coefficient ν in $[\mu]_{X'}$, and w^ν induces a Teichmüller mapping f_0: $X' \to Y'$. The inverse mapping from Y' to X' is also a Teichmüller mapping, so its Beltrami coefficient equals $k|\varphi_\mu|/\varphi_\mu$, with $0 < k < 1$, for a uniquely determined φ_μ of norm one in $Q(Y')$.

Since $[\mu]_{X'}$ belongs to W, ν is not X-extremal. Therefore f_0 and f_0^{-1} cannot be extended to Teichmüller mappings between X and Y, so φ_μ has a pole at $f(x_0)$.

According to Section 6 of [10], the derivative at $[\mu]_{X'}$ of the function $\tau \mapsto d([0]_{X'}, \tau)$ on $T(X')$ is the map $l \mapsto \operatorname{Re}(-l(\varphi_\mu))$ from $Q(Y')^*$ to $\mathbf{R}$. Therefore $\Psi'([\mu]_{X'})$ is the $\mathbf{R}$-linear map from $Q(Y')^*$ to $Q(Y)^* \oplus \mathbf{R}$ defined by

$$\text{(9)} \qquad \Psi'([\mu]_{X'})(l) = \big(l \mid Q(Y), \operatorname{Re}(-l(\varphi_\mu))\big) \qquad \text{for all } l \in Q(Y')^*.$$

Since φ_μ has a pole at y_0, (9) implies that $\Psi'([\mu]_{X'})$ is a surjective map whose kernel is the real one-dimensional subspace of $Q(Y)^*$ generated by the linear functional that equals zero on $Q(Y)$ and maps φ_μ to i. Every finite dimensional subspace has a closed complement, and $[\mu]_{X'}$ is an arbitrary point of W, so the restriction of Ψ to W is a proper C^1 split submersion.

Finally, we must examine the fibers $\Psi^{-1}(p)$, $p \in \Psi(W)$. Each fiber is a compact one-dimensional real submanifold of W. It therefore has finitely many connected components, each of which is a simple closed curve in W. It follows readily from the implicit function theorem and the properness of Ψ that the number $n(p)$ of components of $\Psi^{-1}(p)$ is a locally constant function of p in $\Psi(W)$. The homeomorphism $([\mu]_X, r) \mapsto \big([\mu]_X, r - d_X([0]_X,[\mu]_X)\big)$ of $\Psi(W)$ onto $T(X)\times(0,\infty)$ shows that $\Psi(W)$ is connected, so $n = n(p)$ is independent of p.

Evaluating n requires a more detailed study of $\Psi^{-1}(p)$. Choose any μ in $M(\Gamma)$ and any positive number $r > r_\mu = d_X([0]_X,[\mu]_X)$. Then $p = ([\mu]_X, r)$ belongs to $\Psi(W)$, and $\Psi^{-1}(p)$ is contained in the submanifold $D^\mu = P_{x_0}^{-1}([\mu]_X)$ of $T(X')$. Define the function $h\colon D^\mu \to \mathbf{R}$ by $h([\nu]_{X'}) = d_{X'}([0]_{X'},[\nu]_{X'})$, $[\nu]_{X'}$ in D^μ. The definition (7) of Ψ implies that $\Psi^{-1}(p) = h^{-1}(r)$. Since Ψ is a C^1 split submersion in W, (7) also implies that h is C^1 with a non-vanishing gradient in the set $D^\mu \cap W = D^\mu \setminus V^\mu(x_0)$. Therefore h has no local maximum, and its only local minima are located at the points of $V^\mu(x_0)$.

Recall from Section 2 that D^μ is biholomorphically equivalent to the simply connected region $w^\mu(\mathscr{H})$, so we may think of D^μ as an open disk in $\mathbf{C}$. Let C be a component of $h^{-1}(r)$. Since C is a simple closed curve in D^μ, it bounds a closed disk D in D^μ. Since h has no local maximum, we must have $h(\tau) < r$ for all τ in the interior of D. Since the only local minima of h are at points of $V^\mu(x_0)$, the interior of D must contain at least one point of $V^\mu(x_0)$.

We conclude that there are n disjoint closed disks $D_1, D_2, \dots, D_n$ in D^μ such that $h(\tau) \le r$ for τ in $\bigcup_{1\le j\le n} D_j$, $h(\tau) \ne r$ for τ in $D^\mu \setminus \bigcup_{1\le j\le n} D_j$, and each D_j contains at least one point of $V^\mu(x_0)$. Since $D^\mu \setminus \bigcup_{1\le j\le n} D_j$ is connected and, by (8), h takes arbitrarily large values in D^μ, we can conclude further that $h(\tau) > r$ for τ in $D^\mu \setminus \bigcup_{1\le j\le n} D_j$.

To see that $n = 1$ we observe that $V^\mu(x_0)$ contains at least n points for any μ in $M(\Gamma)$ and that $V^0(x_0)$ consists of the single point $[0]_{X'}$. □

Remark 2. The proof of Proposition 4(A) in [4] shows that $\Psi\colon W \to \Psi(W)$ is topologically a locally trivial fibration. Since $\Psi(W)$ is contractible, the fibration is globally trivial and W is homeomorphic to the product of $\Psi(W)$ and the unit circle. As we do not need this result here, we shall not go into detail.

6. Proof of Theorem 1

Lemma 1 reduces the proof of Theorem 1 to showing that for each μ in $M(\Gamma)$ the set $V^\mu(x_0)$ is compact and the sets $V^\mu(x_0)$ and $P_{x_0}^{-1}([\mu]_X) \setminus V^\mu(x_0)$ are connected.

These properties of $V^\mu(x_0)$ follow easily from Theorem 3 and its proof. Consider again the fiber $D^\mu = P_{x_0}^{-1}([\mu]_X)$ and the function $h\colon D^\mu \to \mathbf{R}$ obtained by restricting the distance function $[\nu]_{X'} \mapsto d_{X'}([0]_{X'},[\nu]_{X'})$ to D^μ. By its definition (4), $V^\mu(x_0)$ is the set of points in D^μ where h attains its minimum value $r_\mu = d_X([0]_X,[\mu]_X)$.

In the proof of Theorem 3 we showed that for each $r > r_\mu$ the set $h^{-1}([r_\mu, r])$ is a closed Jordan domain whose boundary is the level set $h^{-1}(r)$. Since $V^\mu(x_0)$ is the intersection of these domains it is compact and connected. Since $D^\mu \setminus V^\mu(x_0)$ is the union of the annular regions $D^\mu \setminus h^{-1}([r_\mu, r])$, $r > r_\mu$, it is also connected. □

7. The Teichmüller metric on the fibers of the forgetful map

Lemma 1 and its proof made essential use of the fact that for each μ in $M(\Gamma)$ the Bers isomorphism induces a biholomorphic map between the region $w^\mu(\mathscr{H})$ and the fiber $P_{x_0}^{-1}([\mu])$ of the forgetful map. The question whether that map is an isometry with respect to the Poincaré metric ϱ_μ on $w^\mu(\mathscr{H})$ and the Teichmüller metric $d_{X'}$ on $P_{x_0}^{-1}([\mu])$ has been thoroughly investigated.

Kra [9] and Nag [16] independently showed that the answer is negative when $T(X)$ has positive finite dimension. Liu [14] extended the Kra–Nag result to all infinite dimensional Teichmüller spaces $T(X)$ with three exceptions: the cases when X is Δ, Δ with one puncture, or an annulus. In Section 8 we shall give a more explicit and elementary proof of the Kra, Nag, and Liu results, and in Section 9 we shall extend them to the remaining exceptional cases, thus obtaining the following theorem.

Theorem 4 (Kra–Nag–Liu). *Given μ in $M(\Gamma)$, the map $\phi\colon w^\mu(\mathscr{H}) \to T(X')$ defined by $\phi(\zeta) = \mathscr{B}^{-1}([\mu]_X, \zeta)$, $\zeta \in w^\mu(\mathscr{H})$, is not an isometry unless X is conformally equivalent to $\mathbf{C} \setminus \{0,1\}$.*

In the finite dimensional case part one of the following corollary was proved in [9] and [16], and the second part was proved in [6]. The infinite dimensional case follows immediately from Theorem 4 above and Theorem 5 of [6], as Liu and Yang pointed out in [15].

Corollary 2. *Given μ in $M(\Gamma)$, let $\phi\colon w^\mu(\mathscr{H}) \to T(X')$ be the map in Theorem 4. If X is not conformally equivalent to $\mathbf{C} \setminus \{0,1\}$, then*

$$d_{X'}\big(\phi(\zeta_1), \phi(\zeta_2)\big) < \varrho_\mu(\zeta_1, \zeta_2) \tag{10}$$

for any pair of distinct points ζ_1 and ζ_2 in $w^\mu(\mathscr{H})$. In addition,

$$\lim_{h\to 0+} \frac{d_{X'}\big(\phi(\zeta), \phi(\zeta + hv)\big)}{h} < \lim_{h\to 0+} \frac{\varrho_\mu(\zeta, \zeta + hv)}{h} \tag{11}$$

for any ζ in $w^\mu(\mathscr{H})$ and any nonzero v in $\mathbf{C}$.

Notice that h approaches zero through positive real values in (11). The metrics ϱ_μ on $w^\mu(\mathscr{H})$ and $d_{X'}$ on $T(X')$ are arc length metrics. Inequality (11) says that f strictly decreases the infinitesimal length of nonzero tangent vectors.

8. Proof of Theorem 4: the generic case

We begin with some general remarks. Let f be a quasiconformal map of X onto Y, and let $Y' = f(X')$. The biholomorphic maps from $T(X)$ onto $T(Y)$ and from $T(X')$ to $T(Y')$ that f induces (see [5]) preserve Teichmüller distances and respect the forgetful maps. Therefore, in the proof of Theorem 4 we can assume that μ is identically zero, and we can also choose any convenient basepoint x_0 on X.

When $\mu = 0$ the map $\phi\colon \mathscr{H} \to T(X')$ in Theorem 4 satisfies the equation $\mathscr{B}\big(\phi(\zeta)\big) = ([0]_X, \zeta)$ for all ζ in $\mathscr{H}$. Hence, if ζ in $\mathscr{H}$ and ν in $M(\Gamma)$ are given, then $\phi(\zeta) = [\nu]_{X'}$ if and only if $[\nu]_X = [0]_X$ and $w^\nu(i) = \zeta$.

Suppose ν in $M(\Gamma)$ satisfies

$$\nu \in [0]_X \qquad \text{and} \qquad \tfrac{1}{2}\log K(\nu) < \varrho_0\big(i, w^\nu(i)\big), \tag{12}$$

where ϱ_0 is the Poincaré metric on $\mathscr{H}$. Put $\zeta = w^\nu(i) (\in \mathscr{H})$. Then

$$d_{X'}\big(\phi(i), \phi(\zeta)\big) = d_{X'}([0]_{X'}, [\nu]_{X'}) \le \tfrac{1}{2}\log K(\nu) < \varrho_0(i, \zeta).$$

Therefore Theorem 4 will be proved as soon as we find ν in $M(\Gamma)$ satisfying (12).

In this section we shall consider hyperbolic Riemann surfaces X on which there is a simple closed geodesic C. We require the point $x_0 = \varpi(i)$ to lie on C and we choose the covering map ϖ so that the component of $\varpi^{-1}(C)$ that contains the point i is the positive imaginary axis.

The stabilizer of ∞ in the group Γ of covering transformations is the cyclic subgroup Γ_∞ of Γ generated by a transformation $z \mapsto cz$ with $c > 1$. Since the image of the imaginary axis under ϖ is a simple closed curve, the collar lemma (see for instance Theorem 11.7.1 in [1]) gives us a number α such that $0 < \alpha \le \frac{1}{2}\pi$ and the subregion

$$\Omega_\alpha = \big\{z = re^{i\theta} : r > 0 \text{ and } |\theta - \tfrac{1}{2}\pi| < \alpha\big\}$$

of $\mathscr{H}$ is precisely Γ_∞-invariant with respect to Γ. This means that $\gamma(\Omega_\alpha) = \Omega_\alpha$ for γ in Γ_∞ and $\gamma(\Omega_\alpha) \cap \Omega_\alpha$ is empty for γ in $\Gamma \setminus \Gamma_\infty$.

Now, given $t > 0$ we define a quasiconformal map f_t of Ω_α onto itself by

$$f_t(re^{i\theta}) = \begin{cases} re^{i\theta + t(\alpha + \theta - \pi/2)}, & r > 0 \text{ and } 0 \le \frac{1}{2}\pi - \theta < \alpha, \\ re^{i\theta + t(\alpha - \theta + \pi/2)}, & r > 0 \text{ and } 0 \le \theta - \frac{1}{2}\pi < \alpha. \end{cases} \tag{13}$$

It is obvious that $f_t(cz) = cf_t(z)$ for all z in Ω_α, so $f_t \circ \gamma = \gamma \circ f_t$ in Ω_α for all γ in Γ_∞. Since Ω_α is precisely Γ_∞-invariant, the formula

$$f_t\big(\gamma(z)\big) = \gamma\big(f_t(z)\big), \qquad z \in \Omega_\alpha \text{ and } \gamma \in \Gamma,$$

extends f_t to a well-defined quasiconformal map of $\bigcup_{\gamma\in\Gamma}\gamma(\Omega_\alpha)$ onto itself. We extend f_t to a quasiconformal map of $\mathbf{C}$ onto itself by setting $f_t(z) = z$ for z in the complement of $\bigcup_{\gamma\in\Gamma}\gamma(\Omega_\alpha)$.

Since the extended quasiconformal map $f_t\colon \mathbf{C} \to \mathbf{C}$ equals the identity in the complement of $\mathscr{H}$, $f_t = w^\nu$ for some ν in $M(\{I\})$. Since $w^\nu \circ \gamma = \gamma \circ w^\nu$ for all γ in Γ, ν belongs to $M(\Gamma)$. We shall verify that ν satisfies condition (12) for sufficiently large values of t.

First, $\nu \in [0]_X$ because $w^\nu = f_t$ is the identity on $\mathbf{R}$. Second, $w^\nu(i) = e^{\alpha t} i$ by (13), so $\varrho_0\bigl(i, w^\nu(i)\bigr) = \frac{1}{2}\alpha t$. Third, we must compute $K(\nu)$.

The definition of w^ν shows that ν has the same L^∞ norm on $\mathscr{H}$ and Ω_α. An elementary calculation using (13) shows that in Ω_α

$$\nu(re^{i\theta}) = \begin{cases} \left(\dfrac{ti}{2 - ti}\right) e^{2i\theta}, & r > 0 \text{ and } 0 \le \frac{1}{2}\pi - \theta < \alpha, \\ \left(\dfrac{-ti}{2 + ti}\right) e^{2i\theta}, & r > 0 \text{ and } 0 \le \theta - \frac{1}{2}\pi < \alpha. \end{cases}$$

Therefore $\|\nu\| = t/\sqrt{4 + t^2}$, $K(\nu) = \frac{1}{4}\bigl(\sqrt{4 + t^2} + t\bigr)^2$, and condition (12) holds if

$$2 \log\bigl(\tfrac{1}{2}\bigl(\sqrt{4 + t^2} + t\bigr)\bigr) < \alpha t. \tag{14}$$

Elementary calculus shows that the set of $t > 0$ where (14) holds is a nonempty open interval (t_0, ∞), so Theorem 4 holds for all hyperbolic Riemann surfaces X that have simple closed geodesics.

Remark 3. The maps f_t induce quasiconformal mappings of X onto itself that are closely related to the spins about C used in the proofs of Theorem 4 in [9], [14], and [16].

9. Proof of Theorem 4: the remaining cases

The only hyperbolic Riemann surfaces with no simple closed geodesics are conformally equivalent to Δ, $\Delta \setminus \{0\}$, or $\mathbf{C} \setminus \{0, 1\}$. Since Theorem 4 excludes the case when X is $\mathbf{C} \setminus \{0, 1\}$, we need only consider Δ and Δ' $(= \Delta \setminus \{0\})$. It is easy to handle these cases directly (see Section 11), but it is even easier to deduce them from the generic case. First we shall prove two elementary results.

Lemma 4. *Let X and Y be hyperbolic Riemann surfaces with basepoints x_0 and y_0 respectively, and let $f\colon X \to Y$ be a holomorphic covering map with $f(x_0) = y_0$. Let $\varpi_X\colon \mathscr{H} \to X$ be a holomorphic universal covering map with $\varpi_X(i) = x_0$, let $\varpi_Y = f \circ \varpi_X$, and let Γ_X and Γ_Y be the groups of covering transformations of ϖ_X and ϖ_Y respectively. Then $M(\Gamma_Y) \subset M(\Gamma_X)$, and*

$$d_{X'}([0]_{X'}, [\nu]_{X'}) \le d_{Y'}([0]_{Y'}, [\nu]_{Y'}) \qquad \text{for all } \nu \text{ in } M(\Gamma_Y), \tag{15}$$

where $X' = X \setminus \{x_0\}$ and $Y' = Y \setminus \{y_0\}$.

Proof. Since Γ_X is clearly a subgroup of Γ_Y, the inclusion $M(\Gamma_Y) \subset M(\Gamma_X)$ and the inequality (15) follow immediately from the definitions. □

Corollary 3. *Under the conditions of Lemma 4 let $\mathscr{B}_{X'}\colon T(X') \to F(\Gamma_X)$ and $\mathscr{B}_{Y'}\colon T(Y') \to F(\Gamma_Y)$ be the Bers isomorphisms associated with Γ_X and Γ_Y respectively. For ζ in $\mathscr{H}$, let $\phi_{X'}(\zeta) = \mathscr{B}_{X'}^{-1}([0]_X,\zeta)$ and $\phi_{Y'}(\zeta) = \mathscr{B}_{Y'}^{-1}([0]_Y,\zeta)$. Then*

$$d_{X'}\big(\phi_{X'}(i),\phi_{X'}(\zeta)\big) \le d_{Y'}\big(\phi_{Y'}(i),\phi_{Y'}(\zeta)\big) \qquad \text{for all } \zeta \text{ in } \mathscr{H}. \tag{16}$$

Proof. If $\zeta \in \mathscr{H}$, $\nu \in M(\Gamma_Y)$, and $\phi_{Y'}(\zeta) = [\nu]_{Y'}$, then $\phi_{X'}(\zeta) = [\nu]_{X'}$. Therefore (15) implies (16). □

We shall apply these results with Y equal to the quotient of $\mathscr{H}$ by the Fuchsian group Γ_Y generated by the transformations $z \mapsto z+1$ and $z \mapsto z/(5z+1)$. We take X to be the quotient space $\mathscr{H}/\Gamma_X$, where Γ_X is either the trivial subgroup of Γ_Y or the subgroup generated by $z \mapsto z+1$. We denote the quotient maps from $\mathscr{H}$ to X and Y by ϖ_X and ϖ_Y respectively, give X and Y the basepoints $\varpi_X(i)$ and $\varpi_Y(i)$, and define $f\colon X \to Y$ by setting $f\big(\varpi_X(z)\big) = \varpi_Y(z)$, z in $\mathscr{H}$.

Since Y is conformally equivalent to a twice punctured disk, it contains a simple closed geodesic. Therefore Theorem 4 and Corollary 2 hold for Y.

Choose any $\zeta \ne i$ in $\mathscr{H}$, and let ϱ_0 be the Poincaré metric on $\mathscr{H}$. Inequality (10) in Corollary 2 gives

$$d_{Y'}\big(\phi_{Y'}(i),\phi_{Y'}(\zeta)\big) < \varrho_0(i,\zeta).$$

Combining this inequality with the inequality (16) we obtain

$$d_{X'}\big(\phi_{X'}(i),\phi_{X'}(\zeta)\big) < \varrho_0(i,\zeta),$$

which proves Theorem 4 for X. Since our choices of Γ_X allow us to make X conformally equivalent to either Δ or $\Delta \setminus \{0\}$, the proof of Theorem 4 is complete.

10. Proof of Theorem 2

Using appropriate biholomorphic maps between Teichmüller spaces as in the proof of Theorem 4, we may assume that a is the basepoint $[0]_{X'}$ of $T(X')$. We choose an X'-extremal μ in $M(\Gamma)$ so that $b = [\mu]_{X'}$. By hypothesis $P_{x_0}(a) = P_{x_0}(b)$ and $a \ne b$, so $[\mu]_X = [0]_X$ and $[\mu]_{X'} \ne [0]_{X'}$. Therefore Lemma 3 implies that $[\mu]_{X'}$ is a Strebel point of $T(X')$. Since μ is X'-extremal, Strebel's frame mapping theorem (see [8]) implies that μ is the unique X'-extremal Beltrami coefficient in $[\mu]_{X'}$ and that μ has the special Teichmüller form described in the proof of Theorem 3 in Section 5.

Now let ν in $M(\Gamma)$ be X'-extremal and let $c = [\nu]_{X'}$ be distinct from a and b. Since μ is uniquely X'-extremal and has constant absolute value, a calculation due to Li Zhong (see the proof of Theorem 3 in [12]) shows that

$$d_{X'}(a,b) = d_{X'}(a,c) + d_{X'}(c,b)$$

if and only $\nu = r\mu$ for some real r with $0 < r < 1$. We shall now obtain a contradiction from the assumption that $P_{x_0}^{-1}([0]_X)$ contains infinitely many such points $c = [r\mu]_{X'}$.

Consider first the holomorphic map $f(t) = [t\mu/\|\mu\|]_{X'}$ from Δ into $T(X')$. Let g be any holomorphic function on $T(X)$ such that $g([0]_X) = 0$. By our assumption there are infinitely many numbers r in $[0,1]$ such that $g\big(P_{x_0}\big(f(r\|\mu\|)\big)\big) = 0$. Hence $g \circ P_{x_0} \circ f$ has infinitely many zeros in the closed disk $\overline{D}(0; \|\mu\|)$, so it is identically zero. The Bers embedding (see [8] or [17]) and the Hahn–Banach theorem imply that the holomorphic functions on $T(X)$ separate points, so our assumption implies that $P_{x_0}\big(f(t)\big) = [0]_X$ for all t in Δ.

Now recall the biholomorphic map $\phi(\zeta) = \mathscr{B}^{-1}([0]_X, \zeta)$ from $\mathscr{H}$ to $P_{x_0}^{-1}([0]_X)$ that we studied in Theorem 4. Since f maps Δ into $P_{x_0}^{-1}([0]_X)$, there is a holomorphic map h from Δ to $\mathscr{H}$ such that $f = \phi \circ h$. Let ϱ_Δ be the Poincaré metric on Δ. Inequality (10) and the Schwarz–Pick lemma imply that

$$(17) \quad d_{X'}([0]_{X'}, [\mu]_{X'}) = d_{X'}\big(\phi\big(h(0)\big), \phi\big(h(\|\mu\|)\big)\big) < \varrho_\Delta(0, \|\mu\|) = \tfrac{1}{2}\log K(\mu).$$

Since μ is X'-extremal, the inequality (17) provides the desired contradiction. □

11. An example

Finally, we shall use some explicit extremal quasiconformal mappings to reprove Theorem 4 for the nongeneric cases $X = \Delta$ and $Y = \Delta \setminus \{0\}$. Our computations will also show that the ratio of the two sides of the inequality (15) can be arbitrarily large.

Let Γ_Y be the cyclic Fuchsian group generated by $z \mapsto z+1$, Γ_X be the trivial subgroup of Γ_Y, and ϱ_0 be the Poincaré metric on $\mathscr{H}$. For each positive integer n let ν_n be the constant function $\nu_n(z) = -n/(n+2i)$, z in $\mathscr{H}$.

Observe that ν_n belongs to $M(\Gamma_Y)$ ($\subset M(\Gamma_X)$). Since $w^{\nu_n}(z) = z + \mathrm{Im}(nz)$ for all z in $\mathscr{H}$, we have $[\nu_n]_X = [0]_X$, $[\nu_n]_Y = [0]_Y$, and $w^{\nu_n}(i) = i+n$ for all n. We shall compare the numbers $\varrho_0\big(i, w^{\nu_n}(i)\big)$, $d_{Y'}([0]_{Y'}, [\nu_n]_{Y'})$, and $d_{X'}([0]_{X'}, [\nu_n]_{X'})$.

The number $\varrho_0\big(i, w^{\nu_n}(i)\big)$ is easily seen to equal $\log\big(\frac{1}{2}\big(\sqrt{n^2+4}+n\big)\big)$, which is asymptotic to $\log n$ as $n \to \infty$.

The number $d_{X'}([0]_{X'}, [\nu_n]_{X'})$ equals $\frac{1}{2}\log K_n$, where K_n is the maximal dilatation of the extremal quasiconformal mapping of $\mathscr{H}$ onto itself that fixes the extended real axis pointwise and maps i to $i+n$. Both Teichmüller [23] and Reich [18] give elegant geometric constructions of that map. We shall follow Section 4 of [18] because of misprints on the last page of [23], where every K or D in a displayed formula must be replaced by its square root.

The first step is to map $\mathscr{H}$ one-to-one and conformally onto Δ so that i and $i+n$ go to 0 and $r_n = |n/(n+2i)|$ respectively. Next we map the region $\Delta \setminus [0, r_n]$ one-to-one and conformally to an annulus $\mathscr{A}_n = \{\zeta : 1 < |\zeta| < R_n\}$, sending 0 and r_n to -1 and 1 respectively. The map $w = \frac{1}{2}(\zeta + \zeta^{-1})$ carries $\mathscr{A}_n$

to $E_n\setminus[-1,1]$, where E_n is the interior of the ellipse whose axes are the segments $[-\frac{1}{2}(R_n+R_n^{-1}),\frac{1}{2}(R_n+R_n^{-1})]$ and $[-\frac{1}{2}i(R_n-R_n^{-1}),\frac{1}{2}i(R_n-R_n^{-1})]$. The composed map from $\Delta\setminus[0,r_n]$ to $E_n\setminus[-1,1]$ extends to a conformal homeomorphism f_n of Δ onto E_n. Composing f_n with the original map from $\mathscr{H}$ to Δ we obtain a conformal homeomorphism g_n of $\mathscr{H}$ onto E_n with $g_n(i)=-1$ and $g_n(i+n)=1$.

It is easy to verify (see Theorem 4 of Reich [18]) that the desired extremal map of $\mathscr{H}$ onto itself is $g_n^{-1}\circ h_n\circ g_n$, where h_n is the mapping of E_n onto itself given by

$$h_n(w)=\frac{(R_n^2+1)^2}{(R_n^2-1)^2}\left((w+1)-\frac{4R_n}{R_n^2+1}|w+1|+\frac{4R_n^2}{(R_n^2+1)^2}(\overline{w+1})\right)+1.$$

An easy calculation shows that the Beltrami coefficient of h_n is

$$-\frac{2R_n}{R_n^2+1}\frac{w+1}{|w+1|},$$

so

$$K_n=\left(\frac{R_n+1}{R_n-1}\right)^2 \quad\text{and}\quad d_{X'}([0]_{X'},[\nu_n]_{X'})=\frac{1}{2}\log K_n=\log\left(\frac{R_n+1}{R_n-1}\right). \tag{18}$$

A study of R_n as a function of r_n can be found in Section 2 of Chapter II of [11], where $\log R_n$ is denoted by $\mu(r_n)$. According to equations (2.7) and (2.11) in that chapter

$$\mu(r_n)\mu\big(\sqrt{1-r_n^2}\,\big)=\pi^2/4 \qquad\text{and}\qquad \lim_{n\to\infty}\big(\mu\big(\sqrt{1-r_n^2}\,\big)-\log\big(4/\sqrt{1-r_n^2}\,\big)\big)=0.$$

These equations, together with the definition of r_n and equation (18) above, imply that $d_{X'}([0]_{X'},[\nu_n]_{X'})$ is asymptotic to $\log\log n$ as $n\to\infty$.

It remains to study $d_{Y'}([0]_{Y'},[\nu_n]_{Y'})$. Our universal covering map from $\mathscr{H}$ to $Y=\Delta\setminus\{0\}$ will be $\varpi(z)=e^{2\pi iz}$, the basepoint of Y will be $\varpi(i)=e^{-2\pi}$, and Y' will equal $Y\setminus\{e^{-2\pi}\}$. The restriction of w^{ν_n} to $\mathscr{H}$ is a lift of the quasiconformal self-mapping

$$f_n(\zeta)=\zeta e^{-ni\log|\zeta|},\quad \zeta\in Y,$$

of Y. Clearly f_n maps Y' onto itself and $d_{Y'}([0]_{Y'},[\nu_n]_{Y'})$ equals $\frac{1}{2}\log K_n$, where K_n is the maximal dilatation of the extremal quasiconformal self-mapping of Y' that is Teichmüller equivalent to f_n in Y'.

Construction of the extremal mapping again involves mapping a slit disk onto an annulus. Let $w=g(\zeta)$ map $\Delta\setminus[0,e^{-2\pi}]$ one-to-one and conformally to the annulus $\mathscr{A}=\{w:1<|w|<R\}$, sending 0 and $e^{-2\pi}$ to -1 and 1 respectively.

The quasiconformal self-mapping

$$h_n(w)=we^{-2\pi in\log|w|/\log R},\qquad w\in\mathscr{A}, \tag{19}$$

of $\mathscr{A}$ fixes the boundary of $\mathscr{A}$ pointwise, so $g^{-1}\circ h_n\circ g$ extends to a quasiconformal self-mapping of Δ that fixes the segment $[0,e^{-2\pi}]$ pointwise. The restriction of that mapping to Y' is the desired extremal mapping, for it is easily seen to be Teichmüller equivalent to f_n in Y' and, as we shall now verify, it is a Teichmüller mapping of Y' onto itself.

First consider the mapping h_n. By (19) its Beltrami coefficient is the function

$$\sigma_n(w)=\left(\frac{-\pi in}{\log R-\pi in}\right)\frac{w}{\overline{w}},\qquad w\in\mathscr{A}, \tag{20}$$

so h_n is a Teichmüller mapping whose associated quadratic differential is a (complex) scalar multiple of $-dw^2/w^2$. The self-mapping $g^{-1}\circ h_n\circ g$ of $\Delta\setminus[0,e^{-2\pi}]$ is therefore a Teichmüller mapping, and its associated quadratic differential is a scalar multiple of $q=-dg^2/g^2$.

Now q is determined up to a positive multiple by the properties that it has no zeros in $\Delta\setminus[0,e^{-2\pi}]$ and all its horizontal trajectories in $\Delta\setminus[0,e^{-2\pi}]$ are simple closed curves (see Section 9 of [21]). The quadratic differential

$$q_0=\frac{d\zeta^2}{\zeta(\zeta-e^{-2\pi})(e^{-2\pi}\zeta-1)},\qquad \zeta\in\mathbf{C}\setminus\{0,e^{-2\pi},e^{2\pi}\},$$

has all these properties. Indeed, either direct calculation or a symmetry argument will show that the unit circle is a horizontal trajectory of q_0, and all non-critical trajectories of q_0 are simple closed curves because the only critical horizontal trajectories of q_0 in the extended plane are the segments $[0,e^{-2\pi}]$ and $[e^{2\pi},\infty]$ on the extended real axis (see Section 12.2 of [21]). Therefore q is a positive multiple of q_0.

Since q extends to an integrable holomorphic quadratic differential on Y' the extension of $g^{-1}\circ h_n\circ g$ to Y' is a Teichmüller mapping, as we claimed. Since the extremal mapping equals $g^{-1}\circ h_n\circ g$ almost everywhere in Y' we can use equation (20) to compute its maximal dilatation K_n. We find that

$$d_{Y'}([0]_{Y'},[\nu_n]_{Y'})=\frac{1}{2}\log K_n=\log\left(\frac{\sqrt{(\pi n)^2+(\log R)^2}\,+\pi n}{\log R}\right).$$

Thus $d_{Y'}([0]_{Y'},[\nu_n]_{Y'})$ and $\varrho_0\bigl(i,w^{\nu_n}(i)\bigr)$ are both asymptotic to $\log n$ as $n\to\infty$, while $d_{X'}([0]_{X'},[\nu_n]_{X'})$ is asymptotic to $\log\log n$.

Finally, since the module $(1/2\pi)\log R$ of $\Delta\setminus[0,e^{-2\pi}]$ is greater than one, our formulas for $d_{Y'}([0]_{Y'},[\nu_n]_{Y'})$ and $\varrho_0\bigl(i,w^{\nu_n}(i)\bigr)$ imply that

$$d_{Y'}([0]_{Y'},[\nu_n]_{Y'})<\varrho_0\bigl(i,w^{\nu_n}(i)\bigr)\qquad\text{for each } n\ge 1.$$

References

[1] Beardon, A.F.: The Geometry of Discrete Groups. - Springer-Verlag, New York–Berlin–Heidelberg, 1983.

[2] Bers, L.: Fiber spaces over Teichmüller spaces. - Acta Math. 130, 1973, 89–126.
[3] Bulatovic, A.: Forgetful mappings between Teichmüller spaces. - Ph.D. Dissertation, Graduate School and University Center, CUNY, New York, NY, 1999.
[4] Earle, C.J., and J. Eells, Jr.: Foliations and fibrations. - J. Differential Geom. 1, 1967, 33–41.
[5] Earle, C.J., and F.P. Gardiner: Geometric isomorphisms between infinite dimensional Teichmüller spaces. - Trans. Amer. Math. Soc. 348, 1996, 1163–1190.
[6] Earle, C.J., I. Kra, and S.L. Krushkal′: Holomorphic motions and Teichmüller spaces. - Trans. Amer. Math. Soc. 343, 1994, 927–948.
[7] Earle, C.J., and Z. Li: Isometrically embedded polydisks in infinite dimensional Teichmüller spaces. - J. Geom. Anal. 9, 1999, 51–71.
[8] Gardiner, F.P., and N. Lakic: Quasiconformal Teichmüller Theory. - American Mathematical Society, Providence, RI, 2000.
[9] Kra, I.: On the Nielsen–Thurston–Bers type of some self-maps of Riemann surfaces. - Acta Math. 146, 1981, 231–270.
[10] Lakic, N.: Strebel points. - In: Lipa's Legacy, Contemp. Math. 211, 1997, 417–431.
[11] Lehto, O., and K.I. Virtanen: Quasiconformal Mappings in the Plane. - Springer-Verlag, Berlin-Heidelberg–New York, 1973.
[12] Li, Z.: Nonuniqueness of geodesics in infinite dimensional Teichmüller spaces. - Complex Variables Theory Appl. 16, 1991, 261–272.
[13] Li, Z.: Strebel differentials and Hamilton sequences. - Sci. China Ser. A (to appear).
[14] Liu, L.X.: Invariant metrics in infinite-dimensional Teichmüller space. - Complex Variables Theory Appl. 25, 1994, 337–349.
[15] Liu, L.X., and X. Yang: On some properties of infinite dimensional Bers fiber space. - Northeast. Math. J. 15, 1999, 203–208.
[16] Nag, S.: Nongeodesic discs embedded in Teichmüller spaces. - Amer. J. Math. 104, 1982, 399–408.
[17] Nag, S.: The Complex Analytic Theory of Teichmüller Spaces. - John Wiley and Sons, New York, 1988.
[18] Reich, E.: On the mapping with complex dilatation $ke^{i\theta}$. - Ann. Acad. Sci. Fenn. Math. 12, 1987, 261–267.
[19] Strebel, K.: Zur Frage der Eindeutigkeit extremaler quasikonformer Abbildungen des Einheitskreises. - Comment. Math. Helv. 36, 1962, 306–323.
[20] Strebel, K.: On the existence of extremal Teichmüller mappings. - J. Analyse Math. 30, 1976, 464–480.
[21] Strebel, K.: Quadratic Differentials. - Springer-Verlag, Berlin–Heidelberg–New York, 1984.
[22] Strebel, K.: Point shift differentials and extremal quasiconformal mappings. - Ann. Acad. Sci. Fenn. Math. 23, 1998, 475–494.
[23] Teichmüller, O.: Ein Verschiebungssatz der quasikonformen Abbildung. - Deutsche Math. 7, 1944, 336–343.

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