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ON THE CONNECTEDNESS OF THE LOCUS OF REAL RIEMANN SURFACES

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Abstract. It is well known that the functorial equivalence between pairs (X,σ), where X is a Riemann surface which admits an antiholomorphic involution (symmetry) $\sigma\colon X \to X$, and real algebraic curves. We shall refer to such Riemann surfaces as real Riemann surfaces, following Klein's terminology. We consider the sets $\mathcal{M}_g^{\mathcal{R}}$ and $\mathcal{M}_g^{2\mathcal{R}}$ of real curves and real hyperelliptic curves, respectively in the moduli space $\mathcal{M}_g$ of complex algebraic curves of genus g.

In this paper we prove that any real hyperelliptic Riemann surface can be quasiconformally deformed, preserving the real and hyperelliptic character, to a real hyperelliptic Riemann surface (X,σ), such that X admits a symmetry τ, where $\mathrm{Fix}(\tau)$ is connected and non-separating. As a consequence, we obtain the connectedness of the sets $\mathcal{M}_g^{2\mathcal{R}}(\subset \mathcal{M}_g)$ of all real hyperelliptic Riemann surfaces of genus g and $\mathcal{M}_g^{\mathcal{R}}(\subset \mathcal{M}_g)$ of all real Riemann surfaces of given genus g using a procedure different from the one given by Seppälä for $\mathcal{M}_g^{2\mathcal{R}}$ and Buser, Seppälä and Silhol for $\mathcal{M}_g^{\mathcal{R}}$.

A Riemann surface X is called a p-gonal Riemann surface, where p is a prime, if there exists a p-fold covering map from X onto the Riemann sphere. We prove in this paper that the subset of real p-gonal Riemann surfaces, $p \geq 3$, is not a connected subset of $\mathcal{M}_g$ in general. This generalizes a result of Gross and Harris for real trigonal algebraic curves.

1. Introduction

Let X_g be a compact Riemann surface of genus $g \geq 2$. A *symmetry* σ of X_g is an anticonformal involution of X_g. The topological type of a symmetry is determined by properties of its fixed-point set $\mathrm{Fix}(\sigma)$. By Harnack's theorem the fixed-point set of σ consists of $k \leq g+1$ Jordan curves, called *ovals*. The space $X_g - \mathrm{Fix}(\sigma)$ consists of one component if the quotient Klein surface $X_g/\langle\sigma\rangle$ is non-orientable and of two components if it is orientable. Let σ be a symmetry

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of X_g with k ovals, then the *species* of σ is $+k$ or $-k$ according to whether $X_g - \mathrm{Fix}(\sigma)$ has two or one component, respectively.

Whereas a compact Riemann surface corresponds to a complex algebraic curve, a compact Riemann surface X_g with a symmetry σ corresponds to a real algebraic curve. Each conjugacy class of symmetries in $\mathrm{Aut}(X_g)$ corresponds to an equivalence class, under real birational isomorphisms, of real algebraic curves, a real form of the complex algebraic curve. The ovals of the symmetry correspond to the graph components of the real form. So, if a Riemann surface X_g admits two non-conjugate symmetries σ_1, σ_2 with k_1 and k_2 ovals, respectively, then the complex algebraic curve corresponding to X_g has two real forms with k_1 and k_2 components, respectively.

A Riemann surface X_g is called a *cyclic p-gonal Riemann surface*, where p is a prime, if X_g is a cyclic p-fold covering of the Riemann sphere. When $p=2$ the surface X_g is called hyperelliptic.

We study in this paper the sets $\mathscr{M}_g^{p\mathscr{R}}$ of complex isomorphism classes of real cyclic p-gonal curves of genus g by means of their uniformization groups. The study of moduli spaces of real algebraic curves was initiated by Felix Klein [21]. Seppälä [23] proved that $\mathscr{M}_g^{2\mathscr{R}}$ is connected and Buser, Seppälä and Silhol [8] proved that $\mathscr{M}_g^{\mathscr{R}}$ is connected, using the fact that $\mathscr{M}_g^{2\mathscr{R}}$ is a connected subset of $\mathscr{M}_g$. There is another proof of this fact in [12] with the techniques described in [13]. We also give, for the sake of completeness, a proof of the connectedness of $\mathscr{M}_g^{\mathscr{R}}$ different from the ones given in [8] and in [12], and following the ideas in [10].

Let $\mathscr{M}_{\varepsilon k}^g$ be the subset of $\mathscr{M}_g^{\mathscr{R}}$ formed by all Riemann surfaces admitting a symmetry with species εk, where $\varepsilon = \pm$ and k is the number of ovals. The spaces $\mathscr{M}_{\varepsilon k}^g$ and $\mathscr{M}_{\varepsilon k}^g \cap \mathscr{M}_g^{2\mathscr{R}}$ are connected (see [11], [8] and Theorems 2.1 and 2.2). In this paper we prove not only that $\mathscr{M}_g^{2\mathscr{R}}$ is connected, but that $\mathscr{M}_{-1}^g \cap \mathscr{M}_g^{2\mathscr{R}}$, the subset formed by all real hyperelliptic Riemann surfaces admitting a non-separating symmetry with one oval, cuts every $\mathscr{M}_{\varepsilon k}^g \cap \mathscr{M}_g^{2\mathscr{R}}$ for any possible species εk for a given genus g. We shall say that $\mathscr{M}_{-1}^g \cap \mathscr{M}_g^{2\mathscr{R}}$ is a spine for $\mathscr{M}_g^{2\mathscr{R}}$. The above property not only implies the connectedness of $\mathscr{M}_g^{2\mathscr{R}}$, but also gives a way to connect any pair of points in $\mathscr{M}_g^{2\mathscr{R}}$. In the same way we show that $\mathscr{M}_{-1}^g$ is a spine for the space $\mathscr{M}_g^{\mathscr{R}}$ (see [10]).

The above result has been inspired by the following fact on elliptic curves: the set of real elliptic curves defined by rombic lattices, i.e., admitting a symmetry with one nonseparating oval, is a spine for the locus of real elliptic curves (see [20]).

On the contrary, we shall prove that the set $\mathscr{M}_g^{p\mathscr{R}}$, $p \geq 3$, of real cyclic p-gonal Riemann surfaces is not connected in general. This generalizes a result of Gross and Harris for real, trigonal algebraic curves [16]. As a consequence the set of real p-gonal Riemann surfaces is not connected using Lemma 2.1 in [1].

The results presented in this work imply the following fact on equations of algebraic curves. Given two algebraic curves admitting polynomial equations $y^p = P(x)$ and $y^p = Q(x)$ with real coefficients, we shall consider two types of allowed deformations for such equations. The first type of deformation is to modify continuously the real coefficients of $P(x)$ and $Q(x)$. The second type is to change the real form of a fixed complex algebraic curve. Then if $p = 2$ it is always possible to go from a curve to the other one, but this is not the case in general if $p > 2$.

2. NEC groups and moduli spaces of Riemann surfaces

Let X_g be a compact Riemann surface of genus $g \geq 2$. The surface X_g can be represented as a quotient $X_g = \mathscr{H}/\Gamma$ of the upper half plane $\mathscr{H}$ under the action of a surface Fuchsian group Γ, that is, a cocompact orientation-preserving subgroup of the group $\mathscr{G} = \mathrm{Aut}(\mathscr{H})$ of conformal and anticonformal automorphisms of $\mathscr{H}$ without elliptic elements. A discrete, cocompact subgroup Γ of $\mathrm{Aut}(\mathscr{H})$ is called an *NEC* (*non-euclidean crystallographic*) *group*. The subgroup of Γ consisting of the orientation-preserving elements is called the *canonical Fuchsian subgroup of* Γ; it is denoted by Γ^+. The algebraic structure of an NEC group and the geometric structure of its quotient orbifold are given by the signature of Γ:

$$s(\Gamma) = \big(h; \pm; [m_1, \dots, m_r]; \big\{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\big\}\big). \tag{2.1}$$

The orbit space $\mathscr{H}/\Gamma$ is an orbifold with underlying surface of genus h, having r cone points and k boundary components, each with $s_j \geq 0$ corner points. The signs "$+$" and "$-$" correspond to orientable and non-orientable orbifolds, respectively. The integers m_i are called the proper periods of Γ and they are the orders of the cone points of $\mathscr{H}/\Gamma$. The brackets $(n_{i1}, \dots, n_{is_i})$ are the period cycles of Γ and the integers n_{ij} are the link periods of Γ and the orders of the corner points of $\mathscr{H}/\Gamma$. The group Γ is called the *fundamental group* of the orbifold $\mathscr{H}/\Gamma$.

A group Γ with signature (2.1) has a *canonical presentation* with generators:

$$x_1, \dots, x_r, e_1, \dots, e_k, c_{ij}, \qquad 1 \leq i \leq k,\ 1 \leq j \leq s_i + 1,$$

$$\text{and } a_1, b_1, \dots, a_h, b_h \text{ if } \mathscr{H}/\Gamma \text{ is orientable or } d_1, \dots, d_h \text{ otherwise,}$$

and relators:

$$x_i^{m_i}, i = 1, \dots, r, \quad c_{ij}^2, (c_{ij-1}c_{ij})^{n_{ij}}, c_{i0}e_i^{-1}c_{is_i}e_i, \quad i = 1, \dots, k,\ j = 2, \dots, s_i + 1,$$

and $x_1 \cdots x_r e_1 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_h b_h a_h^{-1} b_h^{-1}$ or $x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_h^2$, according to whether $\mathscr{H}/\Gamma$ is orientable or not.

This last relation is called the long relation.

The hyperbolic area of the orbifold $\mathscr{H}/\Gamma$ coincides with the hyperbolic area of an arbitrary fundamental region of Γ and equals:

$$(2.2) \qquad \mu(\Gamma) = 2\pi\left(\varepsilon h - 2 + k + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right)\right),$$

where $\varepsilon = 2$ if there is a "+" sign and $\varepsilon = 1$ otherwise. If Γ' is a subgroup of Γ of finite index then Γ' is an NEC group and the following Riemann–Hurwitz formula holds:

$$(2.3) \qquad [\Gamma : \Gamma'] = \mu(\Gamma')/\mu(\Gamma).$$

An NEC group Γ without elliptic elements is called a *surface group* and it has signature $\big(h; \pm; [-], \{(-), \overset{k}{\ldots}, (-)\}\big)$. In such a case $\mathscr{H}/\Gamma$ is a *Klein surface*, i.e., a surface with a dianalytic structure of topological genus h, orientable or not according to the sign "+" or "−", and having k boundary components. Conversely, a Klein surface whose complex double has genus greater than one can be expressed as $\mathscr{H}/\Gamma$ for some NEC surface group Γ. Furthermore, given a Riemann (respectively, Klein) surface represented as the orbit space $X = \mathscr{H}/\Gamma$, with Γ a surface group, a finite group G is a group of automorphisms of X if and only if there exists an NEC group Δ and an epimorphism $\theta\colon \Delta \to G$ with $\ker(\theta) = \Gamma$ (see [6]). The NEC group Δ is the lifting of G to the universal covering $\pi\colon \mathscr{H} \to \mathscr{H}/\Gamma$ and is called the *the universal covering transformation group* of (X, G).

Given an NEC group Γ, we denote by $\mathbf{R}(\Gamma)$ the set of monomorphisms $r\colon \Gamma \to \mathrm{Aut}(\mathscr{H})$ such that $r(\Gamma)$ is discrete and cocompact. Two elements $r_1, r_2 \in \mathbf{R}(\Gamma)$ are said to be equivalent if there exists $g \in \mathrm{Aut}(\mathscr{H})$ such that for each $\gamma \in \Gamma$, $r_1(\gamma) = gr_2(\gamma)g^{-1}$. The orbit space $\mathbf{T}(\Gamma)$ is called the *Teichmüller space of* Γ and it is homeomorphic to a real ball. Notice that $g \in \mathrm{Aut}(\mathscr{H})$, where $\mathrm{Aut}(\mathscr{H})$ are conformal and anticonformal automorphisms, then the groups uniformising the same orientable (but not oriented) Riemann surface appear in the same Teichmüller space. Let $\Gamma \leq \Delta$ be NEC groups, the inclusion mapping $i\colon \Gamma \to \Delta$ induces an embedding $m_i\colon \mathbf{T}(\Delta) \to \mathbf{T}(\Gamma)$ defined by $m_i[r] = [r \circ i]$ (see [6]).

Let $A(\Gamma)$ denote the automorphism group of Γ ($A(\Gamma)^+$ the orientation-preserving automorphism group if Γ is a Fuchsian group), and $I(\Gamma)$ the subgroup of inner automorphisms. The modular group $M(\Gamma) = A(\Gamma)/I(\Gamma)$ ($M(\Gamma)^+ = A(\Gamma)^+/I(\Gamma)$ if Γ is a Fuchsian group), acts on $\mathbf{T}(\Gamma)$. The *moduli space of* Γ is the quotient space

$$\mathscr{M}_g = \mathbf{T}(\Gamma)/M(\Gamma)$$

and $\mathscr{M}_g = \mathbf{T}(\Gamma)/M(\Gamma)^+$ if Γ is a Fuchsian group. For Γ being a Fuchsian group, $\mathscr{M}_g$ is the space of conformal classes of Riemann surfaces of genus g, or equivalently, the space of isomorphism classes of smooth complex projective algebraic curves.

Let now $\mathscr{M}_g^{\mathscr{R}}$ denote the subspace of $\mathscr{M}_g$ consisting of real Riemann surfaces of genus g, that is, Riemann surfaces X_g admitting a symmetry (an anticonformal involution) $\sigma\colon X_g \to X_g$. If Γ is a uniformizing Fuchsian group of X_g, then σ induces an element of $M(\Gamma)$ acting on $\mathbf{T}(\Gamma)$. If we denote σ^* the action of σ on $\mathbf{T}(\Gamma)$, then $[X_g] \in \mathbf{T}(\Gamma)_{\sigma^*}$, where $\mathbf{T}(\Gamma)_{\sigma^*}$ is the set of fixed points under σ^*, and the converse holds. Now, $\mathscr{M}_g^{\mathscr{R}}$ is the projection of $\mathbf{T}_{\mathbf{R}} = \bigcup_{\sigma^*} \mathbf{T}(\Gamma)_{\sigma^*}$ on $\mathscr{M}_g$, where σ runs over the different topological types of involutions of Riemann surfaces of genus g.

Let $\mathscr{M}_{\varepsilon k}^g$ denote the space of real Riemann surfaces of genus g admitting a symmetry with species εk, $-g \le \varepsilon k \le g+1$. Then we have the following theorem:

Theorem 2.1. *Let X_g be a Riemann surface of genus g. Given a symmetry $\sigma\colon X_g \to X_g$ with species εk, then $\mathscr{M}_{\varepsilon k}^g$ is connected.*

Proof. Let $X_g = \mathscr{H}/\Gamma$ and $X_g/\langle\sigma\rangle = \mathscr{H}/\Gamma'$, then Γ is an index two subgroup of Γ' and let $i\colon \Gamma \to \Gamma'$ be the inclusion map. Thus $\mathscr{M}_{\varepsilon k}^g$ is $m_i(\mathbf{T}(\Gamma'))/M(\Gamma)$ and since $\mathbf{T}(\Gamma')$ is connected then $\mathscr{M}_{\varepsilon k}^g$ is connected.

Let (X_g, ϕ) be a hyperelliptic Riemann surface of genus g. Given a symmetry σ of X_g with species εk, consider the finite group $G = \langle \phi, \sigma \rangle$. Then with the same proof as in Theorem 2.1 we have

Theorem 2.2. *Let X_g be a hyperelliptic Riemann surface of genus g. Given a symmetry $\sigma\colon X_g \to X_g$ with species εk, then $\mathscr{M}_{\varepsilon k}^g \cap \mathscr{M}_g^{2\mathscr{R}}$ is connected.*

Theorems 2.1 and 2.2 are well-known facts going back to Klein and Earle [21] and [11] (cf. [8]).

We have a natural decomposition of $\mathscr{M}_g^{\mathscr{R}}$ and $\mathscr{M}_g^{2\mathscr{R}}$ into the connected subspaces $\mathscr{M}_{\varepsilon k}^g$ and $\mathscr{M}_{\varepsilon k}^g \cap \mathscr{M}_g^{2\mathscr{R}}$. We shall prove in the next two sections that the subspaces $\mathscr{M}_{-1}^g$ and $\mathscr{M}_{-1}^g \cap \mathscr{M}_g^{2\mathscr{R}}$ of real and real hyperelliptic Riemann surfaces having a non-separating symmetry with one oval intersect any other subspace $\mathscr{M}_{\varepsilon k}^g$ and $\mathscr{M}_{\varepsilon k}^g \cap \mathscr{M}_g^{2\mathscr{R}}$, respectively. To prove this we have to find a (hyperelliptic) Riemann surface admitting two symmetries σ_1, σ_2 with species -1 and εk, respectively for every possible species εk. Notice that the possible species εk for symmetries of hyperelliptic Riemann surfaces are: $\varepsilon = -1$, $1 \le k \le g$, $\varepsilon = 1$, $k = g+1$, and $\varepsilon = 1$, $k = 1$ for g even, and $\varepsilon = 1$, $k = 2$ for g odd.

In the following we shall consider hyperelliptic Riemann surfaces (X_g, ϕ) with uniformizing surface Fuchsian group Γ, and where ϕ is the hyperelliptic involution.

Let σ_1, σ_2 be symmetries of a hyperelliptic Riemann surface $X_g = \mathscr{H}/\Gamma$ with species $\varepsilon_1 k_1$ and $\varepsilon_2 k_2$, respectively. The involutions σ_1, σ_2 and ϕ generate a finite group G. The group G is isomorphic either to D_n or to $D_n \times C_2$, with n the order of $\sigma_1\sigma_2$. Notice that $G = D_n$ if and only if $\phi = (\sigma_1\sigma_2)^{n/2}$. Then there exist an NEC group Δ (the universal covering transformation group) and an epimorphism $\theta\colon \Delta \to G$ with $\ker(\theta) = \Gamma$. If $\theta^{-1}(\langle\sigma_1\rangle) = \Lambda_1$, $\theta^{-1}(\langle\sigma_2\rangle) = \Lambda_2$, and $\theta^{-1}(\langle\phi\rangle) = \Lambda_h$ then $s(\Lambda_1) = (h_1, \varepsilon_1, [-], \{(-)^{k_1}\})$, $s(\Lambda_2) = (h_2, \varepsilon_2, [-], \{(-)^{k_2}\})$

and $s(\Lambda_h) = (0, +, [2, \overset{2g+2}{\ldots}, 2], \{-\})$, where $|\Delta : \Lambda_1| = |\Delta : \Lambda_2| = |\Delta : \Lambda_h| = n$ (or $2n$) and $|\Lambda_1 : \Gamma| = |\Lambda_2 : \Gamma| = |\Lambda_h : \Gamma| = 2$.

The aim is to give signatures Δ and epimorphisms $\theta\colon \Delta \to G$ such that $s(\Lambda_1) = (h_1, -, [-], \{(-)\})$, and $s(\Lambda_2) = (h_2, \varepsilon, [-], \{(-)^k\})$ where εk ranges over the possible species for a hyperelliptic Riemann surface.

In order to know the signature of Λ_i from the epimorphism θ we use the following procedure.

Let G' be the set of generators of a canonical presentation of Δ, the Schreier $\langle\sigma_i\rangle$-coset graph S_i is the graph with vertices the $\langle\sigma_i\rangle$-cosets in G and labelled directed edges joining $\langle\sigma_i\rangle\alpha_i$ with $\langle\sigma_i\rangle\alpha_j$ with label $g \in G'$ if $\langle\sigma_i\rangle\alpha_i\theta(g) = \langle\sigma_i\rangle\alpha_j$. Let $c \in G'$ be a reflection such that θ maps c to a conjugate of σ_1 or σ_2 in G. The action of c on the Λ_i-cosets is the same as the action of $\theta(c)$ on the $\langle\sigma_i\rangle$-cosets. Each coset $\Lambda_i\alpha$ fixed by c gives a reflection $c_\alpha = \alpha c \alpha^{-1}$ in Λ_i, called a *reflection induced* by c. In this way we obtain representatives of all conjugacy classes of reflections in Λ_i. Now each period cycle in $(s\Lambda_i)$ gives one conjugacy class of reflections in Λ_i. Suppose that d is another reflection in Δ and that cd has finite order. Two induced reflections (c_α and d_β, c_α and c_β or d_α and d_β) are conjugate in Λ_i if $\langle\sigma_i\rangle\alpha$ and $\langle\sigma_i\rangle\beta$ are in the same orbit under the action of $\theta(cd)$ on the $\langle\sigma_i\rangle$-cosets ([17]). In terms of the Schreier graph: two reflection loops c_α and d_β define conjugate reflections in Λ_i if the vertices of these reflection loops are joined by a path with the sides alternatively labelled by c and d. Finally, assume c is the reflection generator and e is the hyperbolic generator corresponding to an empty period cycle in $s(\Delta)$ (i.e., a period cycle without link periods). If $\sigma_1\sigma_2$ has even order, then c can induce several reflections in Λ_i. In this case all the reflections c_α and c_β are conjugate in Λ_i if and only if $G = D_n$ and $\theta(e) \neq 1$ (see [15] and [17]).

The sign of $s(\Lambda_i)$, $i = 1, 2$, is determined by the following fact: the sign of $s(\Lambda_i)$ is $+$ if and only if in the Schreier graph S_i the product of the labels of each cycle (not containing reflection loops) is an orientation preserving element of Δ (see [18]).

3. The connection of the locus of real hyperelliptic Riemann surfaces

Notice that the possible species εk for symmetries of hyperelliptic Riemann surfaces are: $\varepsilon = -1$, $1 \le k \le g$, $\varepsilon = 1$, $k = g+1$, and $\varepsilon = 1$, $k = 1$ for g even, and $\varepsilon = 1$, $k = 2$ for g odd. With the notation above:

Theorem 3.1. *We have $\mathscr{M}^g_{-1} \bigcap \mathscr{M}^g_{-k} \bigcap \mathscr{M}^{2\mathscr{R}}_g \neq \emptyset$ for $1 \le k \le g$ for every genus $g \ge 2$.*

Proof. We divide the proof in two cases according to the parity of $g-k$.

(1) $g-k$ *odd.* Let Δ be an NEC group with signature

$$s(\Delta) = \Big(0; +; [2^{(g-k+1)/2}]; \Big\{\Big(4, 4, \overbrace{2, \ldots, 2}^{k-1}\Big)\Big\}\Big). \tag{3.1}$$

and canonical presentation as given in Section 2. Let $\theta\colon \Delta \to D_4$ be the epimorphism defined by $\theta(x_j) = \phi = (\sigma_1\sigma_2)^2$, for all j, $\theta(c_{11}) = \theta(c_{1k+2}) = \sigma_1$, $\theta(c_{1i}) = \sigma_2$ for i even, and $\theta(c_{1i}) = \sigma_2\phi$ for i odd, $2 \le i \le k-1$, $\theta(e_1) = \phi^i$, where $i \in \{0,1\}$ in order to fulfill the long relation.

The Riemann formula applied to $\ker\theta \le \Delta$ yields:

$$8\left(-1 + \frac{g-k+1}{4} + \frac{k-1}{4} + \frac{3}{4}\right) = 2(g-4+3) = 2(g-1), \tag{3.2}$$

Then $\ker\theta$ is a Fuchsian surface group of genus g. Let $X_g = \mathscr{H}/\ker\theta$, $\Lambda_1 = \theta^{-1}(\langle\sigma_1\rangle)$ and $\Lambda_2 = \theta^{-1}(\langle\sigma_2\rangle)$. The reflection c_{11} induces two reflections in Λ_1 that are conjugate because $\theta(c_{11}c_{12}) = \sigma_1\sigma_2$; then there is only one conjugacy class of reflections and one empty period-cycle in $s(\Lambda_1)$. In the Schreier graph of $\langle\sigma_1\rangle$-cosets there is an orientation-reversing cycle with labelled edges c_{12}, c_{13} and c_{11}. Hence there is a $-$ sign and one empty period cycle in $s(\Lambda_1)$, then the species of σ_1 is -1.

Each generating reflection c_{1i}, $k+1 \ge i \ge 2$, induces one conjugacy class of reflections in Λ_2. Hence there are $k+1-1 = k$ empty period-cycles in the signature of $\theta^{-1}(\langle\sigma_2\rangle) = \Lambda_2$. In the Schreier graph of $\langle\sigma_2\rangle$-cosets there is a cycle with labelled edges x_1 and c_{12} and there is a $-$ sign in $s(\Lambda_2)$. The symmetry σ_2 has species $-k$.

Notice that $X_g/\langle\phi\rangle$ is an orbifold with $4\big(\frac{1}{2}(g-k+1)\big) + 2(k) = 2g+2$ conic points of order 2 and genus $\frac{1}{2}(g-4+3-g-1+2) = 0$.

(2) $g-k$ *even.* Let Δ be a group with signature

$$s(\Delta) = \Big(0; +; [2^{(g-k)/2}]; \Big\{\Big(\overbrace{2,\ldots,2}^{k+3}\Big)\Big\}\Big). \tag{3.3}$$

Let $\theta\colon \Delta \to D_2 \times C_2$ be the epimorphism defined by $\theta(x_j) = \phi$, for all j, $\theta(c_{11}) = \theta(c_{1k+4}) = \sigma_1$, $\theta(c_{1i}) = \sigma_2$ for i even, and $\theta(c_{1i}) = \sigma_2\phi$ for i odd, $2 \le i \le k+2$, $\theta(c_{1k+3}) = \sigma_1\phi$, $\theta(e_1) = \phi^i$, where $i \in \{0,1\}$ in order to fulfil the long relation.

The Riemann formula applied to $\ker\theta \le \Delta$ gives

$$8\left(-1 + \frac{k+3}{4} + \frac{g-k}{4}\right) = 2(g-4+3) = 2(g-1). \tag{3.4}$$

Then $\ker\theta$ is a Fuchsian surface group of genus g. We define $X_g = \mathscr{H}/\ker\theta$, $\Lambda_1 = \theta^{-1}(\langle\sigma_1\rangle)$ and $\Lambda_2 = \theta^{-1}(\langle\sigma_2\rangle)$. The same argument as in the case (1) shows that there is only one empty period-cycle in $s(\Lambda_1)$. In the Schreier graph of $\langle\sigma_1\rangle$-cosets there is an orientation-reversing 3-cycle with edges labelled x_1, c_{1k+3} and c_{11}. Then the signature of Λ_1 is $\big(w, -, [-], \{(-)\}\big)$ and the species of σ_1 is -1.

The generating reflection c_{12} induces one conjugacy class of reflections in Λ_2. Each generating reflection $c_{1,2i}$, $2 \geq i \geq \frac{1}{2}(k-1)$, induces two conjugacy classes of reflections in Λ_2. Hence there are $\frac{1}{2}\big(2(k-1)\big)+1=k$ empty period-cycles in the signature of $\theta^{-1}(\langle\sigma_2\rangle)=\Lambda_2$. In the Schreier graph of $\langle\sigma_2\rangle$-cosets there is an orientation-reversing 3-cycle with edges labelled x_1, c_{12} and c_{13}. The symmetry σ_2 has species $-k$.

Again, $X_g/\langle\phi\rangle$ is an orbifold with $4\big(\frac{1}{2}(g-k)\big)+2(k+1)=2(g+1)$ conic points of order 2 and genus $\frac{1}{2}(g-4+3-g-1+2)=0$, that is, X_g is a hyperelliptic surface.

The surface X_g is a point in $\mathscr{M}^g_{-1}\bigcap\mathscr{M}^g_{-k}\cap\mathscr{M}^{2\mathscr{R}}_g$, so then $\mathscr{M}^g_{-1}\bigcap\mathscr{M}^g_{-k}\cap\mathscr{M}^{2\mathscr{R}}_g\neq\emptyset$.

Theorem 3.2 ([3]). *We have $\mathscr{M}^g_{-1}\bigcap\mathscr{M}^g_{g+1}\bigcap\mathscr{M}^{2\mathscr{R}}_g\neq\emptyset$ for every genus $g\geq 2$.*

Proof. The proof is as for Theorem 3.1, Case 1, but now taking as Δ an NEC group with signature

$$s(\Delta)=\Big(0;+;[-];\Big\{\Big(4,4,\overbrace{2,\dots,2}^{g}\Big)\Big\}\Big), \tag{3.5}$$

and an epimorphism $\theta\colon\Delta\to D_4$ defined by $\theta(c_{1,1})=\sigma_1$, $\theta(c_{1,2j})=\sigma_2$, $\theta(c_{1,2j+1})=\sigma_2\phi$, $1\leq j\leq\big\lfloor\frac{1}{2}(g+1)\big\rfloor$, $\theta(c_{1,0})=\sigma_2\phi^i$, where $i=0$ for even g and $i=1$ for odd g and $\theta(e)=1$.

Theorem 3.3. *We have $\mathscr{M}^g_{-1}\bigcap\mathscr{M}^g_0\bigcap\mathscr{M}^g_{+k}\bigcap\mathscr{M}^{2\mathscr{R}}_g\neq\emptyset$, for $k=1$ for even genera g, and $k=2$ for odd genera $g\geq 2$.*

Proof. First of all, notice that a symmetry of a hyperelliptic Riemann surface of genus g that separates, satisfies the fact that the number of its ovals is either $g+1$ or 1 if the genus g is even, and either $g+1$ or 2 if the genus g is odd.

Consider an NEC group with signature $s(\Delta)=\big(0;+;[-];\{(2,2g{+}2,2g{+}2)\}\big)$, and an epimorphism $\theta\colon\Delta\to D_{2(g+1)}\times C_2$ defined by $\theta(c_{11})=\sigma_1$, $\theta(c_{12})=\sigma_1\phi$, $\theta(c_{13})=\sigma_2$.

The Riemann formula applied to $\ker\theta\leq\Delta$ gives:

$$8(g+1)\left(-1+\frac{1}{4}+\frac{2g+1}{2(g+1)}\right)=8(g+1)\left(\frac{-3(g+1)+4g+2}{4(g+1)}\right)=2(g-1). \tag{3.6}$$

We define $X_g=\mathscr{H}/\ker\theta$, $\Lambda_1=\theta^{-1}(\langle\sigma_1\rangle)$, $\Lambda_2=\theta^{-1}(\langle\sigma_2\rangle)$ and $\Lambda_3=\theta^{-1}(\langle\sigma_2\phi\rangle)$. The same argument as in Theorem 3.1 shows that there is only one empty period-cycle in $s(\Lambda_1)$. In the Schreier graph of $\langle\sigma_1\rangle$-cosets there is an orientation-reversing 5-cycle with edges labelled c_{12}, c_{13}, c_{11}, c_{12} and c_{13}. Then the signature of Λ_1 is $(w,-,[-],\{(-)\})$ and the species of σ_1 is -1.

The generating reflection c_{12} induces four reflections in Λ_2. These reflections are conjugated by $\theta\big((c_{12}c_{13})^{g+1}\big) = (\sigma_1\sigma_2)^{g+1}\phi^{g+1}$ and $\theta\big((c_{13}c_{11})^{g+1}\big) = (\sigma_1\sigma_2)^{g+1}$. Then Λ_2 contains two conjugacy classes of reflections induced by c_{12} if g is odd, and one if g is even. The Schreier graph of $\langle\sigma_2\rangle$ is bipartite: the set of vertices $\{(\sigma_2)(\sigma_1\sigma_2)^i\phi^\varepsilon \mid 1 \le i \le 2g+2,\ \varepsilon \in \{0,1\}\}$ can be separated in the sets $A = \{(i,0),(i,1),\ 1 \le i \le g+1\}$ and $B = \{(i,0),(i,1),\ g+2 \le i \le 2g+2\}$, where the $\langle\sigma_2\rangle$-cosets are represented by the corresponding exponent of $\sigma_1\sigma_2$. The symmetry σ_2 has species $+2$ if the genus g is odd, and σ_2 has species $+1$ if the genus g is even.

The Schreier graph of $\langle\sigma_2\phi\rangle$ has no reflection loops, then $\sigma_2\phi$ has species 0.

Again, $\mathscr{H}/\theta^{-1}(\langle\sigma_1,\sigma_2\rangle)$ is an orbifold with $(2g{+}2)(1) = 2(g+1)$ conic points of order 2 and genus 0.

The surface X_g defined by (Δ,θ) admits the required three symmetries σ_1, σ_2, and $\sigma_2\phi$.

We can now establish some direct consequences of the results 3.1, 3.2 and 3.3.

Two real hyperelliptic Riemann surfaces (X_i,σ_i) $i=1,2$, are *quasiconformally equivalent in the hyperelliptic locus* if there is a quasiconformal deformation F_t from X_1 to X_2 such that $X_t = F_t(X_1)$ is a hyperelliptic surface and $F_t \circ \sigma_1 \circ F_t^{-1}$ is a symmetry of X_t. Now we have the result quoted in the abstract:

Theorem 3.4. *Every real hyperelliptic Riemann surface is quasiconformally equivalent in the hyperelliptic locus to a real hyperelliptic Riemann surface (X,σ), such that X admits an antiholomorphic involution τ, where $\mathrm{Fix}(\tau)$ has one non-separating connected component.*

Proof. Each set $\mathscr{M}^g_{\varepsilon k} \bigcap \mathscr{M}^{2\mathscr{R}}_g$ corresponds to the real hyperelliptic Riemann surfaces with the same topological type. By Lemma 1.2 in [22] the set $\mathscr{M}^g_{\varepsilon k} \bigcap \mathscr{M}^{2\mathscr{R}}_g$ is a quasiconformal class of real hyperelliptic Riemann surfaces (this is a consequence of deep results in Teichmüller theory); then the theorem follows from Theorems 3.1, 3.2 and 3.3.

A consequence of the above theorem is the following:

Corollary 3.5. *The space $\mathscr{M}^{2\mathscr{R}}_g$ of real hyperelliptic algebraic curves is a connected subspace of the moduli space $\mathscr{M}_g$ of complex algebraic curves. Furthermore the subset formed by all real hyperelliptic Riemann surfaces admitting a non-separating symmetry with one oval $\mathscr{M}^g_{-1} \bigcap \mathscr{M}^{2\mathscr{R}}_g$ cuts every subset $\mathscr{M}^g_{\varepsilon k} \bigcap \mathscr{M}^{2\mathscr{R}}_g$ for any possible species εk for a given g, i.e., $\mathscr{M}^g_{-1}$ is a spine for $\mathscr{M}^{2\mathscr{R}}_g$.*

4. On the existence of spines in $\mathscr{M}^{\mathscr{R}}_g$

We have proved in the previus section that $\mathscr{M}^g_{-1} \bigcap \mathscr{M}^g_{\varepsilon k} \neq \emptyset$ for εk being negative, 0, $g+1$, and $+1$ if g is even or $+2$ if g is odd. To prove that $\mathscr{M}^g_{-1}$ is a spine of $\mathscr{M}^{\mathscr{R}}_g$ we must find surfaces X_g admitting two symmetries σ_1, σ_2

such that σ_1 has species -1 and σ_2 has species $+k$, with $k = g+1-2t$ and $1 \le t \le \left\lfloor \frac{1}{2}(g-2) \right\rfloor$.

Let σ_1, σ_2 be symmetries of a Riemann surface X_g with species $+1$ and $+k$, respectively. The involutions σ_1, σ_2 generate a finite group G. $G = D_n = \langle \sigma_1, \sigma_2 \rangle$, with n the order of $\sigma_1\sigma_2$.

Theorem 4.1 ([10]). *We have* $\mathscr{M}^g_{-1} \bigcap \mathscr{M}^g_{+k} \neq \emptyset$, *for* $k = g+1-2t$, $1 \le t \le \frac{1}{2}(g-2)$ *for even genera* g, *and* $1 \le t \le \frac{1}{2}(g-3)$ *for odd genera* g.

Proof. The method of the proof is similar to the one used in Theorems 3.1, 3.2 and 3.3. We consider suitable NEC groups and epimorphisms θ from them to dihedral groups $D_n = \langle \sigma_1, \sigma_2 \rangle$. The surface $X = \mathscr{H}/\ker\theta$ admits two symmetries given by $X \to X/\langle \sigma_i \rangle$, $i = 1, 2$, with species -1 and $+k$.

We divide the proof in three cases according to the parity of g and k. We shall only give the signature of the groups Δ and the epimorphisms $\theta\colon \Delta \to D_n$ in each case.

(1) g *even*, $k \equiv g+1 \pmod 2$ and $k \le g-1$ ([19]). Let Δ be a group with signature

$$s(\Delta) = \left(\tfrac{1}{2}(g+1-k); -; [-]; \{(2,2)(-)^{(k-1)/2}\}\right), \tag{4.1}$$

and $\theta\colon \Delta \to D_2$ the epimorphism defined by $\theta(d_j) = \sigma_1$, for all j, $\theta(c_{11}) = \theta(c_{13}) = \sigma_2$, $\theta(c_{12}) = \sigma_1$, $\theta(e_1) = 1$, $\theta(c_{i1}) = \sigma_2$, $\theta(e_i) = 1$, for all $i \ge 2$.

(2) g *odd*, $k \equiv g+1 \pmod 4$. Let Δ be a group with signature

$$s(\Delta) = \left(\tfrac{1}{4}(g+1-k); -; [-]; \left\{\left(4, \overbrace{2, \ldots, 2}^{k-1}, 4\right)\right\}\right). \tag{4.2}$$

Let $\theta\colon \Delta \to D_4$ be the epimorphism defined by $\theta(d_j) = \sigma_1$, for all j, $\theta(c_{1,1}) = \theta(c_{1,k+2}) = \sigma_1$, $\theta(c_{1,2i}) = \sigma_2$, $\theta(c_{1,2i+1}) = \sigma_2(\sigma_1\sigma_2)^2$, $1 \le i \le \frac{1}{2}k$, $\theta(e_1) = 1$.

(3) g *odd*, $k \equiv g-1 \pmod 4$, $3 < k$. Let Δ be a group with signature either

$$s(\Delta) = \left(\tfrac{1}{4}(g-1-k); -; [-]; \left\{\left(4, \overbrace{2, \ldots, 2}^{k-3}, 4\right), (-)\right\}\right), \qquad \text{if } k < g-1 \tag{4.3}$$

or

$$s(\Delta) = \left(0; +; [-]; \left\{\left(4, \overbrace{2, \ldots, 2}^{k-3}, 4\right), (-)\right\}\right), \qquad \text{if } k = g-1. \tag{4.4}$$

Let $\theta\colon \Delta \to D_4$ be the epimorphism defined by $\theta(d_j) = \sigma_1$, for all j, $\theta(c_{1,1}) = \theta(c_{1,k}) = \sigma_1$, $\theta(c_{1,2i}) = \sigma_2$, $\theta(c_{1,2i+1}) = \sigma_2(\sigma_1\sigma_2)^2$, $1 \le i \le \frac{1}{2}k - 1$, $\theta(e_1) = 1$, $\theta(c_{2,1}) = \sigma_2$ and $\theta(e_2) = 1$.

We can now establish

Theorem 4.2. *Every real Riemann surface is quasiconformally equivalent in the real locus to a real Riemann surface* (X, σ), *such that* X *admits an antiholomorphic involution* τ, *where* $\mathrm{Fix}(\tau)$ *has one non-separating connected component.*

Proof. Each set $\mathscr{M}^g_{\varepsilon k}$ corresponds to the real Riemann surfaces with the same topological type. By Lemma 1.2 in [22] there is a quasiconformal class of real Riemann surfaces. Thus the theorem follows from Theorems 3.1, 3.2, 3.3 and 4.1.

A consequence of the above theorem is the following:

Corollary 4.3. *The space* $\mathscr{M}^{\mathscr{R}}_g$ *of real algebraic curves is a connected subspace of the moduli space* $\mathscr{M}_g$ *of complex algebraic curves. Furthermore the subset formed by all real Riemann surfaces admitting a non-separating symmetry with one oval,* $\mathscr{M}^g_{-1}$, *cuts every subset* $\mathscr{M}^g_{\varepsilon k}$ *for any possible species* εk *for a given* g, *i.e.,* $\mathscr{M}^g_{-1}$ *is a spine for* $\mathscr{M}^{\mathscr{R}}_g$.

Given the decomposition $\mathscr{M}^{\mathscr{R}}_g = \bigcup \mathscr{M}^g_k$, we try to find another spine different from $\mathscr{M}^g_{-1}$, i.e., we are looking for $\varepsilon' k' \neq -1$ such that $\mathscr{M}^g_{\varepsilon k} \bigcap \mathscr{M}^g_{\varepsilon' k'} \neq \emptyset$ for all possible species εk.

Theorem 4.4 ([10]). *Let* $g > 2$ *be an even integer. Then the only spine in the decomposition of the subset* $\mathscr{M}^{\mathscr{R}}_g = \bigcup \mathscr{M}^g_k$ *of real Riemann surfaces of* $\mathscr{M}_g$ *is the subspace* $\mathscr{M}^g_{-1}$.

Proof. Assume that g is an integer and $\varepsilon' k' \neq -1$ is such that $\mathscr{M}^g_{\varepsilon k} \bigcap \mathscr{M}^g_{\varepsilon' k'} \neq \emptyset$ for all possible species εk for g. First of all, by Theorem 3.3 in [3], if $\mathscr{M}^g_{g+1} \bigcap \mathscr{M}^g_{\varepsilon' k'} \neq \emptyset$, then the species $\varepsilon' k' \geq -1$. By Theorem 3.4 in [3], if $\mathscr{M}^g_{-g} \bigcap \mathscr{M}^g_{\varepsilon' k'} \neq \emptyset$, then the species $\varepsilon' k' \leq 0$. On the other hand, by Theorem 3.2 in [19] $\mathscr{M}^g_0 \bigcap \mathscr{M}^g_{\varepsilon k} = \emptyset$ for even genera g and even $k \neq 0$. Then if g is even there is no such a $\varepsilon' k'$.

With the same proof we can show the following

Remark 4.5. Let $g > 2$ be an even integer. Then the only spine in the decomposition of the subset $\mathscr{M}^{2\mathscr{R}}_g$ of real hyperelliptic Riemann surfaces of $\mathscr{M}_g$ is the subspace $\mathscr{M}^g_{-1} \bigcap \mathscr{M}^{2\mathscr{R}}_g$.

Notice that in the proof of the above theorems the hypothesis on the parity of g is only used to avoid the possibility $k' = 0$, then the only possible spine for g odd is $\mathscr{M}^g_0$. The results of [19] assert that $\mathscr{M}^g_0$ is in fact a spine.

Theorem 4.6 ([10]). *Let* g *be an odd integer. Then a real algebraic curve* C_g *of genus* g *is quasiconformally equivalent to a real curve* C'_g *such that the complexification of* C'_g *admits a real form which is purely imaginary.*

Proof. We have to prove that there exist real Riemann surfaces admitting two symmetries with species 0 and εk, where εk runs over all possibilities. The signatures and epimorphisms listed below give us the required real Riemann surfaces ([19]).

(1.1) $\varepsilon = -$, k *odd.* Let Δ be a group with signature

$$s(\Delta) = \Big(0; +; \Big[4, \overbrace{2, \dots, 2}^{(g-k)/2}\Big]; \{(2^k)\}\Big). \tag{4.5}$$

We construct an epimorphism $\theta\colon \Delta \to D_4 = \langle \sigma_1, \sigma_2 \rangle$ by sending the elliptic generator of order 4 to $\sigma_1\sigma_2$, the elliptic generators of order 2 to $(\sigma_1\sigma_2)^2$, the generating reflection generators alternately to σ_2 and $\sigma_2(\sigma_1\sigma_2)^2$, and e to $\sigma_1\sigma_2$ or $(\sigma_1\sigma_2)^3$ (depending on the parity of $\frac{1}{2}(g-k)$).

(1.2) $\varepsilon = -$, k *even.* Let Δ be a group with signature

$$s(\Delta) = \Big(0; +; \big[2^{(g+3-k)/2}\big]; \Big\{\Big(\overbrace{2, \dots, 2}^{k}\Big)\Big\}\Big). \tag{4.6}$$

We define an epimorphism θ from Δ to $D_2 \times C_2 = \langle \sigma_1, \sigma_2, \phi \rangle$ by mapping all but one elliptic generators to ϕ, one elliptic generator to $\sigma_1\sigma_2\phi$, the generating reflections and glide reflection to σ_2 alternatively $\sigma_2\phi$ and e to $\sigma_1\sigma_2\phi^h$, where $h = 0$ if $\frac{1}{2}(g+3-k)$ is even and $h = 1$ if $\frac{1}{2}(g+3-k)$ is odd.

(2) $\varepsilon = +$. In this case k is even, since $k \equiv g + 1 (\operatorname{mod} 2)$. Let Δ be a group with signature

$$s(\Delta) = \big(0; +; [2^{g+3-k}]; \{(-)^{k/2}\}\big). \tag{4.7}$$

We define an epimorphism θ from Δ to $D_2 = \langle \sigma_1, \sigma_2 \rangle$ by mapping the elliptic generators to $\sigma_1\sigma_2$, the generating reflections to σ_2, and the connecting generators e_i to 1.

Notice that the Schreier graph of $\langle \sigma_1 \rangle$ in D_2 contain no reflection loops, so the species of σ_1 is 0.

The above proof is different from the one given in [10] in the cases 1.1 and 1.2. The surfaces in the cases 1.1 and 1.2 of Theorem 4.6 are hyperelliptic, being the hyperelliptic involution $\phi = (\sigma_1\sigma_2)^2$ in the case 1.1 and ϕ in 1.2. Then we have the following result:

Remark 4.7. Let g be an odd integer. Then a real hyperelliptic algebraic curve C_g of genus g is quasiconformally equivalent in the hyperelliptic locus to a real hyperelliptic curve C'_g such that the complexification of C'_g admits a real form which is purely imaginary.

5. The locus of real cyclic p-gonal Riemann surfaces

Let p be a prime integer. A cyclic p-gonal Riemann surface is a closed Riemann surface which can be realized as a cyclic p-fold covering space of the Riemann sphere (see [9] and [16]). We will denote by $\mathscr{M}_g^p$ the subset of cyclic p-gonal Riemann surfaces in $\mathscr{M}_g$. The complexification of a smooth cyclic p-gonal real algebraic curve gives rise to a cyclic p-gonal Riemann surface X_g with a symmetry, we shall call such a surface X_g *a real cyclic p-gonal Riemann surface.*

Let $\mathscr{M}_g^{p\mathscr{R}}$ be the locus of real cyclic p-gonal Riemann surfaces in the moduli space of Riemann surfaces of genus g. In Section 3 we proved that $\mathscr{M}_g^{2\mathscr{R}}$ is connected, and now we want to show that the situation is essentially different for $\mathscr{M}_g^{p\mathscr{R}}$, $p > 2$. The set of cyclic p-gonal Riemann surfaces is actually disconnected in general. Let $P^n(F_p)$ denote the n-dimensional projective space over the finite field with p elements, and define the following subset D_p^r of $P^{r-1}(F_p)$:

$$D_p^r = \left\{ m = (m_1, \dots, m_r) \;\middle|\; \sum_1^r m_i = 0,\ \Pi_1^r m_i \neq 0 \right\}.$$

The symmetric group $\sum_r$ acts on D_p^r in the natural way and we write $D_p^{(r)}$ for $D_p^r / \sum_r$. Let $m = (m_1, \dots, m_r) \in \overline{m} \in D_p^{(r)}$. Consider the abstract Fuchsian group Γ with signature $s(\Gamma) = \big(0, [p, \overset{r}{\dots}, p]\big)$ and epimorphism $\varphi_m \colon \Gamma \to F_p$ (F_p as an additive group) defined by $\varphi_m(x_i) = m_i$. The group $\ker \varphi_m$ is a surface Fuchsian group that uniformizes a cyclic p-gonal Riemann surface. In fact, the map $\pi \colon \mathscr{H} / \ker \varphi_m \to \mathscr{H} / \Gamma$ is a cyclic covering map of the Riemann sphere. There is a natural decomposition $\mathscr{M}_g^p = \bigcup \mathscr{M}_g^p(\overline{m})$, where $\mathscr{M}_g^p(\overline{m})$ is the set of Riemann surfaces uniformized by pairs (Γ, φ_t), with $t \in \overline{m}$ in $D_p^{(r)}$.

Let $p > 2$. In general the action of $\sum_r$ on D_p^r is not transitive. So, for $r = 2p$, the elements $m_1 = \big(1, \overset{2p}{\dots}, 1\big)$ and $m_2 = (1, p-1, \overset{p}{\dots}, 1, p-1\big)$ belong to distinct classes under the action of $\sum_r$.

Theorem 5.1 (Theorem 2 in [14]). *The union $\mathscr{M}_g^p$ is the disjoint union of the sets $\mathscr{M}_g^p(\overline{m})$, where $\overline{m}$ ranges on the whole set $D_p^{(r)}$.*

The integers p, g and r in the statement are related by the Riemann–Hurwitz formula $2g = (p-1)(r-2)$.

We shall see that any connected component of $\mathscr{M}_g^p$ contains real, cyclic p-gonal Riemann surfaces. As a consequence, we find that $\mathscr{M}_g^{p\mathscr{R}}$ is not connected in general.

Theorem 5.2. *Let $\mathscr{M}_g^p = \coprod \mathscr{M}_g^p(\overline{m})$. Each connected component $\mathscr{M}_g^p(\overline{m})$ contains a real, cyclic p-gonal Riemann surface.*

Proof. Let now Δ be an NEC group with signature

$$s(\Delta) = \big(0; +; [-]; \big\{\big(p \overset{r}{.\,.\,.}, p\big)\big\}\big)$$

and canonical presentation $\Delta = \langle e, c_0, c_1, \ldots, c_r \mid c_0^2 = c_i^2 = (c_{i-1}c_i)^p = ec_0e^{-1}c_r = 1,\ 1 \le i \le r\rangle$. Consider the epimorphism $\theta_m\colon \Delta \to D_p = \langle \sigma_1, \sigma_2 \mid \sigma_i^2 = (\sigma_1\sigma_2)^p = 1,\ i = 1, 2\rangle$ defined recursively by $\theta_m(c_0) = \sigma_2$, $\theta_m(c_i) = \theta_m(c_{i-1})(\sigma_1\sigma_2)^{m_i}$. Hence $\mathscr{H}/\ker\theta_m$ is a real Riemann surface since $\sum m_i = 0$. For instance, σ_2 represents a symmetry of $\mathscr{H}/\ker\theta_m$. Finally, $\mathscr{H}/\ker\theta_m$ belongs to $\mathscr{M}_g^p$ since $\theta_m^{-1}(\langle\sigma_1\sigma_2\rangle)$ is a Fuchsian group with signature $\big(0, \big[p \overset{r}{.\,.\,.}, p\big]\big)$ and $\ker\theta_m = \ker\theta_m/\theta_m^{-1}(\langle\sigma_1\sigma_2\rangle)$.

The surface $\mathscr{H}/\ker\theta_m$ constructed above has a dihedral group D_p of automorphisms with all the symmetries conformally conjugate since p is an odd prime. Using [17], and [18], we find that the species of any of the symmetries in D_p is $+1$, for all $\overline{m} \in D_p^{(r)}$. Hence each connected component of $\mathscr{M}_g^p$ contains a real, cyclic p-gonal Riemann surface with a symmetry with species $+1$.

As a particular case we have:

Theorem 5.3. *The locus $\mathscr{M}_{(p-1)^2}^{p\mathscr{R}}$ of a real, cyclic p-gonal Riemann surface in $\mathscr{M}_{(p-1)^2}$ is disconnected for prime integers $p > 2$.*

Gross and Harris proved Theorem 5.3 in the case $p = 3$ (see [16]).

To prove that $\mathscr{M}_g^{p\mathscr{R}}$ is disconnected we used the fact that $\mathscr{M}_g^p$ is disconnected. The final paragraph of this work is to remark that, in general, given a connected component $\mathscr{M}_g^p(\overline{m})$ of $\mathscr{M}_g^p$, the set $\mathscr{M}_g^p(\overline{m}) \cap \mathscr{M}_g^{p\mathscr{R}}$ of real p-gonal Riemann surfaces contained in $\mathscr{M}_g^p(\overline{m})$ is disconnected in general. The reason is that the representations (Γ, φ_m) of cyclic p-gonal Riemann surfaces admit an action of $\sum_r$, while the representations $\theta_m\colon \Delta \to D_p$, $s(\Delta) = \big(0; +; [-]; \big\{\big(p \overset{r}{.\,.\,.}, p\big)\big\}\big)$, of real, cyclic p-gonal Riemann surfaces admit only actions of cyclic groups.

Example 5.4. $\mathscr{M}_{16}^5\big(\overline{(1,1,1,1,1,4,4,4,4,4)}\big) \cap \mathscr{M}_{16}^{5\mathscr{R}}$ is not connected.

Let now Δ be an NEC group with signature $s(\Delta) = \big(0; +; [-]; \big\{(5^{10})\big\}\big)$ and presentation $\Delta = \langle e, c_0, c_1, \ldots, c_r \mid c_0^2 = c_i^2 = (c_{i-1}c_i)^5 = ec_0e^{-1}c_r = 1,\ 1 \le i \le 10\rangle$. We construct epimorphisms

$\theta_1, \theta_2\colon \Delta \to D_5 = \langle \sigma_1, \sigma_2 \mid \sigma_i^2 = (\sigma_1\sigma_2)^5 = 1,\ i = 1, 2\rangle$ as follows:

θ_1:

$\theta_1(c_0) = \sigma_2$ $\quad\theta_1(c_1) = \sigma_2(\sigma_1\sigma_2)$ $\quad\theta_1(c_2) = \sigma_2$ $\quad\theta_1(c_3) = \sigma_2(\sigma_1\sigma_2)$

$\theta_1(c_4) = \sigma_2$ $\quad\theta_1(c_5) = \sigma_2(\sigma_1\sigma_2)$ $\quad\theta_1(c_6) = \sigma_2$ $\quad\theta_1(c_7) = \sigma_2(\sigma_1\sigma_2)$

$\theta_1(c_8) = \sigma_2(\sigma_1\sigma_2)^2$ $\quad\theta_1(c_9) = \sigma_2(\sigma_1\sigma_2)$ $\quad\theta_1(c_{10}) = \sigma_2.$

θ_2:

$\theta_2(c_0) = \sigma_2$ $\quad\theta_2(c_1) = \sigma_2(\sigma_1\sigma_2)$ $\quad\theta_2(c_2) = \sigma_2$ $\quad\theta_2(c_3) = \sigma_2(\sigma_1\sigma_2)$

$\theta_2(c_4) = \sigma_2$ $\quad\theta_2(c_5) = \sigma_2(\sigma_1\sigma_2)^4$ $\quad\theta_2(c_6) = \sigma_2$ $\quad\theta_2(c_7) = \sigma_2(\sigma_1\sigma_2)$

$\theta_2(c_8) = \sigma_2(\sigma_1\sigma_2)^2$ $\quad\theta_2(c_9) = \sigma_2(\sigma_1\sigma_2)$ $\quad\theta_2(c_{10}) = \sigma_2.$

We define the surfaces $X_1 = \mathscr{H}/\ker\theta_1$ and $X_2 = \mathscr{H}/\ker\theta_2$. The transposition $(5,6) \in \sum_{10}$ maps the representation $\varphi_1\colon \Gamma_1 = \theta_1^{-1}(\langle\sigma_1\sigma_2\rangle) \to F_5$ to the representation $\varphi_2\colon \Gamma_2 = \theta_2^{-1}(\langle\sigma_1\sigma_2\rangle) \to F_p$. Then the surfaces X_1 and X_2 belong to the same connected component of $\mathscr{M}_{16}^5$.

Finally we see that X_1 and X_2 lie in different components of $\mathscr{M}_{16}^{5\mathscr{R}}$. Assume that X_1 and X_2 are in the same component of $\mathscr{M}_{16}^{5\mathscr{R}}$. Let τ_1 be a symmetry of X_1 contained in D_5. Since the action of D_5 on X_1 is not topologically equivalent to the action of D_5 on X_2, then there exists a Riemann surface X admitting an action of D_5 which is topologically equivalent to the action of D_5 on X_1 and such that X has a symmetry $\tau \notin D_5$, not conformally equivalent to τ_1 in $\mathrm{Aut}(X)$. So the order of $\tau\tau_1$ is even and τ induces a symmetry of $X/\langle\tau_1\rangle$. Now, let α be an automorphism of order 5 in D_5. Since α and $\tau\alpha^j\tau$ are conjugate in $\mathrm{Aut}(X)$ for $1 \le j \le 4$, then the markings $1,4,1,4,1,4,1,1,4,4$ corresponding to the components of $m \in \overline{(1,1,1,1,1,4,4,4,4,4)}$ are situated symmetrically with respect to τ in $\mathrm{Fix}(\tau_1)$ giving the rotation indices of α. This does not happen for the representation $\theta_1\colon \Delta \to D_5$.

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