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BOUNDARY CORRESPONDENCE UNDER HARMONIC QUASICONFORMAL HOMEOMORPHISMS OF THE UNIT DISK

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Abstract. Let f be a harmonic homeomorphism of the unit disk onto itself. The following conditions are equivalent: (a) f is quasiconformal; (b) f is bi-Lipschitz in the Euclidean metric; (c) the boundary function is bi-Lipschitz and the Hilbert transformation of its derivative is in L^∞.

1. Introduction

Throughout the paper we denote by φ a continuous increasing function on $\mathbf{R}$ such that $\varphi(t+2\pi)-\varphi(t)\equiv 2\pi$, so that the function

$$\gamma(t)=e^{i\varphi(t)}$$

is 2π-periodic and continuous, and of bounded variation on $[0,2\pi]$. We consider the harmonic mapping f defined on $\mathbf{D}=\{z:|z|<1\}$ by

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi}P(r,\theta-t)\gamma(t)\,dt \qquad (z=re^{i\theta}), \tag{1.1}$$

where P is the Poisson kernel,

$$P(r,t)=\frac{1-r^2}{1+r^2-2r\cos t}.$$

By the fundamental result of Choquet [2], f is a homeomorphism of $\overline{\mathbf{D}}$ onto $\overline{\mathbf{D}}$. Conversely, every orientation-preserving homeomorhism $f\colon\overline{\mathbf{D}}\mapsto\overline{\mathbf{D}}$, harmonic in $\mathbf{D}$, can be represented in the form (1.1). A consequence of the Choquet theorem and a result of Lewy [8] is that the Jacobian of f is strictly positive in $\mathbf{D}$, i.e.,

$$J_f(z)=|\partial f(z)|^2-|\bar\partial f(z)|^2>0 \qquad (z\in\mathbf{D}). \tag{1.2}$$

Being harmonic, the mapping f can be represented as

$$f(z)=h(z)+\overline{g(z)}, \qquad g(0)=0,$$

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where h and g are analytic in $\mathbf{D}$ and uniquely determined by f. We can rewrite (1.2) as

$$\left|\frac{g'(z)}{h'(z)}\right| < 1 \qquad (z \in \mathbf{D}). \tag{1.3}$$

In this paper we characterize those φ for which f is quasiconformal, i.e., for which (1.3) can be improved to

$$k = \sup_{z\in\mathbf{D}}\left|\frac{g'(z)}{h'(z)}\right| < 1. \tag{1.4}$$

Martio [9] was the first who posed and studied this question. He proved that, if $\varphi \in C^1(\mathbf{R})$, the following two conditions are sufficient for quasiconformality of f: $\min \varphi' > 0$ and

$$\int_0^\pi \frac{\omega(t)}{t}\,dt < \infty, \tag{1.5}$$

where ω is the modulus of continuity of φ',

$$\omega(t) = \sup\bigl\{|\varphi'(x) - \varphi'(y)| : |x-y| < t\bigr\}.$$

Condition (1.5), known as the Dini condition (applied to φ'), is sufficient, but not necessary, for the Hilbert transformation $H\varphi'$ of φ' to belong to L^∞. Our idea is to replace the Dini condition by $H\varphi' \in L^\infty$.

Theorem 1.1. *The mapping f is quasiconformal if and only if the function φ is bi-Lipschitz and the Hilbert transformation of φ' is essentially bounded on $\mathbf{R}$. In other words, f is quasiconformal if and only if φ is absolutely continuous and satisfies the conditions*:

$$\operatorname{ess\,inf} \varphi' > 0, \tag{1.6}$$

$$\operatorname{ess\,sup} \varphi' < \infty, \tag{1.7}$$

$$\operatorname*{ess\,sup}_{\theta\in\mathbf{R}}\left|\int_{+0}^\pi \frac{\varphi'(\theta+t) - \varphi'(\theta-t)}{t}\,dt\right| < \infty. \tag{1.8}$$

The proof that these three conditions are sufficient is short; we simply compute the radial limits of the modulus of the *bounded* analytic function g'/h' and apply the maximum modulus principle (see the end of Section 2).

The necessity proof, given in Section 3, is more intriguing and depends on Mori's theorem in the theory of quasiconformal mappings (cf. Ahlfors [1]):

If Φ *is a quasiconformal homeomorphism of* $\mathbf{D}$*, then*

$$|\Phi(z_1) - \Phi(z_2)| \le C|z_1 - z_2|^\alpha, \qquad \alpha = \frac{1-k}{1+k} \qquad (z_1,\, z_2 \in \mathbf{D}), \tag{1.9}$$

where

$$k = \sup_{z\in\mathbf{D}} \left|\frac{\bar\partial\Phi(z)}{\partial\Phi(z)}\right|$$

and C *depends only on* $f(0)$. (Note that $C = 16$ if $\Phi(0) = 0$.)

The mapping $|z|^\alpha(z/|z|)$ shows that the exponent α is optimal in the class of arbitrary k-quasiconformal homeomorphisms. However, it follows from our proof (see (3.5)) that if Φ is harmonic, then it satisfies the ordinary Lipschitz condition (with Lipschitz constant depending on k). On the other hand, from (1.4) and the inequality

$$|h'(z)|^2 + |g'(z)|^2 \ge 1/\pi^2 \qquad (z \in \mathbf{D}),$$

due to Heinz [7], it follows that $\inf_{z\in\mathbf{D}}(|\partial f(z)| - |\bar\partial f(z)|) > 0$, which implies that the inverse mapping, f^{-1}, satisfies a Lipschitz condition. Therefore we have the following.

Theorem 1.2. *If the mapping* f *is quasiconformal, then it is bi-Lipschitz, i.e., there is a constant* $L < \infty$ *such that*

$$\frac{1}{L} \le \left|\frac{f(z_1) - f(z_2)}{z_1 - z_2}\right| \le L \qquad (z_1,\, z_2 \in \mathbf{D})$$

and consequently

$$\frac{1}{L} \le \frac{1 - |f(z)|}{1 - |z|} \le L \qquad (z \in \mathbf{D}).$$

Note that an arbitrary bi-Lipschitz homeomorphism is quasiconformal.

2. Boundary values of the derivatives

We recall that the (periodic) Hilbert transformation of a 2π-periodic function $\psi \in L^1$ is defined by

$$\begin{aligned}(H\psi)(\theta) &= -\frac{1}{\pi}\int_{+0}^{\pi} \frac{\psi(\theta+t) - \psi(\theta-t)}{2\tan(t/2)}\,dt \\ &= -\frac{1}{\pi}\int_{+0}^{\pi} \frac{\Psi(\theta+t) + \Psi(\theta-t) - 2\Psi(\theta)}{4\sin^2(t/2)}\,dt,\end{aligned} \tag{2.1}$$

where Ψ is the indefinite integral of ψ. The integrals are improper and converge for almost all $\theta \in \mathbf{R}$; this and other facts concerning the operator H used in our paper can be found in Zygmund [11, Chapter VII]. We note the connection of H with harmonic conjugates.

If a function u, harmonic in $\mathbf{D}$, is given by

$$u(re^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta},$$

then its harmonic conjugate v is defined by

$$v(re^{i\theta}) = \sum_{n=-\infty}^{\infty} m_n c_n r^{|n|} e^{in\theta},$$

where $m_n = -i \operatorname{sign} n$; in particular $v(0) = 0$. If u is the Poisson integral of ψ, then v has radial limits almost everywhere and there holds the relation

$$v(e^{i\theta}) := \lim_{r\to 1^-} v(re^{i\theta}) = (H\psi)(\theta) \qquad \text{(a.e.)}.$$

In calculating the boundary values of the analytic functions h' and g' we use the formulae

$$h'(z) = \partial f(z) = \frac{1}{2} e^{-i\theta}\left(f_r(z) - i\frac{f_\theta(z)}{r}\right) \tag{2.2}$$

and

$$\overline{g'(z)} = \bar{\partial} f(z) = \frac{1}{2} e^{i\theta}\left(f_r(z) + i\frac{f_\theta(z)}{r}\right), \tag{2.3}$$

where

$$f_\theta = \frac{\partial f}{\partial \theta}, \qquad f_r = \frac{\partial f}{\partial r}.$$

The derivatives f_r and f_θ are connected by the simple but fundamental fact that

the function rf_r is equal to the harmonic conjugate of f_θ.

It follows from (1.1) that f_θ equals the Poisson–Stieltjes integral of $\gamma = e^{i\varphi}$:

$$f_\theta(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta - t)\, d\gamma(t).$$

Hence, by Fatou's theorem, the radial limits of f_θ exist almost everywhere and $\lim_{r\to 1^-} f_\theta(re^{i\theta}) = \gamma_0'(\theta)$ a.e., where γ_0 is the absolutely continuous part of γ.

It turns out that if γ is absolutely continuous, then

$$\lim_{r\to 1^-} f_r(re^{i\theta}) = H(\gamma')(\theta) \qquad \text{(a.e.)}.$$

The function γ, of course, need not be absolutely continuous. However:

If

$$\sup_{\rho<1} \frac{1}{2\pi} \int_0^{2\pi} \bigl|f_r(\rho e^{i\theta})\bigr| \, d\theta \ < \infty, \tag{2.4}$$

then γ *is absolutely continuous and, moreover, the functions* $h(e^{i\theta})$ *and* $g(e^{i\theta})$ *are absolutely continuous.*

This is one of possible formulations of a classical theorem of Riesz (cf. Zygmund [11, Chapter VII, Section (8.3)]). Usually this theorem is stated in the following way (cf. [4]):

If the derivative of an analytic function ϕ *belongs to the Hardy space* H^1, *then* $\phi(e^{i\theta})$ *is absolutely continuous.*

In view of the formulae (2.2) and (2.3) condition (2.4) implies that h' and g' are in H^1.

Using these formulae one can easily show that (1.4) implies

$$\frac{1-k}{1+k} \le \left|\frac{r f_r(z)}{f_\theta(z)}\right| \le \frac{1+k}{1-k} \qquad (z \in \mathbf{D}). \tag{2.5}$$

Thus:

If f *is quasiconformal, then* φ *is absolutely continuous.*

From now on we will suppose that φ is absolutely continuous. Then there hold the formulae

$$f_\theta(e^{i\theta}) = \gamma'(\theta) = i\varphi'(\theta)e^{i\varphi(\theta)} \tag{2.6}$$

and

$$f_r(e^{i\theta}) = H(\gamma')(\theta) = -\frac{1}{\pi} \int_{+0}^{\pi} \frac{\gamma(\theta+t)+\gamma(\theta-t)-2\gamma(\theta)}{4\sin^2(t/2)} \, dt. \tag{2.7}$$

By straightforward computation we find that

$$e^{-i\varphi(\theta)} f_r(e^{i\theta}) = A(\theta) + iB(\theta), \tag{2.8}$$

where

$$\begin{aligned} A(\theta) &= \frac{1}{\pi} \int_{+0}^{\pi} \frac{2-\cos\bigl(\varphi(\theta+t)-\varphi(\theta)\bigr)-\cos\bigl(\varphi(\theta-t)-\varphi(\theta)\bigr)}{4\sin^2(t/2)} \, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\sin\bigl(\varphi(\theta+t)/2-\varphi(\theta)/2\bigr)}{\sin(t/2)}\right)^2 dt \end{aligned} \tag{2.9}$$

and

$$B(\theta) = -\frac{1}{\pi}\int_{+0}^{\pi} \frac{\sin\bigl(\varphi(\theta+t)-\varphi(\theta)\bigr)+\sin\bigl(\varphi(\theta-t)-\varphi(\theta)\bigr)}{4\sin^2(t/2)}\,dt. \tag{2.10}$$

Then using (2.2) and (2.3) we get

$$|h'(e^{i\theta})|^2 = \tfrac{1}{2}\bigl((A(\theta)+\varphi'(\theta))^2 + B(\theta)^2\bigr) \tag{2.11}$$

and

$$|g'(e^{i\theta})|^2 = \tfrac{1}{2}\bigl((A(\theta)-\varphi'(\theta))^2 + B(\theta)^2\bigr). \tag{2.12}$$

Since the function g'/h' is analytic and bounded, by (1.3), we find that

$$k^2 = \sup_{z\in\mathbf{D}}\left|\frac{g'(z)}{h'(z)}\right|^2 = \operatorname*{ess\,sup}_{\theta} \frac{\varphi'(\theta)^2 + A(\theta)^2 + B(\theta)^2 - 2\varphi'(\theta)A(\theta)}{\varphi'(\theta)^2 + A(\theta)^2 + B(\theta)^2 + 2\varphi'(\theta)A(\theta)}.$$

Hence:

The mapping f is quasiconformal if and only if

$$K := \operatorname*{ess\,sup}_{\theta\in\mathbf{R}} \frac{\varphi'(\theta)^2 + A(\theta)^2 + B(\theta)^2}{2\varphi'(\theta)A(\theta)} < \infty. \tag{2.13}$$

There holds the formula

$$k = \left(\frac{K-1}{K+1}\right)^{1/2}.$$

Now it is easy to show that conditions (1.6), (1.7) and (1.8) imply that f is quasiconformal. We have only to note that condition (1.7) implies

$$\|B - H\varphi'\|_\infty \le C\|\varphi'\|_\infty^2, \tag{2.14}$$

where C is an absolute constant; this inequality is deduced from (2.1) by using the relation $x - \sin x = O(x^3)$.

3. The necessity proof

Let f be quasiconformal. Then $K < \infty$ (see (2.13), i.e.,

$$\varphi'(\theta)^2 + A(\theta)^2 + B(\theta)^2 \le 2K\varphi'(\theta)A(\theta). \tag{3.1}$$

It follows that $A(\theta)^2 \le 2K\varphi'(\theta)A(\theta)$ and therefore

$$\varphi'(\theta) \ge \frac{1}{2K}A(\theta). \tag{3.2}$$

Since

$$\begin{aligned} A(\theta) &\ge \frac{1}{4\pi}\int_{-\pi}^{\pi}\big(1-\cos(\varphi(\theta+t)-\varphi(\theta))\big)\,dt \\ &= \tfrac{1}{2}\big(1-\mathrm{Re}\big(e^{-i\varphi(\theta)}f(0)\big)\big) \ge \tfrac{1}{2}(1-|f(0)|), \end{aligned}$$

we get $\operatorname{ess\,inf}\varphi'(\theta) > 0$. Thus condition (1.6) is satisfied.

In order to verify (1.7) we use the inequality

$$\varphi'(\theta) \le C\int_{-\pi}^{\pi}\left(\frac{\varphi(\theta+t)-\varphi(\theta)}{t}\right)^2 dt \tag{3.3}$$

(C is an absolute constant) which is obtained from (3.1). Assume first that φ is of class C^2 and choose θ so that $\varphi'(\theta) = \max\varphi' =: M$. Let $0 < \beta < 1$. It follows from (3.3) that

$$M \le C\int_{-\pi}^{\pi}\left(\frac{\varphi(\theta+t)-\varphi(\theta)}{t}\right)^{2-\beta} M^\beta\,dt,$$

whence

$$M^{1-\beta} \le C\int_{-\pi}^{\pi}\left(\frac{\varphi(\theta+t)-\varphi(\theta)}{t}\right)^{2-\beta} dt.$$

Now we apply Mori's inequality (1.9) to deduce that

$$M^{1-\beta} \le C_1\int_0^{\pi}(t^{\alpha-1})^{2-\beta}\,dt, \qquad \alpha = \frac{1-k}{1+k}.$$

Choose β so that $(\alpha-1)(2-\beta) > -1$, which is possible because $(\alpha-1)(2-\beta) \to \alpha-1 > -1$ as $\beta \to 1^-$, to get

$$\max\varphi' \le C_2, \tag{3.4}$$

where C_2 depends only on K. From this and (3.2) we get $A(\theta) \le 2KC_2$ and hence, by (3.1) and (2.11), $|h'(e^{i\theta})| \le C_3$. The function $h'(z)$ is continuous on the closed disk because the function $\gamma = e^{i\varphi}$ is C^2, so we have

$$|h'(z)| \le C_3 \qquad (z \in \mathbf{D}), \tag{3.5}$$

and the constant C_3 depends only on K.

In the general case we proceed as in [3]: we consider the mappings f_n, of $\mathbf{D}$ onto $\mathbf{D}$, defined by

$$f_n(z) = f\bigl(w_n(z)\bigr)/r_n = h_n(z) + \overline{g_n(z)} \qquad (r_n = 1 - 1/n,\ n \ge 2),$$

where w_n is the conformal mapping of $\mathbf{D}$ onto $G_n = f^{-1}(r_n\mathbf{D})$, $w_n(0) = 0$, $w_n'(0) > 0$. Since the boundary of G_n, for n large enough, is an analytic Jordan curve, the mapping w_n can be continued analytically across $\partial\mathbf{D}$, which implies that f_n has a harmonic extension across $\partial\mathbf{D}$. Since also

$$\left|\frac{g_n'}{h_n'}\right| = \left|\frac{(g' \circ w_n)w_n'}{(h' \circ w_n)w_n'}\right| \le k,$$

we can appeal to the preceding special case to conclude that $\bigl|h'\bigl(w_n(z)\bigr)\bigr|\,|w_n'(z)|/r_n \le C_3$, where C_3 is independent of n and z. And since $G_n \subset G_{n+1}$ and $\cup G_n = \mathbf{D}$, we can apply the Carathéodory convergence theorem (cf. [5]): $w_n(z)$ tends to z, uniformly on compacts, whence $w_n'(z) \to 1$ $(n \to \infty)$. Thus inequality (3.5) holds in the general case. Using this and (2.11) we get $\varphi'(\theta) + |B(\theta)| \le C_4$. Finally, it remains to apply (2.14).

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