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WALL PROPERTIES OF DOMAINS IN EUCLIDEAN SPACES

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Abstract. Wall theorems give local lower bounds for the p-measure of the boundary of a domain in the euclidean n-space. We improve earlier results by replacing the euclidean metric by the inner metric of the domain and also the Hausdorff p-content by the projectional p-measure.

1. Introduction

Wall properties of domains in the euclidean n-space R^n were first considered by J. Heinonen [He] in 1996 in connection with quasiconformal maps. Suppose that $G \subset \mathsf{R}^n$ is a domain, K-quasiconformally equivalent to a ball. Let $a \in G$ and set $r = d(a, \partial G)$. Heinonen made the following *quasiconformal wall conjecture*:

$$m_{n-1}\bigl(\partial G \cap B(a, 2r)\bigr) \geq r^{n-1}/c, \tag{1.1}$$

where m_{n-1} is the $(n-1)$-dimensional Hausdorff measure, and the constant c depends only on K and n. The case $n = 2$ is trivial, and the case $n = 3$ was proved in [He, Section 5].

The author [Vä] proved a result that implied the wall conjecture in all dimensions. Instead of quasiconformality, the result was formulated in terms of homological [Vä, 1.3] or homotopical [Vä, 6.2] connectivity properties of the domain G. Moreover, the result gave a lower bound for the Hausdorff p-measure m_p, $1 \leq p \leq n-1$, and in fact, for the Hausdorff p-content m_p^∞. We recall the homotopical version.

1.2. Theorem ([Vä, 6.2]). *Let $n \geq 2$ and $1 \leq p \leq n-1$ be integers and let $c \geq 1$. Suppose that $G \subsetneq \mathsf{R}^n$ is a domain with the following properties*:

(1) *Every map $f\colon S^{n-1} \to G$ is null-homotopic.*

(2) *If $x \in \mathsf{R}^n$, $t > 0$, $n-p \leq k \leq n-2$, then every map $f\colon S^k \to G \cap B(x,t)$ is null-homotopic in $G \cap B(x, ct)$.*

Let $a \in G$ and set $r = d(a, \partial G)$. Then

$$m_p^\infty\bigl(\partial G \cap B(a, 2r)\bigr) \geq r^p/c_1$$

with $c_1 = c_1(c, n, p)$.

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A K-quasiconformal ball satisfies these conditions for $1 \le p \le n-2$ with $c = c(K,n)$, and (1.1) follows. In the homological version, (1) and (2) are replaced by

(1) $H_{n-1}(G) = 0$,

(2) the natural homomorphisms $H_k\big(G \cap B(x,t)\big) \to H_k\big(G \cap B(x,ct)\big)$ are zero for $n-p \le k \le n-2$.

In a recent paper [BK, 6.5], M. Bonk and P. Koskela proved an *inner version* of (1.1), where the euclidean ball $B(a,2r)$ was replaced by a ball in the inner length metric of G (and m_{n-1} by m_{n-1}^{∞}).

The purpose of this paper is to prove inner versions of the results in [Vä]. We find it convenient to replace the Hausdorff content m_p^{∞} by the *projectional measure* μ_p, defined in 2.3. Since $\mu_p \le m_p^{\infty} \le m_p$, the new results are somewhat stronger than those in [Vä] also in the euclidean metric.

I thank M. Bonk for calling my attention to this problem.

2. Preliminaries

2.1. Notation. Open balls in R^n are written as $B(x,r)$ where x is the center and r is the radius. For $a, b \in \mathsf{R}^n$, we let $[a,b]$ denote the closed line segment with endpoints a, b, and we set $[a,b) = [a,b] \setminus \{b\}$. For $1 \le p \le n$, the family of all p-dimensional linear subspaces of R^n is written as $\mathsf{G}_p(\mathsf{R}^n)$. For $E \in \mathsf{G}_p(\mathsf{R}^n)$, we let $\pi_E\colon \mathsf{R}^n \to E$ denote the orthogonal projection. The distance between nonempty subsets A, B of R^n is $d(A,B)$, and the diameter of a set $A \subset \mathsf{R}^n$ is $d(A)$. The volume of the unit ball in R^n is $\alpha(n)$.

2.2. Hausdorff measure and content. For $A \subset \mathsf{R}^n$ we let $m_p(A)$ denote the Hausdorff (outer) p-measure of A, defined and normalized as in [Fe, 2.10.2], and the Hausdorff p-content $m_p^{\infty}(A)$ is defined similarly but without any restrictions on the diameters of the covering sets. Then $m_p^{\infty}(A) \le m_p(A)$. We consider only the case where p is an integer, $0 \le p \le n$, and the normalization means that $m_p(A) = m_p^{\infty}(A)$ is the Lebesgue (outer) p-measure of A whenever A lies in a p-dimensional affine subspace of R^n.

2.3. Projectional measure. Let $n \ge p \ge 1$ be integers. The *projectional p-measure* of a set $A \subset \mathsf{R}^n$ is the number

$$\mu_p(A) = \sup\{m_p(\pi_E A)\colon E \in \mathsf{G}_p(\mathsf{R}^n)\}.$$

Since the diameter of a set is decreased in each π_E, we have always

$$\mu_p(A) \le m_p^{\infty}(A) \le m_p(A),$$

but there are, for example, sets $A \subset \mathsf{R}^2$ with $\mu_1(A) = 0 < m_1^{\infty}(A)$; see [Ma, 9.2]. It is easy to see that the set function μ_p is monotone and countably subadditive, hence a measure in the sense of [Fe, 2.1.2].

3. Projectional versions of the wall theorems

We show that one can replace the Hausdorff content m_p^∞ by the projectional measure μ_p in the wall theorems of [Vä].

3.1. Wall theorem. *The homological* [Vä, 1.3], *homotopical* (1.2 *and* [Vä, 6.2]) *and the quasiconformal* [Vä, 1.11] *wall theorems are true with* m_p^∞ *replaced by* μ_p.

Proof. Inspection of the proofs of [Vä] shows that it suffices to prove a projectional version of the Grid lemma [Vä, 2.3]. This is done in 3.2 below. We recall the notation of [Vä, 2.2].

For $s > 0$ let $\mathscr{K}_s$ be the natural decomposition of R^n into closed cubes of side s. For $0 \le k \le n$ we let K_s^k denote the union of all k-dimensional faces of the members of $\mathscr{K}_s$.

3.2. Grid lemma. *Suppose that* $1 \le p \le n$, *that* $s > 0$, *and that* $A \subset \mathsf{R}^n$ *is a set with* $\mu_p(A) < s^p/N$, *where* N *is the binomial coefficient* $\binom{n}{p}$. *Then there is* $a \in \mathsf{R}^n$ *such that* $A \cap (K_s^{n-p} + a) = \varnothing$.

Proof. We may assume that $s = 1$. Let F be one of the N linear subspaces of R^n spanned by p vectors of the standard basis $(e_1, \dots, e_n)$. Set $J = [0,1)^n$, $J_F = \pi_F J$ and

$$D_F = \bigcup \bigl\{ J_F \cap (\pi_F A - w) \colon w \in \mathsf{Z}^n \cap F \bigr\}.$$

Since the p-cubes $J_F + w$ are disjoint and m_p measurable, we obtain (see [HS, 10.9])

$$\begin{aligned} m_p(D_F) &\le \sum_w m_p\bigl(J_F \cap (\pi_F A - w)\bigr) = \sum_w m_p\bigl((J_F + w) \cap \pi_F A\bigr) \\ &= m_p(\pi_F A) \le \mu_p(A) < 1/N. \end{aligned}$$

Set $D_F^* = J \cap \pi_F^{-1} D_F$ and $D = \bigcup_F D_F^*$. By Fubini's theorem we obtain $m_n(D_F^*) < 1/N$, and hence $m_n(D) < 1$. Since K_1^{n-p} is the union of the sets $\pi_F^{-1}[\mathsf{Z}^n \cap F]$, the lemma holds with any $a \in J \setminus D$. □

3.3. Remarks. 1. Lemma 3.2 has three advantages compared with [Vä, 2.3]: (1) Since $\mu_p \le m_p^\infty$, it is stronger, (2) the constant N is smaller than the corresponding constant $\beta(n,p)$ of [Vä, 2.3], (3) the proof is simpler.

2. In [Vä, 2.3] there is a misprint: s/β should be s^p/β.

4. Wall properties and inner wall properties

4.1. Wall properties. Let $c > 0$, and let $n \ge 2$ and $1 \le p \le n-1$ be integers. We say that a domain $G \subsetneq \mathsf{R}^n$ has the (c,p)-*wall property* if

$$\mu_p\big(\partial G \cap B(a,2r)\big) \ge r^p/c$$

whenever $a \in G$ and $r = d(a,\partial G)$. Observe that since $\mu_p\big(B(a,2r)\big) = \alpha(p)(2r)^p$, we always have $c \ge 2^{-p}/\alpha(p)$.

4.2. Inner wall properties. Let $G \subset \mathsf{R}^n$ be a domain. We let $\lambda = \lambda_G$ denote the inner length metric of G, defined by

$$\lambda(a,b) = \inf_\gamma l(\gamma),$$

where $l(\gamma)$ is the length of γ and the infimum is taken over all rectifiable arcs γ joining a and b in G. The distance $\lambda(a,b)$ is also defined if $a \in G$ and b is a boundary point of G, accessible from G by a rectifiable arc. For $a \in G$ and $r > 0$, we let $B_\lambda(a,r)$ denote the set of all $x \in \overline{G}$ such that $\lambda(a,x)$ is defined and $\lambda(a,x) < r$. Since $\lambda(a,b) \le |a-b|$, we always have $B_\lambda(a,r) \subset B(a,r)$.

We say that a domain $G \subsetneq \mathsf{R}^n$ has the *inner* (c,p)-*wall property* if

$$\mu_p\big(\partial G \cap B_\lambda(a,2r)\big) \ge r^p/c$$

whenever $a \in G$ and $r = d(a,\partial G)$. Again $c \ge 2^{-p}/\alpha(p)$.

Trivially, the (c,p)-wall property implies the inner (c,p)-wall property. The purpose of this section is to prove the converse where, however, c must be replaced by a larger constant $c'(c,p)$. This implies the inner version of the Wall Theorem 3.1, given in 4.9.

4.3. Lemma. *Let $G \subsetneq \mathsf{R}^n$ be a domain, let $p \in [1,n-1]$ be an integer and let $s \ge 2$. Suppose that*

$$\mu_p\big(\partial G \cap B_\lambda(a,sr)\big) \ge r^p/c$$

whenever $a \in G$ and $r = d(a,\partial G)$. Then G has the inner (c',p)-wall property with $c' = (s-1)^p c$.

Proof. Let $a \in G$ and $r = d(a,\partial G)$. Choose a point $b \in \partial G$ with $|a-b| = r$, and set

$$e = (b-a)/r, \quad t = r/(s-1), \quad y = b - te.$$

Then $d(y,\partial G) = t$. It is easy to see that $B_\lambda(y,st) \subset B_\lambda(a,2r)$. Hence

$$\mu_p\big(\partial G \cap B_\lambda(a,2r)\big) \ge \mu_p\big(\partial G \cap B_\lambda(y,st)\big) \ge t^p/c = r^p/c'. \ \square$$

4.4. Theorem. *If a domain $G \subsetneq \mathsf{R}^n$ has the (c,p)-wall property, it has the inner (c',p)-wall property with $c' = c'(c,p)$.*

Proof. Let $a \in G$ and $r = d(a, \partial G)$. It suffices to find an estimate

$$\mu_p\big(\partial G \cap B_\lambda(a, sr)\big) \ge r^p/c_1, \tag{4.5}$$

where c_1 and $s \ge 2$ depend only on c and p, since the theorem then follows from 4.3 with $c' = (s-1)^p c_1$. We shall prove (4.5) with the universal constant $s = \frac{9}{4}$ and with $c_1 = 4 \cdot 8^p c$, which yield $c' = 4 \cdot 10^p c$.

We shall several times make use of the elementary inequality

$$|\pi_E x| + |x - \pi_E x| \le |x|\sqrt{2}\,, \tag{4.6}$$

valid for all $x \in \mathsf{R}^n$ and for all linear subspaces $E \subset \mathsf{R}^n$.

We may assume that $a = 0$. Choose a point $b \in \partial G$ with $|b| = r$, and set

$$e = b/|b|, \quad t = r/8, \quad y = b - te = 7te, \quad A = \partial G \cap B(y, 2t).$$

Since $\mu_p(A) \ge t^p/c$, there is a subspace $F \in \mathsf{G}_p(\mathsf{R}^n)$ such that $m_p(\pi_F A) \ge t^p/2c$. Set $\pi = \pi_F$, $\pi' = \pi_{F^\perp}$. By (4.6) we have either $|\pi y| \le |y|/\sqrt{2}$ or $|\pi' y| \le |y|/\sqrt{2}$.

Case 1: $|\pi y| \le |y|/\sqrt{2}$. For each $x \in A$ we have

$$|\pi x| \le |\pi y| + 2t \le 7t/\sqrt{2} + 2t < 7t < r,$$

and hence $[0, \pi x] \subset G$. Since $x \in \partial G$, there is a point $z_x \in \partial G \cap [\pi x, x]$ with $[\pi x, z_x) \subset G$. By (4.6) we obtain

$$\begin{aligned} \lambda(0, z_x) &\le l\big([0, \pi x] \cup [\pi x, z_x]\big) \le |\pi x| + |\pi x - x| \le |x|\sqrt{2} \\ &\le (|y| + 2t)\sqrt{2} = 9t\sqrt{2} < 2r. \end{aligned}$$

Setting $A_1 = \{z_x : x \in A\}$ we thus have $A_1 \subset B_\lambda(0, 2r)$ and $\pi A_1 = \pi A$. Hence

$$\mu_p\big(\partial G \cap B_\lambda(0, 2r)\big) \ge m_p(\pi A_1) = m_p(\pi A) \ge t^p/2c,$$

which implies (4.5) with $s = 2$, $c_1 = 2 \cdot 8^p c$.

Case 2: $|\pi' y| \le |y|/\sqrt{2}$. Set $q = n - p$ and

$$B = B(y, 2t), \quad Z = \pi B + \pi' B, \quad D_x = \pi x + \pi' B$$

for $x \in A$. Then $x \in D_x$, and D_x is a q-disk of radius $2t$ with $\pi' D_x = \pi' B$. For $z \in Z$ we have

$$|\pi' z| \le |\pi' y| + 2t \le |y|/\sqrt{2} + 2t = \big(7/\sqrt{2} + 2\big)t < 7t < r. \tag{4.7}$$

Furthermore, by (4.6) we get

$$|\pi z| + |\pi' z| \le |\pi y| + |\pi' y| + 4t \le |y|\sqrt{2} + 4t < 14t < 2r \tag{4.8}$$

for all $z \in Z$.

We consider two subcases, each of which contains two subsubcases.

Subcase 2a: $p \le q$.

Subsubcase 2a1. There is a point $x \in A$ such that ∂G meets $[w, \pi' w]$ for each $w \in D_x$. Let $w \in D_x$. Then $|\pi' w| < r$ by (4.7). Now there is a point $z_w \in \partial G \cap [\pi' w, w]$ with $[\pi' w, z_w) \subset G$. By (4.8) we obtain

$$\lambda(0, z_w) \le |\pi' w| + |\pi' w - z_w| \le |\pi' w| + |\pi' w - w| < 2r.$$

Setting $D = \{z_w : w \in D_x\}$ we have $D \subset \partial G \cap B_\lambda(0, 2r)$, and $\pi' D = \pi' D_x$ is the q-disk $F^\perp \cap B(\pi' y, 2t)$.

Since $p \le q$, we can choose a subspace $E \in \mathsf{G}_p(\mathsf{R}^n)$ with $E \subset F^\perp$. Then $\pi_E D$ is a p-disk of radius $2t$, and thus

$$\mu_p\bigl(\partial G \cap B_\lambda(0, 2r)\bigr) \ge m_p(\pi_E D) = \alpha(p) 2^p t^p.$$

Since $2^p \alpha(p) \ge 1/c$ by 4.1, this gives (4.5) with $s = 2$, $c_1 = 8^p c$.

Subsubcase 2a2. For each $x \in A$ there is a point $w_x \in D_x$ such that $\partial G \cap [w_x, \pi' w_x] = \varnothing$. Then $[w_x, \pi' w_x] \subset G$. Since $x \in \partial G$, there is a point $z_x \in \partial G \cap [x, w_x]$ with $[w_x, z_x) \subset G$. Since $|w_x - z_x| \le |w_x - x| \le d(D_x) = 4t$, we get by (4.8)

$$\lambda(0, z_x) \le |\pi' w_x| + |\pi' w_x - w_x| + |w_x - z_x| < 14t + 4t = 18t.$$

Hence the set $A_1 = \{z_x : x \in A\}$ lies in $\partial G \cap B_\lambda(0, 9r/4)$. Since $\pi A_1 = \pi A$, we obtain

$$\mu_p(A_1) \ge m_p(\pi A_1) = m_p(\pi A) \ge t^p / 2c,$$

and (4.5) follows with $s = \frac{9}{4}$, $c_1 = 2 \cdot 8^p c$.

Subcase 2b: $p > q$. Since

$$\dim(F^\perp \cup \{y\})^\perp = p - 1 \ge p - q,$$

we can choose $F_1 \in \mathsf{G}_{p-q}(\mathsf{R}^n)$ with $F_1 \subset (F^\perp \cup \{y\})^\perp$. Setting $E = F^\perp + F_1$ we have $\dim E = p$ and $\pi_E y = \pi' y$.

Recall the notation $D_x = \pi x + \pi' B$ for $x \in A$. Let C be the union of those D_x for which $\partial G \cap [w, \pi_E w] \ne \varnothing$ for all $w \in D_x$. Then $C \subset Z \subset B\bigl(y, 2t\sqrt{2}\bigr)$.

Subsubcase 2b1: $m_p(\pi_E C) \ge t^p / 4c$. For each $x \in C$ we have

$$|\pi_E x| \le |\pi_E y| + 2t\sqrt{2} \le 7t/\sqrt{2} + 2t\sqrt{2} < 8t = r;$$

hence $[0, \pi_E x] \subset G$. Choose a point $z_x \in \partial G \cap [\pi_E x, x]$ such that $[\pi_E x, z_x) \subset G$. By (4.6) we obtain

$$\begin{aligned} \lambda(0, z_x) &\le |\pi_E x| + |\pi_E x - z_x| \le |\pi_E x| + |\pi_E x - x| \le |x|\sqrt{2} \\ &\le \bigl(7t + 2t\sqrt{2}\bigr)\sqrt{2} < 14t < 2r. \end{aligned}$$

Hence the set $C_1 = \{z_x : x \in C\}$ lies in $\partial G \cap B_\lambda(0, 2r)$. Since

$$\mu_p(C_1) \ge m_p(\pi_E C_1) = m_p(\pi_E C) \ge t^p / 4c,$$

we get (4.5) with $s = 2$, $c_1 = 4 \cdot 8^p$.

Subsubcase 2b2: $m_p(\pi_E C) \le t^p/4c$. For $v \in \pi_E C$ set $Q(v) = \pi_E^{-1}\{v\} \cap Z$. Since $E^\perp \subset F$, we have for each $x \in Q(v)$

$$|\pi_{E^\perp} x - \pi_{E^\perp} y| \le |\pi x - \pi y| < 2t.$$

Thus $Q(v)$ is contained in a q-disk of radius $2t$. By Fubini's theorem we obtain

$$m_n(C) \le \int_{\pi_E C} m_q\bigl(Q(v)\bigr)\, dm_p(v) \le \alpha(q)(2t)^q m_p(\pi_E C) \le \alpha(q)(2t)^q t^p/4c.$$

On the other hand, Fubini's theorem also gives

$$m_n(C) = m_q(D_x) m_p(\pi C) = \alpha(q)(2t)^q m_p(\pi C),$$

and hence $m_p(\pi C) \le t^p/4c$. Since $m_p(\pi A) \ge t^p/2c$, we obtain $m_p(\pi[A \setminus C]) \ge t^p/4c$.

Let $x \in A \setminus C$. There is a point $w_x \in D_x$ such that $\partial G \cap [w_x, \pi_E w_x] = \varnothing$. Since $|\pi_E w_x| \le |\pi_E y| + 3t = |\pi' y| + 3t < r$, we have $[w_x, \pi_E w_x] \subset G$. Since $x \in \partial G$, there is a point $z_x \in \partial G \cap [x, w_x]$ with $[w_x, z_x) \subset G$. Then $|w_x - z_x| \le |w_x - x| \le d(D_x) \le 4t$. By (4.6) we get

$$\begin{aligned}\lambda(0, z_x) &\le |\pi_E w_x| + |\pi_E w_x - w_x| + |w_x - z_x| \\ &\le |w_x|\sqrt{2} + 4t \le \bigl(7 + 2\sqrt{2}\bigr)t\sqrt{2} + 4t < 18t.\end{aligned}$$

Hence the set $A_1 = \{z_x : x \in A \setminus C\}$ lies in $B_\lambda(0, 9r/4)$. Since

$$\mu_p(A_1) \ge m_p(\pi_F A_1) = m_p(\pi_F[A \setminus C]) \ge t^p/4c,$$

we get (4.5) with $s = 9r/4$, $c_1 = 4 \cdot 8^p c$. □

4.9. Inner wall theorem. *Suppose that a domain $G \subsetneq \mathsf{R}^n$ satisfies the hypotheses of one of the wall theorems* [Vä, 1.3], [Vä, 6.2] (*see* 1.2) *with a constant $c > 0$ and an integer $p \in [1, n-1]$. Then G has the inner (c', p)-wall property with $c' = c'(c, n, p)$. If G is K-quasiconformally equivalent to a ball, then G has the inner $(c', n-1)$-wall property with $c' = c'(K, n)$.*

Proof. The theorem follows from Theorems 3.1 and 4.4. □

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