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LIMIT FUNCTIONS IN WANDERING DOMAINS OF MEROMORPHIC FUNCTIONS

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Abstract. Let f be a function which is meromorphic outside a sufficiently small, non-empty, totally-disconnected compact set of essential singularities, and let U be a wandering component of the Fatou set of f. We prove that any limit function of a subsequence of iterates of f in U is a constant which lies in the derived set of the forward orbit of the set of singular points of the inverse f. This extends results of Bergweiler et al. and Zheng about transcendental entire or meromorphic functions in $\mathbf{C}$.

1. Introduction

Let $g\colon \mathbf{C} \to \widehat{\mathbf{C}}$ be a transcendental meromorphic function and g^n the n-th iterate of g, $n = 0, 1, 2, \dots$ (where $g^0 = \mathrm{Id}$). The Fatou set $F(g) = \{z : \{g^n\}$ is meromorphic and normal in some neighbourhood of $z\}$ and the Julia set $J(g)$ is $\widehat{\mathbf{C}} \setminus F(g)$. See [2] for the basic results about meromorphic iteration.

It is sometimes inconvenient that the iterates of a function such as g are not in general meromorphic. In [1] a more general theory is developed without this disadvantage. For a set A, let $A^c = \widehat{\mathbf{C}} \setminus A$. We consider the class M of functions f for which there is a compact, totally-disconnected set $E = E(f) \subset \widehat{\mathbf{C}}$ such that f is meromorphic in E^c, and for each $z_0 \in E$ the cluster set $C(f, E^c, z_0) = \{w : w = \lim_{n\to\infty} f(z_n)$ for some $z_n \in E^c$ with $z_n \to z_0\}$ is $\widehat{\mathbf{C}}$. If $E = \emptyset$ we make the further assumption that f is neither constant nor univalent in $\widehat{\mathbf{C}}$.

For technical reasons we introduce a subset MA of M. We say that $f \in M$ has the k-island property at z_0 in $E(f)$ if, given any neighbourhood U of z_0 and k simply-connected domains Δ_i in $\widehat{\mathbf{C}}$ which have disjoint closures and which are bounded by sectionally analytic Jordan curves, there is a simply-connected subdomain D in $U \setminus E(f)$ which maps univalently under f onto one of the Δ_i. MA_k is the set of $f \in M$ such that $E(f) \neq \emptyset$ and for each $z_0 \in E(f)$ the function f has the k-island property at z_0. MA is the union of all MA_k, $k \in \mathbf{N}$.

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A simpler but smaller class of functions introduced by A. Bolsch [4] is $K = \{f \in M : E(f)$ is compact and countable$\}$. We have $K \subset MA_k \subset M$, and all three classes are closed under composition of functions. Thus K includes any function of the type $f_1 \circ f_2 \circ \cdots \circ f_n$, where f_i are meromorphic in $\mathbf{C}$.

It is shown in [1] that applying the definitions given above for $F(g)$, $J(g)$ to functions f in M yields a completely invariant $F(f)$ and a perfect set $J(f)$. Moreover $f \in M$ implies that $f^m \in M$ and $F(f^m) = F(f)$. If, in addition, $f \in MA$, then repelling cycles are dense in $J(f)$.

If U is a component of $F(f)$, $f \in M$, then for each $k \in \mathbf{N}$, $f^k(U)$ is contained in some component U_k of $F(f)$. If the sequence U_k is eventually periodic we have only to understand the behaviour of iterates in some periodic component, indeed by replacing f by an iterate we could study only invariant components. Here only a few cases arise and are reasonably well understood (see below). The other case is when all U_k are different and U is said to be a wandering component. In such a component any convergent sequence of iterates has a limit function which is a constant in $J(f)$ (see [1]).

The possible constant limits are connected with the singular values of the inverse f^{-1}. If $f \in M$, the set $S(f)$ of singular values of some branch of f^{-1} consists of the critical values $f(c)$, where $f'(c) = 0$, together with the set of all asymptotic values of f: w is an asymptotic value of f if there is some $z_0 \in E(f)$ and a path $\gamma(t)$, $0 \le t < 1$, in $E(f)^c$ such that $\gamma(t) \to z_0$ and $f\big(\gamma(t)\big) \to w$ as $t \to 1$. Further, $E_j(f) = \bigcup_{k=0}^{j-1} f^{-k}\big(E(f)\big)$ is the set of essential singularities of f^j, and the set where for some $n \in \mathbf{N}$ some branch of f^{-n} has a singularity is

$$P(f) = \bigcup_{j=0}^{\infty} f^j\big(S(f) \setminus E_j(f)\big), \qquad \text{where } E_0(f) = \emptyset.$$

Thus $P(f)$ consists of the forward orbit of $S(f)$, so far as this is defined. For a set A, the derived set of A is denoted by A'.

Theorem 1. *If $f \in MA$ and U is a wandering component of $F(f)$, then any limit function of a sequence of iterates in U is a constant which lies in $P(f)'$.*

Remarks. MA includes K and in particular any (non-constant, non-univalent) function meromorphic in $\mathbf{C}$. Rational functions have no wandering components but transcendental ones may do so. The result was proved for transcendental entire functions by Bergweiler et al. [3]. Zheng [9] extended it to transcendental meromorphic functions but with an additional hypothesis. In [10] he has removed this restriction.

The theorem can be used to prove the non-existence of wandering domains given suitable information about $S(f)$ and its forward orbit.

Recall that a fixed point a of f satisfies $f(a) = a$. If a is finite the multiplier is $\lambda(a) = f'(a)$. If a is ∞ we define the multiplier by conjugating f so that a

becomes finite. The fixed point is attracting if $|\lambda(a)| < 1$ and parabolic if $\lambda(a)$ is a primitive p-th root of unity for some $p \in \mathbf{N}$. Concerning the behaviour of iterates in a non-wandering component it is enough to know the following.

If $f \in M$ and U is a component of $F(f)$ such that $f(U) \subset U$, then precisely one of the following is true; see [1].

(i) *U contains an attracting fixed point a. For $z \in U$ we have $f^n(z) \to a$ as $n \to \infty$. U is called the immediate attractive basin of a.*

(ii) *U is a domain of attraction of a parabolic fixed point $a \in \partial U$ and for $z \in U$ we have $f^n(z) \to a$ as $n \to \infty$.*

(iii) *There is an analytic homeomorphism $\psi\colon U \to D$, where D is the unit disc, such that $\psi\big(f\big(\psi^{-1}(z)\big)\big) = e^{2\pi i\alpha}z$, for some $\alpha \in \mathbf{R} \setminus \mathbf{Q}$. In this case U is called a Siegel disc.*

(iv) *There is an analytic homeomorphism $\psi\colon U \to A$ where A is an annulus $A = \{z : 1 < |z| < r\}$ such that $\psi\big(f\big(\psi^{-1}(z)\big)\big) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbf{R} \setminus \mathbf{Q}$. In this case U is called a Herman ring.*

(v) *There exists $a \in \partial U$ such that $f^n(z) \to a$, for $z \in U$ as $n \to \infty$, but f is not meromorphic at a. There exists a path $\gamma \subset U$ leading to a such that $f(\gamma) \subset \gamma$ and the spherical distance between $f(z)$ and z tends to zero as $z \to a$ on γ. In this case U is called essentially parabolic at a or alternatively a 'Baker domain'.*

We may note that if a is a parabolic fixed point whose multiplier is a primitive p-th root of unity, then it has multiplier 1 as a fixed point of f^p. In fact there are then kp domains U_i of $F(f)$ (for some $k \in \mathbf{N}$), each invariant under f^p, in which $f^n \to a$. We can now state the following.

Theorem 2. *If $f \in MA$ and U is a component of $F(f)$ such that $f^n \to a$ in U, where $a \in \widehat{\mathbf{C}}$, then either* (i) *$a$ is an attracting fixed point of f,* (ii) *a is a parabolic fixed point of f, or* (iii) *$a \in E(f) \cap S(f)'$.*

Thus the only cases where f is meromorphic at a are (i) and (ii) which correspond to the first two non-wandering cases described above. In (iii) U is either a wandering domain or a Baker domain. The idea of the proof goes back to Eremenko and Lyubich [5] who deal with the case where f is transcendental entire and $S(f)$ is bounded ($E(f) = \{\infty\}$). Their work was taken up in the meromorphic case by Bergweiler [2], Rippon and Stallard [8] and Zheng [10], who proved Theorem 2 for functions meromorphic in $\mathbf{C}$ (by a somewhat different method). So far as Theorem 2 relates to the case of a Baker domain it is proved in [1, Theorem F].

An advantage of working in the class MA or K is that we can apply our results automatically to f^p without special discussion of periodic cycles etc. Also we have no need to separate finite and infinite limits.

There are of course many examples in which case (iii) of Theorem 2 applies, for example entire functions with wandering components in which $f^n \to \infty$.

2. Proof of Theorem 1

Suppose that $f \in MA$ has a wandering component U of $F(f)$ such that for some integers n_k increasing to ∞ the sequence $f^{n_k} \to a$ in U, where a is a constant which we may assume to be 0 (for f may be conjugated by a Möbius transform). Recall that $0 \in J(f)$. We assume that there is a positive ε such that the set $N = D(0,\varepsilon) \setminus \{0\}$ does not meet $P(f)$ and shall derive a contradiction.

I. Since periodic cycles are dense in $J(f)$ it follows that f has some cycle $\{\alpha_1, \dots, \alpha_j\}$ of length $j \geq 3$ such that no $\alpha_i = 0$. The constant ε will be chosen so that all α_i are outside N. For a disc $D = D(z_0, r)$ with $\overline{D} \subset U$ we have $D_k = f^{n_k}(D) \subset N$ for all sufficiently large k. We may replace U by a suitable $f^j(U)$ and adjust the notation to get $D_k \subset N$ for all k and $D \subset N$. We have $w_k = f^{n_k}(z_0) \to 0$ as $k \to \infty$.

Let g_k be the branch of f^{-n_k} such that $g_k(w_k) = z_0$. Fix a value of $u_k = \log(w_k)$, noting that $w_k \neq 0$ since $w_k \in F(f)$. Then $h_k(t) = g_k(\exp t)$ is analytic near $t = u_k$, $h_k(u_k) = z_0$, and h_k continues throughout $H = \{t : \operatorname{Re} t < \log \varepsilon\}$ to give a single-valued analytic function in H. Further, h_k takes none of the values α_i for $t \in H$. By Montel's theorem $q_k(v) = h_k\big(u_k + (\log\varepsilon - \operatorname{Re} u_k)v\big)$ is a normal family in $D(0,1)$, so by Marty's criterion the spherical derivatives $q_k^{\#}(0) = |q_k'(0)|/\big(1 + |q_k(0)|^2\big)$ are bounded by some constant B. Hence

$$\frac{(\log\varepsilon - \operatorname{Re} u_k)|w_k g_k'(w_k)|}{1 + |z_0|^2} \leq B,$$

so

$$|(f^{n_k})'(z_0)| \geq \frac{|f^{n_k}(z_0)|(\log\varepsilon - \operatorname{Re} u_k)}{B(1 + |z_0|^2)}, \tag{1}$$

where

$$\operatorname{Re} u_k = \log|w_k| = \log|f^{n_k}(z_0)|.$$

Suppose now that there are arbitrarily large k such that every closed path in D_k is null-homotopic in N. Then the analytic continuation of g_k from w_k within D_k is single-valued and maps D_k to D (univalently). Thus f^{n_k} maps D to D_k univalently. By Koebe's $\frac{1}{4}$-theorem D_k contains a disc of centre w_k and radius $\frac{1}{4}r|(f^{n_k})'(z_0)|$. Since zero is not in D_k we have $\frac{1}{4}r|(f^{n_k})'(z_0)| < |w_k|$. Since $\operatorname{Re} u_k \to -\infty$ as $k \to \infty$, this conflicts with (1) for large k.

II. We conclude from the above that, for all large k, D_k and the component U_k of $F(f)$ with $U_k \supset D_k$ contain a simple closed curve γ_k with $\gamma_k \not\sim 0$ in N. We may assume (by replacing n_k by a subsequence) that for $l > k$ the component of γ_k^c which contains zero also contains $U_l \supset D_l$. Fix such k, l with U_l, U_k in N. Write $\gamma = \gamma_k$, $m = n_l - n_k$ and take a point $z' \in \gamma$. Denote by g

the branch of f^{-m} which maps $w' = f^m(z')$ to z'. Take a value t' of $\log w'$. Then $h(t) = g(\exp t)$, where $h(t') = z'$, continues in H to give a single-valued meromorphic function. As shown e.g. in [6, Chapter 11] either (a) h is univalent in H or (b) h has period $2\pi i p$ for some minimal positive integer p. In case (b) $g(w^p)$ is univalent in $N' = \{w : 0 < |w| < \varepsilon^{1/p}\}$.

Now f^m maps γ to a closed path γ' in U_l. In case (a) there is some simple closed path γ'' in H such that $h\colon \gamma'' \to \gamma$ and we have $\gamma' = \exp\gamma''$, so γ'' is a lift of γ'. Hence $\gamma' \sim 0$ in N. If Δ denotes the interior of γ'', then $h(\Delta)$ is either the interior or exterior of γ, in fact the interior because values of $h(\Delta)$ belong to $g(\exp\Delta) \subset g(N)$ and so do not include the points α_i. Thus in $\operatorname{Int}\gamma$, h^{-1} is univalent with values in Δ and $f^m = \exp\circ h^{-1}$ is analytic with $\partial\bigl(f^m(\operatorname{Int}\gamma)\bigr) \subset f^m(\gamma) = \gamma'$ and $f^m(\operatorname{Int}\gamma) = \exp\Delta \subset N$. Thus f^m maps $\operatorname{Int}\gamma$ into itself which implies that $\operatorname{Int}\gamma \subset F(f^m) = F(f)$. This contradicts $0 \in J(f)$.

It remains to discuss case (b). Let $\tilde\gamma$ be the simple curve in N' which $G(w) = g(w^p)$ maps to γ. Since G is univalent in N', zero is a removable singularity. Assume that $G(0)$ has been defined so as to make G analytic at zero. Then G remains univalent in $N'' = N' \cup \{0\}$. Now $\tilde\Delta = \operatorname{Int}\tilde\gamma$ maps under G into a subset of $g(N) \cup \{G(0)\}$ with boundary values in γ and omits at least one of the points α_i. Thus $G(\tilde\Delta) = \operatorname{Int}\gamma$ and $G^{-1}(\operatorname{Int}\gamma) = \tilde\Delta$. We obtain that $f^m = (G^{-1})^p$ is analytic in $\operatorname{Int}\gamma$ with values in $(\tilde\Delta)^p \subset N \cup \{0\}$ and boundary values in $(\tilde\gamma)^p = \gamma'$ in the interior of γ. Thus f^m maps $\operatorname{Int}\gamma$ into itself and we obtain a contradiction as before. The proof is complete.

3. Proof of Theorem 2

Suppose that $f \in MA$ and $f^n \to a$ in the component U of $F(f)$ as $n \to \infty$. We may suppose that $a = 0$. If f is meromorphic at 0 we have $f(0) = 0$ and $|f'(0)| \le 1$. If $f'(0) = e^{2\pi i\alpha}$, where α is real and irrational, then it is proved in [7] that an orbit $f^n(z) \to 0$ only if $f^n(z) = 0$, for some $n \in \mathbf{N}$. Since there must be an uncountable set of z in U for which $f^n(z) = 0$ with the same n this implies that f^n is identically zero, which is impossible. Thus we have cases (i) or (ii) of the theorem if f is meromorphic at 0, and otherwise $0 \in E(f)$.

Thus we assume that $0 \in E(f)$ and have to prove that $0 \in S(f)'$. Suppose on the contrary that there is some positive ε such that $N = D(0,\varepsilon) \setminus \{0\}$ does not meet $S(f)$. As in the proof of Theorem 1 we may assume that there is a periodic cycle $\{\alpha_1, \dots, \alpha_j\}$ of length $j \ge 3$ outside N. By a suitable conjugation we may also assume that $\alpha_1 = \infty$.

By assumption there is some disc $D = D(z_0, r)$ such that D and all $f^n(D)$ are in N and $f^n \to 0$ uniformly in D as $n \to \infty$. Let $z_n = f^n(z_0)$. For any branch g of f^{-1} analytic at some point w' in N and any choice t' of $\log w'$, the function $h(t) = g(\exp t)$, where $h(t') = g(w')$, can be continued throughout $H = \{t : \operatorname{Re} t < \log\varepsilon\}$. Either (a) h is univalent in H or (b) h has period $2\pi i p$

for some minimal $p \in \mathbf{N}$, and $g(w^p)$ is univalent in $\{0 < |w| < \varepsilon^{1/p}\}$ with a univalent extension G to $D(0, \varepsilon^{1/p})$.

In case (a) $h(H) = W$, where W is a simply-connected domain consisting of those values taken by the continuation of g within N. Thus W contains neither 0 (where f has an essential singularity) nor $\infty = \alpha_1$, since $f(\alpha_1) = \alpha_2 \notin N$.

In case (b) the values of g in N form a subset of $W = G\bigl(\{|w| < \varepsilon^{1/p}\}\bigr)$. The set W is a simply-connected domain bounded by the Jordan curve which is the image under G of $\{|w| = \varepsilon^{1/p}\}$. Again W contains neither 0 nor ∞. So in either case $h(H) \subset W$.

Take g to be the branch of f^{-1} such that $g(z_{n+1}) = z_n$ and define $h(t) = g(\exp t)$ as above, with $w' = z_{n+1}$ and any choice t_{n+1} of $\log w'$. Then, taking a branch of log which is analytic in W, the function $\phi(t) = \log\bigl(h(t)\bigr) = \log\bigl(g(\exp t)\bigr)$ is analytic in H and $\phi(H)$ contains no disc of radius exceeding π. Bloch's theorem states that if ψ is analytic in $D(0,r)$, then $\psi\bigl(D(0,r)\bigr)$ contains a disc of radius $cr|\psi'(0)|$, where c is some positive absolute constant. Thus

$$c(\log\varepsilon - \operatorname{Re} t_{n+1})|\phi'(t_{n+1})| \le \pi,$$

which yields

$$|f'(z_n)| \ge \frac{c|z_{n+1}|}{\pi|z_n|}\log\biggl(\frac{\varepsilon}{|z_{n+1}|}\biggr),$$

so

$$|(f^n)'(z_0)| \ge \Bigl(\frac{c}{\pi}\Bigr)^n \frac{|z_n|}{|z_0|}\prod_{j=1}^{n}\biggl(\log\frac{\varepsilon}{|z_j|}\biggr). \tag{2}$$

Note that each $D_n = f^n(D)$ lies in some domain of the type W so that $\log f^n$ is analytic in D and the image domain $\log f^n(D)$ contains no disc of radius greater than π. But this implies by Bloch's theorem that

$$rc\frac{|(f^n)'(z_0)|}{|f^n(z_0)|} \le \pi. \tag{3}$$

Since the product on the right in (2) tends to ∞ as $n \to \infty$, we have a contradiction between (2) and (3) if n is sufficiently large.

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