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# REGULATED DOMAINS AND BERGMAN TYPE PROJECTIONS

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**Abstract.** We analyze the relations of the geometry of a regulated complex domain $\Omega$ with the existence of Bergman type projections from $L^p_\omega(\Omega)$ onto the Bergman space $\mathscr{A}^p_\omega(\Omega)$. The main technical device is a Muckenhoupt type weight condition. In particular we find bounded Bergman type projections on $\mathscr{A}^p(\Omega)$ even in the case $\Omega$ has arbitrary inward or outward cusps. As a consequence, $\mathscr{A}^p_\omega(\Omega)$ is isomorphic as a Banach space to $l_p$.

## 1. Introduction

Let $\Omega \subset \mathbf{C}$ be a complex domain bounded by a Jordan curve. We want to find Bergman type projections from $L^p_\omega(\Omega)$ onto the corresponding Bergman space. Moreover, we want to analyze the relations of the geometry of $\Omega$ with the boundedness of various Bergman projections. Here, the space $L^p_\omega(\Omega)$ is with respect to a weighted 2-dimensional Lebesgue measure, where the weight $\omega$ is of the simplest possible type: it is a power of the boundary distance,

$$\omega(z) = \bigl(\operatorname{dist}(z, \partial\Omega)\bigr)^\alpha \tag{1.1}$$

for some $\alpha > -1$. By the Bergman space $\mathscr{A}^p_\omega(\Omega)$ we mean the space of analytic mappings $f\colon \Omega \to \mathbf{C}$ endowed with the norm

$$\|f\| := \|f\|_{p,\omega} := \left(\int_\Omega |f|^p \omega \, dm\right)^{1/p} < \infty. \tag{1.2}$$

We only deal with the case $1 < p < \infty$.

As a consequence of the Koebe distortion theorem (see [5, Corollary 1.4]), the weights (1.1) correspond on the open unit disc $\mathbf{D}$ to the weights where the weight

$$\nu(z) = |\psi'(z)|^{2+\alpha}(1 - |z|)^\alpha, \tag{1.3}$$

where $\psi\colon \mathbf{D} \to \Omega$ is a Riemann conformal map. Clearly, $\nu$ is in general a nonradial weight on $\mathbf{D}$, although it emerges in a canonical way from a most natural class of weights on $\Omega$.

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It turns out that certain Muckenhoupt-type conditions will play a central role. Our predecessor is the paper [2, p. 143], where it was shown that the weight $|\psi'|^{2-p}$ on $\mathbf{D}$ satisfies the Békollé–Bonami condition $B_p(0)$, i.e.,

$$\sup_S \int_S |\psi'|^{2-p}\,dm \biggl(\int_S |\psi'|^{(2-p)(-p^*/p)}\,dm\biggr)^{p/p^*} \le Cm(S)^p \tag{1.4}$$

where $p$ and $p^*$ are the usual dual indices of each other, if and only if certain geometric conditions for the boundary of $\Omega$ are satisfied. In (1.4) the sup is taken over all sets

$$S := S(\theta,\varrho) := \bigl\{z = re^{it} \in \mathbf{D}:\ 1-\varrho < r < 1 \text{ and } |\theta - t| < 2\pi\varrho\bigr\}, \tag{1.5}$$

where $0 \le \theta \le 2\pi$ and $0 < \varrho < 1$.

Moreover, in [2] it was shown that the condition (1.4) is equivalent to the boundedness of the Bergman projection on the space $L^p(\Omega)$.

The conditions of [2] for $\Omega$ require more than being just a regulated domain in the sense of [5].

In our paper we generalize the above mentioned result of [2] both qualitatively and quantitatively, see Theorem 3.1. On the technical, quantitative side we show that (1.4) still holds if $|\psi'|$ is raised to more arbitrary powers and $dm$ is replaced by a weighted measure

$$dm_\alpha := (1-|z|)^\alpha\,dm. \tag{1.6}$$

On the qualitative side, our results hold for *arbitrary* regulated domains. In particular we do not need the assumption that the domain should be of bounded boundary rotation type, cf. [2, assumption H2, (ii) on p. 138]. We do not put any restriction for the number of cusps as in H2, (i), on p. 155. On the other hand, we are not able to handle the limiting case (i.e. equality in the condition (3.3)) in this generality. It seems that some additional assumptions on $\Omega$ are needed in that case. A version of that result is contained in a different paper [6]; it is not presented here since the proof is technical and based on a completely different method.

Theorem 3.1 implies boundedness results for a large family of Bergman type projections from $L^p_\omega(\Omega)$ onto $\mathscr{A}^p_\omega(\Omega)$. These are presented in Section 4. It follows from [2] that for example the existence of cusps in $\Omega$ may make the standard Bergman projection unbounded on $L^p(\Omega)$. However, we are able to show that for an arbitrary (regulated) $\Omega$ there are more general, but still of the Bergman type, bounded projections, see Corollary 4.6.

It also follows from these results that $\mathscr{A}^p_\omega(\Omega)$ is linearly homeomorphic to the Banach space $l_p$; see Corollary 4.7.

Concerning notation, $dm$ denotes the two-dimensional Lebesgue measure and $d\mu$ is the one-dimensional Lebesgue measure on the boundary of the unit disc. The notation "dist" and "diam" denotes the euclidean distance of two given points, respectively, the euclidean diameter of a set. We denote by $C$, $C'$ or $c$ strictly positive constants which may vary from place to place but not in the same sequence of inequalities. By writing "$\cong$" between two quantities $A$ and $B$ (which may depend on some variables $z$, $\theta$, $\dots$) we mean that there exists a constant $C > 0$ (independent of $z$, $\theta$, $\dots$) such that $C^{-1}B \le A \le CB$. Otherwise we recommend the references [3], [5] and [8].

## 2. Regulated domains

Before formulating our main result, Theorem 3.1, we recall the properties of regulated domains. Let us start with a simply connected, bounded domain $\Omega \subset \mathbf{C}$ with a locally connected boundary. In this case a Riemann conformal map $\psi\colon \mathbf{D} \to \Omega$ has a continuous extension to $\overline{\mathbf{D}}$ (still denoted by $\psi$). We can thus define the curve $w(t) = \psi(e^{it})$, $0 \le t \le 2\pi$. According to [5, Section 3.5], $\Omega$ is called a regulated domain, if each point of $\partial\Omega$ is attained only finitely often by $\psi$, and if

$$\beta(t) := \lim_{\tau \to t^+} \arg\bigl(w(\tau) - w(t)\bigr) \tag{2.1}$$

exists for all $t$ and defines a regulated function. (Recall that $\beta$ is regulated, if it can be approximated uniformly by step functions, i.e. for every $\varepsilon > 0$ there exist $0 = t_0 < t_1 < \dots < t_n = 2\pi$ and constants $\gamma_1, \dots, \gamma_n$ such that

$$|\beta(t) - \gamma_j| < \varepsilon \qquad \text{for } t_{j-1} < t < t_j,\ j = 1, \dots, n.) \tag{2.2}$$

Geometrically, $\beta$ is the direction angle of the forward tangent of $\partial\Omega$ at $w(t)$. For more details, see [5, Section 3.5].

Regulated domains can be characterized as follows.

**Theorem 2.1.** *Let $\Omega \subset \mathbf{D}$ be a simply connected domain with locally connected boundary. Then $\Omega$ is regulated if and only if, for a Riemann conformal map $\psi\colon \mathbf{D} \to \Omega$,*

$$\log \psi'(z) = \log|\psi'(0)| + \frac{i}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \left(\beta(t) - t - \frac{\pi}{2}\right) dt, \tag{2.3}$$

*where $\beta\colon [0, 2\pi] \to \mathbf{R}$ is a regulated function.*

In the situation of Theorem 2.1 the function $\beta$ coincides with the direction angle defined above. For a proof, see [5].

Let us denote by $\mathrm{BMO}(\partial\mathbf{D})$ the BMO-space on the boundary of the unit disc; for a detailed definition, see [8]. Let $\mathrm{BMO}(\mathbf{D})$ stand for the space of measurable functions $f$ on the disc with norm

$$\sup_Q \frac{1}{m(Q)} \int_Q |f - f_Q|\, dm, \tag{2.4}$$

where $Q$ is running over all open Euclidean discs intersecting $\mathbf{D}$ and $f_Q$ is the mean of $f$ over $Q$. See [3, p. 282].

We shall need the John–Nirenberg inequality for functions in $\mathrm{BMO}(\partial\mathbf{D})$ (cf. [3, Theorem VI.2.1]):

**Lemma 2.2.** *There exists a constant $C>0$ such that for every positive $\lambda$*

$$\sup_I \frac{\mu\big(\{x\in I \mid |f-f_I|>\lambda\}\big)}{\mu(I)} \le e^{-C\lambda/\|f\|_{\mathrm{BMO}(\partial\mathbf{D})}}, \tag{2.5}$$

*where $I$ runs over all subarcs of $\partial\mathbf{D}$, $\mu(I)$ denotes the 1-dimensional Lebesgue measure of $I$, $f\in\mathrm{BMO}(\partial\mathbf{D})$, and $f_I$ denotes the integral mean of $f$ over $I$.*

Given a function in $\mathrm{BMO}(\partial\mathbf{D})$ we shall also need to control some two-dimensional BMO-like properties of its Poisson extension:

**Lemma 2.3.** *There exists a constant $C>0$ as follows. If $f\in\mathrm{BMO}(\partial\mathbf{D})$ with $\|f\|_{\mathrm{BMO}}\le 1$ and $F\colon\mathbf{D}\to\mathbf{C}$ is the harmonic Poisson extension of $f$, then*

$$\sup_Q \frac{m\big(\{z\in Q \mid |F(z)-F_Q|\ge\lambda\}\big)}{m(Q)} \le e^{-C\lambda/\|f\|_{\mathrm{BMO}(\partial\mathbf{D})}}, \tag{2.6}$$

*where the supremum is taken over all Euclidean discs intersecting $\mathbf{D}$ and $F_Q$ is the mean value of $F$ on $Q$.*

*Proof.* It is known that $\mathrm{BMO}(\partial\mathbf{D})$ is mapped continuously into the harmonic Bloch space $\mathscr{B}_h$ (the space of harmonic functions on the unit disc with finite Bloch norm) via the Poisson extension. On the other hand, $\mathscr{B}_h$ is known to have a norm equivalent to that of $\mathrm{BMO}(\mathbf{D})$. See e.g. [8, p. 188] and [3, p. 282] for these statements. Hence $\|F\|_{\mathrm{BMO}(\mathbf{D})}\le C\|F\|_{\mathscr{B}_h}\le C'\|f\|_{\mathrm{BMO}(\partial\mathbf{D})}\le C'$.

On the other hand, the two-dimensional version of the John–Nirenberg inequality implies

$$\sup_Q \frac{m\big(\{z\in Q \mid |F(z)-F_Q|\ge\lambda\}\big)}{m(Q)} \le e^{-C\lambda/\|F\|_{\mathrm{BMO}(\mathbf{D})}}. \tag{2.7}$$

The lemma follows. □

Finally, we need to control the mean $F_Q$ above.

**Lemma 2.4.** *There exists a constant $C>0$ as follows. If $f$, $F$ and $Q$ are as in the previous lemma then*

$$|F_Q|\le C|\log(m(Q))|\,\|f\|_{\mathrm{BMO}(\partial\mathbf{D})}+C|F_{\mathbf{D}}|. \tag{2.8}$$

*Proof.* Let $Q\subset Q_1\subset Q_2\subset\cdots Q_k$ be a sequence (as short as possible) of sets $Q_j$ (as in (2.4)) such that $m(Q_j\cap\mathbf{D})=2m(Q_{j-1}\cap\mathbf{D})$ for all $j$ (here $Q_0:=Q$)

and such that $m(Q_k) \geq 1$. Hence, $k \leq C' \bigl|\log\bigl(m(Q)\bigr)\bigr|$ for an absolute strictly positive constant $C'$.

We obtain $|F_Q| \leq |F_Q - F_{Q_k}| + |F_{Q_k} - F_{\mathbf{D}}| + |F_{\mathbf{D}}|$, and here

$$|F_Q - F_{Q_k}| \leq \sum_{j=0}^{k-1} |F_{Q_j} - F_{Q_{j+1}}| \leq Ck \|F\|_{\mathrm{BMO}(\mathbf{D})}, \tag{2.9}$$

since

$$\begin{aligned} |F_{Q_j} - F_{Q_{j+1}}| &\leq \frac{1}{m(Q_j)} \int_{Q_j} |F - F_{Q_{j+1}}|\, dm \\ &\leq \frac{2}{m(Q_{j+1})} \int_{Q_{j+1}} |F - F_{Q_{j+1}}|\, dm \leq 2\|F\|_{\mathrm{BMO}(\mathbf{D})}. \end{aligned} \tag{2.10}$$

The term $|F_{Q_k} - F_{\mathbf{D}}|$ is estimated in the same way, since the sets $Q_k$ and $\mathbf{D}$ are of comparable 2-dimensional measure. Now, combine (2.9) with the bound for $k$. The estimate in terms of the BMO-norm of $f$ follows from the inequality above (2.7). □

## 3. Main result

Our main result, Theorem 3.1, gives a connection of a Muckenhoupt type condition (for $|\psi'|$) with the geometry of $\Omega$. The result holds for *arbitrary* regulated domains.

**Theorem 3.1.** *Let $\psi$ be a Riemann conformal map from $\mathbf{D}$ onto the bounded regulated domain $\Omega$. Let $a > 0$, $1 < p < \infty$, $1/p + 1/p^* = 1$, and let the numbers $\alpha$, $\sigma$ and $\gamma$ (all $> -1$) satisfy*

$$\alpha = \frac{\sigma}{p} + \frac{\gamma}{p^*}. \tag{3.1}$$

*Then the (weight) function $|\psi'|^a$ satisfies the condition*

$$\sup_S \int_S |\psi'|^a \, dm_\sigma \biggl( \int_S |\psi'|^{-ap^*/p} \, dm_\gamma \biggr)^{p/p^*} \leq C m_\alpha(S)^p, \tag{3.2}$$

*if*

$$2 + \sigma > \frac{a\delta_1}{\pi} \qquad \text{and} \qquad \frac{a\delta_2}{\pi} > -\frac{p}{p^*}(\gamma + 2). \tag{3.3}$$

*Conversely, if $2 + \sigma < a\delta_1/\pi$ or $a\delta_2/\pi < -p(p^*)^{-1}(\gamma + 2)$, then (3.2) fails.*

**Remarks.** 1° Here, the sup is as in (1.4) and

$$\begin{aligned}\delta_1 &:= \sup_t \lim_{\tau\to 0^+} \big(\beta(t+\tau)-\beta(t-\tau)\big) \ge 0 \qquad \text{and} \\ \delta_2 &:= \inf_t \lim_{\tau\to 0^+} \big(\beta(t+\tau)-\beta(t-\tau)\big) \le 0\end{aligned} \tag{3.4}$$

which exist e.g. by (2.2), see also [5].

2° The theorem still holds, if $a < 0$. In this case (3.3) has to be replaced by "$2+\sigma > a\delta_2/\pi$ and $a\delta_1/\pi > -p(\gamma+2)/p^*$" and the reverse condition by "$2+\sigma < a\delta_2/\pi$ or $a\delta_1/\pi < -p(\gamma+2)/p^*$". This can be deduced from the case $a > 0$ by replacing $a$ by $b := -ap^*/p$ and $p$ by $p^*$.

3° The first (respectively second) condition (3.3) means a restriction for outward (respectively inward) pointing corners and cusps in the boundary of $\Omega$.

The method of proof is basically still the same as in [2]. The main difference is that we explicitly use the notion of regulated domains. This allows us to approximate the direction angle of the boundary curve conveniently just by step functions, and this leads to some simplifications and generalizations in the arguments.

*Proof.* 1° Let us assume that (3.3) holds. We first derive in (3.5)–(3.21) a representation for $|\psi'|^a$ which reveals the essential factors in (3.2).

Let us fix $0 < \varepsilon < 1$ such that $2+\sigma - a\delta_1/\pi > 2\max(1, a\delta_1/\pi, |\sigma|)\varepsilon$ and

$$\gamma + 2 + \frac{p^*}{p}\frac{a\delta_2}{\pi} > 2\max\Big(1, \frac{p^*}{p}\Big|\frac{a\delta_2}{\pi}\Big|, |\gamma|\Big)\varepsilon$$

and such that

$$\frac{1}{\varepsilon} > 100A\max\Big(1, \frac{4\pi}{a}\Big(1+\frac{p}{p^*}\Big)\Big), \tag{3.5}$$

where $A > 0$ is the maximum of the constants $C$ occurring in the bound [3, Theorem VI.1.5], and Lemmas 2.2, 2.3 and 2.4.

According to (2.3), the function $|\psi'|^a$ has the representation

$$|\psi'|^a = C\exp\Big(-\frac{a}{2\pi}\operatorname{Re}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\tilde{\beta}(t)\,dt\Big), \tag{3.6}$$

where $\tilde{\beta}$ is the harmonic conjugate of $\beta - t$, $\beta$ defined by (2.3).

By (2.2), there exist finitely many points $0 = t_0 < t_1 < \cdots < t_n = 2\pi$ such that $|\beta(t) - t - \gamma_j| < \varepsilon^5$ for $t_{j-1} < t < t_j$, for some real constants $\gamma_j$, $j = 1, \ldots, n$. We denote by $\beta_1$ and $\beta_2$ the $2\pi$-periodic extensions to $\mathbf{R}$ of the functions

$$\beta_1 = \sum_{j=1}^{n}\gamma_j\chi_j, \qquad \beta_2 = \beta - t - \beta_1, \tag{3.7}$$

where $\chi_j(t) = 1$ for $t_{j-1} < t < t_j$ and zero elsewhere. Clearly, the modulus of $\beta_2$ is bounded by $\varepsilon^5$. By the choice of the points $t_j$ we have (with $\gamma_0 := \gamma_n$)

$$\left|\delta_1 - \max_{0\le j<n}(\gamma_{j+1}-\gamma_j)\right| \le 2\varepsilon^5 \qquad \text{and} \qquad \left|\delta_2 - \min_{0\le j<n}(\gamma_{j+1}-\gamma_j)\right| \le 2\varepsilon^5. \tag{3.8}$$

Let us define for $j = 1, 2$

$$\begin{aligned} \nu_j(z) &:= \exp\biggl(-\frac{a}{2\pi}\operatorname{Re}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\tilde{\beta}_j(t)\,dt\biggr) \\ &= \exp\biggl(-\frac{a}{2\pi}\operatorname{Im}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\beta_j(t)\,dt\biggr); \end{aligned} \tag{3.9}$$

we thus have $|\psi'|^a = e^C\nu_1\nu_2$.

Concerning $\nu_2$, we want to show that, for $\varrho = 1+\varepsilon^{-1}$ (which may be a large number),

$$\int_S \nu_2^{\varrho}\,dm\biggl(\int_S \nu_2^{-\varrho p^*/p}\,dm\biggr)^{p/p^*} \le Cm(S)^p. \tag{3.10}$$

This follows from [2, Théorème 1.2], as soon as we show that (notation as in (2.5))

$$\biggl(\frac{1}{\mu(I)}\int_I \nu_2^{\varrho/2}\,d\mu\biggr)\biggl(\frac{1}{\mu(I)}\int_I \nu_2^{-\varrho p^*/(2p)}\,d\mu\biggr)^{p/p^*} \le C. \tag{3.11}$$

Notice that by the theorem for conjugate functions, [3, Theorem VI.1.5],

$$\|\tilde{\beta}_2\|_{\mathrm{BMO}(\partial\mathbf{D})} \le A\|\beta_2\|_{L^\infty(\partial\mathbf{D})} = A\operatorname*{ess\,sup}_{t\in[0,2\pi]}|\beta_2(t)| < \varepsilon^4, \tag{3.12}$$

see the choice of $\varepsilon$ above. By [3, Lemma VI.6.5], (3.11) is satisfied, if

$$\begin{aligned} &\sup_I \frac{1}{\mu(I)}\int_I e^{\varrho a(\tilde{\beta}_2-\tilde{\beta}_{2,I})/4\pi}\,d\mu < C \qquad \text{and} \\ &\sup_I \frac{1}{\mu(I)}\int_I e^{-\varrho ap(\tilde{\beta}_2-\tilde{\beta}_{2,I})/4\pi p^*}\,d\mu < C. \end{aligned} \tag{3.13}$$

The proof of these is standard: let $I$ be an interval and denote, for the positive real numbers $s < s'$, $I[s,s'] := \bigl\{t \in I \mid s \le |\tilde{\beta}_2(t) - \tilde{\beta}_{2,I}| \le s'\bigr\}$. Then

$$\int_I e^{\varrho a(\tilde{\beta}_2-\tilde{\beta}_{2,I})/4\pi}\,d\mu = \biggl(\int_{I[0,4\pi/(\varrho a)]} + \int_{I[4\pi/(\varrho a),1]} + \sum_{k=1}^{\infty}\int_{I[k,k+1]}\biggr)e^{\varrho a(\tilde{\beta}_2-\tilde{\beta}_{2,I})/4\pi}\,d\mu. \tag{3.14}$$

Here the integrand is bounded by a constant on $I[0,4\pi/(\varrho a)]$. On $I[4\pi/(\varrho a),1]$, the integrand is bounded by $e^{\varrho a/4\pi}$, but the measure of the set $I[4\pi/(\varrho a),1]$ is at most $\mu(I)$ times

$$e^{-C4\pi/\varrho a\|\tilde\beta_2\|_{\mathrm{BMO}}} \le e^{-C4\pi/\varrho a\varepsilon^4} \le e^{-2\varrho a/4\pi}, \tag{3.15}$$

see the John–Nirenberg inequality (2.5), (3.5), and (3.12), and notice also $1/\varepsilon \le \varrho \le 2/\varepsilon$. This integral thus is less than $\frac{1}{2}$. In the same way one finds that the integral over $I[k,k+1]$ is bounded by $2^{-k}$. We thus obtain the first inequality of (3.13). The proof of the second one is similar.

For $\nu_1$ we want to prove

$$\int_S \nu_1^{\varrho^*}\,dm_{\sigma\varrho^*}\biggl(\int_S \nu_1^{-\varrho^* p^*/p} dm_{\gamma\varrho^*}\biggr)^{p/p^*} \le Cm_{\alpha\varrho^*}(S)^p, \tag{3.16}$$

where $1/\varrho + 1/\varrho^* = 1$, hence, $1 < \varrho^* = 1+\varepsilon$. It is enough to prove this for so small sets $S$ that $\operatorname{dist}(S,e^{it_j}) \le 2m(S) < 1$ for at most one $j$. First, $\nu_1$ has a representation

$$\nu_1(z) = \nu_1(re^{i\theta}) = \hat\nu_1(z)\prod_{j=0}^{n-1}\bigl((1-r)^2 + r(\theta-t_j)^2\bigr)^{-a(\gamma_{j+1}-\gamma_j)/(2\pi)}, \tag{3.17}$$

where $\hat\nu_1$ is a bounded function on $\mathbf{D}$ which is also bounded away from zero. One obtains (3.17) easily by taking the convolution of (3.7) times $-a/(2\pi)$ with the conjugate Poisson kernel $-Q_z(t)$, see [3, p. 102]; the product stems from the principal part $2r(\theta-t)/\bigl((1-r)^2 + r(\theta-t)^2\bigr)$ of $-Q_z(t)$. Now

$$\begin{aligned}(1-r)^2 + r(\theta-t_j)^2 &\cong (1-r)^2 + (\theta-t_j)^2 \cong \operatorname{dist}(e^{i\theta}, re^{it_j})^2\\ &\cong \operatorname{dist}(re^{i\theta}, e^{it_j})^2.\end{aligned} \tag{3.18}$$

Keeping this in mind, we want to estimate $\int_S \nu_1^{\varrho^*}\,dm_{\sigma\varrho^*}$.

Because of (3.17) and (3.18), we can replace $\nu_1$ by

$$\operatorname{dist}(re^{i\theta}, e^{it_j})^{-a(\gamma_{j+1}-\gamma_j)/\pi} \tag{3.19}$$

for some $j$; let us also denote $\operatorname{diam}(S)$ by $h$, hence $m(S) \cong h^2$. Let us fix a point $Re^{iT} \in S$, where $1-h < R < 1$, $0 \le T \le 2\pi$. Applying the linear change of variables $L\colon z \to z - Re^{iT}$ we find that $L(S)$ is contained in the set $\{w \in \mathbf{C} \mid |w| \le 10h\}$. Hence, using polar coordinates,

$$\begin{aligned}\int_S \nu_1^{\varrho^*}\,dm_{\sigma\varrho^*} &\cong \int_S \operatorname{dist}(re^{i\theta}, e^{it_j})^{-a\varrho^*(\gamma_{j+1}-\gamma_j)/\pi}\,dm_{\sigma\varrho^*}\\ &\le \int_0^{10h}\int_0^{2\pi} r^{-\varrho^* a(\gamma_{j+1}-\gamma_j)/\pi} r^{\sigma\varrho^*}(\cos\theta)^{\sigma\varrho^*}\,r\,dr\,d\theta.\end{aligned} \tag{3.20}$$

Since (3.3) and (3.8) hold, the total exponent of $r$ in (3.20) becomes larger than $-1$ (we have $\varrho^* = 1+\varepsilon$, see also the choice of $\varepsilon$). Hence, (3.20) converges and is bounded by a constant times

$$h^{-\varrho^* ai(\gamma_{j+1}-\gamma_j)/\pi+\varrho^*\sigma+2}. \tag{3.21}$$

In the same way one proves that

$$\int_S \nu_1^{-\varrho^* p^*/p}\, dm_{\gamma\varrho^*} \le Ch^{\varrho^* p^* a(\gamma_{j+1}-\gamma_j)p\pi+\varrho^*\gamma+2}. \tag{3.22}$$

As a consequence of (3.21) and (3.22) we obtain (3.16):
(3.23)

$$\int_S \nu_1^{\varrho^*}\, dm_{\sigma\varrho^*}\Big(\int_S \nu_1^{-\varrho^* p^*/p}\, dm_{\gamma\varrho^*}\Big)^{p/p^*}$$
$$\le Ch^{\varrho^*((a/\pi)-(a/\pi))(\gamma_{j+1}-\gamma_j)+\varrho^*(\sigma+\gamma p/p^*)+2(1+p/p^*)} = Ch^{\varrho^* p\alpha+2p} \cong m_{\varrho^*\alpha}(S)^p$$

For the last step one uses $m_c\big(S(\theta,h)\big) \cong h^{2+c}$.

Moreover, to see (3.2) we combine (3.10) and (3.16) and use Hölder:

$$\int_S |\psi'|^a dm_\sigma \Big(\int_S |\psi'|^{-ap^*/p} dm_\gamma\Big)^{p/p^*}$$
$$\le \Big(\int_S \nu_1^{\varrho^*}\, dm_{\sigma\varrho^*}\Big)^{1/\varrho^*}\Big(\int_S \nu_2^{\varrho}\, dm\Big)^{1/\varrho}$$
$$\times \Big(\int_S \nu_1^{-\varrho^* p^*/p}\, dm_{\gamma\varrho^*}\Big)^{p/(\varrho^* p^*)}\Big(\int_S \nu_2^{-\varrho p^*/p} dm\Big)^{p/(\varrho p^*)}$$
$$\le Cm(S)^{p/\varrho} m_{\varrho^*\alpha}(S)^{p/\varrho^*} \le C' m_\alpha(S)^p.$$

2° We consider the reverse direction. Assume for example $2+\sigma < a\delta_1/\pi$. This time we choose $\varepsilon > 0$ such that $-(2+\sigma)+a\delta_1/\pi > 2\max(1, a\delta_1/\pi, |\sigma|)\varepsilon$ and such that (3.5) also holds. We again define the functions $\beta_j$ and $\nu_j$ as in (3.6)–(3.9).

Using the methods of part 1° we find an essential lower bound for $\nu_1$ at least in a large enough subset of $\mathbf{D}$, as follows. We again use the representation (3.17) and Lemma 2.2. Let us fix $j$ such that $\gamma_{j+1}-\gamma_j > \delta_1 - 2\varepsilon^5$; consider the sets $S(t_j, \varrho)$, where notation is as in (1.5) and $0 < \varrho \le \frac{1}{2}$ is so small that $\varrho \le \min_\iota(|e^{it_{\iota+1}} - e^{it_\iota}|)/100$. As in (3.19) we see that

$$\nu_1(z) \ge C\,\mathrm{dist}\,(z, e^{it_j})^{-a(\gamma_{j+1}-\gamma_j)/\pi} \ge C\,\mathrm{dist}\,(z, e^{it_j})^{-a(\delta_1-2\varepsilon^5)/\pi} \tag{3.24}$$

for $z \in S(t_j, \varrho)$. Applying again the linear change of variables $z \to e^{i(\pi - t_j)}(z - e^{it_j})$ we thus obtain the bound ($z = re^{i\theta}$)

$$\int_S |\psi'|^a\, dm_\sigma = \widetilde{C}\int_S \nu_1\nu_2\, dm_\sigma \ge C\int_S \mathrm{dist}\,(re^{i\theta}, e^{it_j})^{-a(\delta_1-2\varepsilon^5)/\pi}\nu_2(re^{i\theta})\, dm_\sigma$$
$$\ge \frac{C}{4}\int_0^h\int_{-\pi/4}^{\pi/4} r^{-a(\delta_1-2\varepsilon^5)/\pi} r^\sigma(\cos\theta)^\sigma r\nu_2(re^{i\theta}-e^{it_j})\, dr\, d\theta. \tag{3.25}$$

This is bounded from below by a constant times

$$\int_0^h \int_{-\pi/4}^{\pi/4} r^{a'} \nu_2(re^{i\theta} - e^{it_j})\, dr\, d\theta, \tag{3.26}$$

where $a' = -a(\delta_1 - 2\varepsilon^5)/\pi + \sigma + 1 < -1 - \varepsilon$.

Our aim is now to obtain the lower bound $\infty$ for (3.26). The main difficulty is to prove that $\nu_2$ is large enough on a large enough set. We prove this by taking advantage of the BMO-property of $\tilde\beta_2$. It might be dangerous if $|\tilde\beta_2|$ were large; but the set where this happens is very small. One of the technical difficulties is to extend the estimates on the boundary to the inside of the disc. We claim that

$$m\big(\{z \mid |\nu_2(z)| \le k^{-\varepsilon}\}\big) \le Ck^{-\varepsilon^{-1}} \tag{3.27}$$

for every $k \in \mathbf{N}$. Let us denote by $B\colon \mathbf{D} \to \mathbf{C}$ the Poisson extension of $\tilde\beta_2$. The mean value (with respect to $z$) over the whole disc of the conjugate Poisson kernel $(e^{it} + z)/(e^{it} - z)$ is a bounded function with respect to $t$, with a bound, say, 10. Hence, since the sup-norm of $\beta_2$ is smaller than $\varepsilon^4$, we find that the mean value $B_{\mathbf{D}}$ of $B$ over the unit disc satisfies $|B_{\mathbf{D}}| \le \varepsilon^3$. Let us define a decomposition of $\mathbf{D}$ into subsets $Q'_{n,j}$ as follows. Denote

$$Q'_{n,j} := \big\{z \mid 1 - 2^{-n} \le |z| < 1 - 2^{-n-1},\ 2\pi j 2^{-n} \le \arg(z) \le 2\pi(j+1)2^{-n}\big\} \tag{3.28}$$

for all $n \in \mathbf{N}$, $1 \le j \le 2^n$. Let $Q_{n,j}$ be a Euclidean disc containing $Q'_{n,j}$ such that $m(Q_{n,j}) \le 2m(Q'_{n,j})$. By Lemma 2.4,

$$|B_{Q_{n,j}}| \le Cn\|\tilde\beta_2\|_{\mathrm{BMO}(\partial\mathbf{D})} + CB_{\mathbf{D}} \le C'n\varepsilon^3. \tag{$*$}$$

Moreover, by Lemma 2.3,

$$\begin{aligned} &m\big(\{z \in Q_{n,j} \mid |B(z) - B_{Q_{n,j}}| \ge \tfrac12\varepsilon\log k\}\big) \\ &\qquad \le C'm(Q_{n,j})e^{-C\varepsilon\log k/\|\tilde\beta_2\|_{\mathrm{BMO}(\partial\mathbf{D})}} \le C''m(Q_{n,j})k^{-\varepsilon^{-2}}. \end{aligned} \tag{3.29}$$

Given $k$, denote now by $N(k)$ the set of pairs $(n,j)$, $n \in \mathbf{N}$, $1 \le j \le 2^n$, such that $|B_{Q_{n,j}}| \ge \frac12\varepsilon\log k$ for $(n,j) \in N(k)$. Clearly, $(*)$ implies $n \ge C\varepsilon^{-2}\log k$. Hence, by (3.28),

$$m\bigg(\bigcup_{(n,j)\in N(k)} Q_{n,j}\bigg) \le C2^{-C\varepsilon^{-2}\log k} \le C'k^{-c\varepsilon^{-2}}. \tag{3.30}$$

Moreover, for $(n,j) \notin N(k)$ we have $|B_{Q_{n,j}}| \le \varepsilon\log k/2$. Hence, for such $(n,j)$ and for $z \in Q_{n,j}$, if $|B(z)| \ge \varepsilon\log k$, then $|B(z) - B_{Q_{n,j}}| \ge \varepsilon\log k/2$. This combined with (3.29) yields $m\big(\{z \in Q_{n,j} \mid |B(z)| \ge \varepsilon\log k\}\big) \le Cm(Q_{n,j})k^{-\varepsilon^{-2}}$.

This and (3.30) together imply $m\big(\{z \in \mathbf{D} \mid |B(z)| \geq \varepsilon \log k\}\big) \leq Ck^{-c\varepsilon^{-2}}$. Hence, (3.27) follows.

We denote by $\Lambda_k$ the set consisting of $z$ with the properties $(k+1)^{-1} \leq |z| \leq k^{-1}$ and $|\nu_2(z)| \geq k^{-\varepsilon}$. Now (3.27) clearly implies $m(\Lambda_k) \geq C'k^{-2}$. Hence, by (3.26),

$$\begin{aligned} \int_S |\psi'|^a \, dm_\sigma &= \widetilde{C} \int_S \nu_1 \nu_2 \, dm_\sigma \geq C \sum_{k=0}^{\infty} \int_{\Lambda_k} k^{-a'} k^{-\varepsilon} \\ &= C \sum_{k=0}^{\infty} m(\Lambda_k) k^{-a'} k^{-\varepsilon} \geq C' \sum_{k=0}^{\infty} k^{-2-a'-\varepsilon} = \infty, \end{aligned} \tag{3.31}$$

since $-2 - a' - \varepsilon > -1$. Hence, (3.2) cannot hold. □

## 4. Applications: Bergman type projections

In this section we assume that $\psi$ is a conformal map onto a bounded regulated domain $\Omega$, $\varphi := \psi^{-1}\colon \Omega \to \mathbf{D}$, and $\beta$, $\delta_1$ and $\delta_2$ are as in Theorem 3.1.

1° The standard Bergman projection. Using Theorem 3.1 and [2, Lemma 1.1] we obtain

**Theorem 4.1.** *The Bergman projection*

$$P_\Omega f(z) := \frac{1}{\pi} \int_\Omega \frac{\varphi'(z)\overline{\varphi'(\zeta)}}{(1 - \varphi(z)\overline{\varphi(\zeta)})^2} f(\zeta) \, dm(\zeta), \tag{4.1}$$

*is a bounded operator from $L^p(\Omega)$ onto $\mathscr{A}^p(\Omega)$, if*

$$(2-p)\frac{\delta_2}{\pi} > -\frac{2p}{p^*} \quad (\textit{in case } p \leq 2), \quad \textit{or,} \quad 2 > (2-p)\frac{\delta_2}{\pi} \quad (\textit{in case } p \geq 2). \tag{4.2}$$

*Conversely, if*

$$(2-p)\frac{\delta_2}{\pi} < -\frac{2p}{p^*} \quad (\textit{in case } p \leq 2), \quad \textit{or,} \quad 2 < (2-p)\frac{\delta_2}{\pi} \quad (\textit{in case } p \geq 2), \tag{4.3}$$

*then $P_\Omega$ is unbounded in the given space.*

*Proof.* Take $\alpha = \gamma = \sigma = 0$ and $a = 2 - p$ in Theorem 3.1. Use (3.3) or Remark 2 after Theorem 3.1 depending on whether $p \leq 2$ or $p \geq 2$. Notice that the first condition (3.3) in Theorem 3.1, $2 > (2-p)\delta_1/\pi$, always holds, since $0 \leq \delta_1 \leq \pi$. The same is true for the analogous condition in Remark 2. □

See the case 3° for the relation of $P_\Omega$ and the standard Bergman projection on the disc. The conditions (4.2) and (4.3) are analogous to (i), p. 143 of [2]. Theorem 4.1 is more general than Theorems 2.1 or 3.3 of [2] in the sense that we do not need a 'bounded boundary rotation'-type assumption, cf. H2, (ii), on p. 138 of [2]. Moreover, we do not put any restriction for the number of cusps or for the local geometry around cusps as in H2, (i), on p. 155.

**Corollary 4.2.** *If* $\frac{4}{3} < p < 4$, *then the Bergman projection* (4.1) *is bounded on* $L^p(\Omega)$ *for every bounded regulated domain* $\Omega$.

2° The conjugated Bergman projection. It is clear that the composition operator $C_\psi$: $f \mapsto f \circ \psi$ is a linear homeomorphism from $L^p(\Omega)$ onto $L^p_\nu(\mathbf{D})$ (this space endowed with the norm $\|\cdot\|_{p,\nu}$, see (1.2)) and from $\mathscr{A}^p(\Omega)$ onto $\mathscr{A}^p_\nu(\mathbf{D})$, where the weight $\nu$ on $\mathbf{D}$ is defined by $\nu(z) = |\psi'(z)|^2$. So, the operator $P_\psi := (C_\psi)^{-1} P_{\mathbf{D}} C_\psi$, where $P_{\mathbf{D}} f = \int_{\mathbf{D}} f(\zeta)/(1 - z\bar{\zeta})^2 \, dm(\zeta)$ is the standard Bergman projection on the disc, is again algebraically a projection operator on $L^p(\Omega)$.

It is known by [1] that $P_{\mathbf{D}}$ is bounded on $L^p_\nu(\mathbf{D})$ if and only if

$$\sup_S \int_S |\nu| \, dm \left( \int_S |\nu|^{-p^*/p} \, dm \right)^{p/p^*} \le C m(S)^p, \tag{4.4}$$

where $S$ is as before. From this and Theorem 3.1 we obtain

**Proposition 4.3.** *The conjugated Bergman projection* $P_\psi$ *is bounded on* $L^p(\Omega)$, *if*

$$\frac{\delta_1}{\pi} < 1 \qquad \text{and} \qquad \frac{\delta_2}{\pi} > -\frac{p}{p^*}. \tag{4.5}$$

*The converse statement is analogous to Theorem* 4.1.

Here the first condition (4.5) always excludes outward cusps. The second condition is vacuous if $p > 2$. So in this case arbitrary inward cusps are allowed.

3° A general family of projections. Finally we want to show that given an arbitrary regulated domain $\Omega$, an arbitrary $1 < p < \infty$ and an arbitrary weight

$$\omega(z) = \big(\operatorname{dist}(z, \partial\Omega)\big)^a, \tag{4.6}$$

$a > -1$, one can find bounded Bergman type projections from $L^p_\omega(\Omega)$ onto $\mathscr{A}^p_\omega(\Omega)$. These will be picked out of a family which depends on the indices $n \in \mathbf{Z}$ and $\alpha > -1$. We consider the space $L^p_\omega(\Omega)$ and the operator

$$\begin{aligned} Pf(z) &:= P_{\varphi,\alpha,n} f(z) \\ &:= (\alpha + 1) \int_\Omega \frac{\varphi'(z)^{2-n} \varphi'(\zeta)^{n-1} \overline{\varphi'(\zeta)} \big(1 - |\varphi(\zeta)|^2\big)^\alpha}{\big(1 - \varphi(z)\overline{\varphi(\zeta)}\big)^{2+\alpha}} f(\zeta) \, dm(\zeta). \end{aligned} \tag{4.7}$$

Notice that the standard Bergman projection is the case $\alpha = 0$, $n = 1$ and the conjugated Bergman projection is the case $\alpha = 0$, $n = 2$.

Formally, $P$ reproduces analytic functions, since

$$Pf(z) = (\alpha + 1) \varphi'(z)^{1-n} \int_\Omega K(z, \zeta) \varphi'(\zeta)^{n-1} f(\zeta) \, dm(\zeta), \tag{4.8}$$

and here the kernel

$$K(z,\zeta) := \frac{\varphi'(z)\overline{\varphi'(\zeta)}\big(1-|\varphi(\zeta)|^2\big)^{\alpha}}{\big(1-\varphi(z)\overline{\varphi(\zeta)}\big)^{2+\alpha}} \tag{4.9}$$

is reproducing. To see this, use

(i) the fact that

$$P_{\mathbf{D}}^{(\alpha)} f(z) = (\alpha+1)\int_{\mathbf{D}} \frac{(1-|\zeta|^2)^{\alpha}}{(1-z\bar{\zeta})^{2+\alpha}} f(\zeta)\,dm(\zeta) \tag{4.10}$$

is the orthogonal projection from $L^2_{(1-|z|^2)^{\alpha}}(\mathbf{D})$ onto $\mathscr{A}^2_{(1-|z|^2)^{\alpha}}(\mathbf{D})$, and

(ii) the Hilbert space isomorphism $f \mapsto (f\circ\psi)\psi'$ from $L^2_{(1-|\varphi(z)|^2)^{\alpha}}(\Omega)$ onto $L^2_{(1-|z|^2)^{\alpha}}(\mathbf{D})$, and from $\mathscr{A}^2_{(1-|\varphi(z)|^2)^{\alpha}}(\Omega)$ onto $\mathscr{A}^2_{(1-|z|^2)^{\alpha}}(\mathbf{D})$.

We have the following

**Lemma 4.4.** *The operator $P$, (4.7), can be extended as a bounded projection from $L^p_{\omega}(\Omega)$ onto $\mathscr{A}^p_{\omega}(\Omega)$, if and only if the projection $P_{\mathbf{D}}^{(\alpha)}$ is bounded from $L^p_{\nu}(\mathbf{D})$ onto $\mathscr{A}^p_{\nu}(\mathbf{D})$, where*

$$\nu(z) := |\psi'(z)|^{2-(2-n)p+a}(1-|z|)^a. \tag{4.11}$$

*Proof.* Since $P$ reproduces an analytic function, we only need to worry about its boundedness. But the proof for this is straightforward. Use e.g. the Koebe distortion theorem (see [5, Corollary 1.4]), to see that the weight $\omega\circ\psi(z)$ on $\mathbf{D}$ is equivalent to the weight $(1-|z|)^a|\psi'(z)|^a$. We omit the details. □

It is very useful for us that the boundedness of $P_{\mathbf{D}}^{(\alpha)}$ in the situation of Lemma 4.4 can be characterized in terms of a condition like (3.2), see [1]: $P_{\mathbf{D}}^{(\alpha)}$ is bounded on $L^p_{\nu}(\mathbf{D})$, if and only if

$$\sup_S \int_S |\nu|\,dm_{\alpha}\bigg(\int_S |\nu|^{-p^*/p}\,dm_{\alpha}\bigg)^{p/p^*} \le C m_{\alpha}(S)^p. \tag{4.12}$$

This immediately gives us

**Theorem 4.5.** *Let $\Omega$ be an arbitrary regulated domain, let, for some $a>-1$, $\omega(z) = \big(\operatorname{dist}(z,\partial\Omega)\big)^a$ and $1<p<\infty$. Let $P$ be as in (4.7) and assume $2-(2-n)p+a>0$. Then $P$ is a bounded projection from $L^p_{\omega}(\Omega)$ onto $\mathscr{A}^p_{\omega}(\Omega)$, if*

$$\begin{gathered} 2+\alpha+a > (2-(2-n)p+a)\frac{\delta_1}{\pi} \qquad \text{and} \\ (2-(2-n)p+a)\frac{\delta_2}{\pi} > -\frac{p}{p^*}(\alpha+2)+a. \end{gathered} \tag{4.13}$$

Here $\delta_1$ and $\delta_2$ are as in Theorem 3.1. The converse statement is analogous to Theorem 4.1. Also the formulation of the case $2-(2-n)p+a<0$ is left to the reader.

*Proof.* The boundedness of $P$ is equivalent to (4.12), by Lemma 4.4. This is equivalent, in view of (4.11), to

(4.14)

$$\sup_S \int_S |\psi'|^{2-(2-n)p+a}\, dm_{\alpha+a} \biggl( \int_S |\psi'|^{-\frac{p^*}{p}(2-(2-n)p+a)}\, dm_{\alpha-ap^*/p} \biggr)^{p/p^*} \le C m_\alpha(S)^p,$$

which by Theorem 3.1 is satisfied, if (4.13) holds. □

Taking a large enough $\alpha$ in Theorem 4.5 one obtains

**Corollary 4.6.** *Let $\Omega$ be an arbitrary regulated domain, let, for some $a>-1$, $\omega(z)=\bigl(\operatorname{dist}(z,\partial\Omega)\bigr)^a$ and $1<p<\infty$. There exists a bounded projection from $L^p_\omega(\Omega)$ onto $\mathscr{A}^p_\omega(\Omega)$. The space $\mathscr{A}^p_\omega(\Omega)$ is isomorphic to $l_p$.*

*Proof.* The operator $f\mapsto (f\circ\psi)\psi'$ is a linear homeomorphism from $L^p_\omega(\Omega)$ onto $L^p_\nu(\mathbf{D})$ and from $\mathscr{A}^p_\omega(\Omega)$ onto $\mathscr{A}^p_\nu(\mathbf{D})$, where $\nu(z)=|\psi'(z)|^{2-p+a}(1-|z|)^a$. Taking $n=1$ and $\alpha$ large enough, Lemma 4.4 and Theorem 4.5 show that $P^{(\alpha)}_{\mathbf{D}}$ is a bounded projection $L^p_\nu(\mathbf{D})$ onto $\mathscr{A}^p_\nu(\mathbf{D})$. But in view of [7, proof of Theorem III.A.11], this suffices to establish that $\mathscr{A}^p_\nu(\mathbf{D})$ is isomorphic to $l_p$. □

## References


[1] Békollé, D.: Inégalités à poids pour le projecteur de Bergman dans la boule unité de $\mathbf{C}^n$. - Studia Math. LXXI, 1982.

[2] Békollé, D.: Projections sur des espaces de fonctions holomorphes dans des domains plans. - Canad. J. Math. XXXVIII, 1986, 127–157.

[3] Garnett, J.: Bounded Analytic Functions. - Academic Press, New York, 1981.

[4] Luecking, D.: Representation and duality in weighted spaces of analytic functions. - Indiana Univ. Math. J. 34, 1985, 319–336. Erratum: Indiana Univ. Math. J. 35, 1986, 927–928.

[5] Pommerenke, Ch.: Boundary Behaviour of Conformal Maps. - Grundlehren der mathematischen Wissenschaften 299, Springer-Verlag, 1992.

[6] Taskinen, J.: Muckenhoupt-type condition for regulated domains. - Manuscript, 2001.

[7] Wojtaszczyk, P.: Banach Spaces for Analysts. - Cambridge University Press, 1991.

[8] Zhu, K.: Operator Theory in Function Spaces. - Marcel Decker, New York, 1995.


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