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SHARP ESTIMATES OF THE CURVATURE OF SOME FREE BOUNDARIES IN TWO DIMENSIONS

Björn Gustafsson and Makoto Sakai

Royal Institute of Technology, Mathematics Department
S-10044 Stockholm, Sweden; gbjorn@math.kth.se

Tokyo Metropolitan University, Department of Mathematics
Minami-Ohsawa, Hachioji-shi, Tokyo 192-0397, Japan; sakai@comp.metro-u.ac.jp

Abstract. We prove that the Cauchy transform of a positive measure on the interval $(-1,1) \subset \mathbf{R}$ in the complex plane maps the exterior of the unit disc onto a domain $\Omega \subset \mathbf{C}$ which can be written as a union of discs centered on the real axis. This is applied to the obstacle problem, partial balayage, quadrature domains and Hele-Shaw flow moving boundary problems, and we obtain sharp estimates of the curvature of free boundaries appearing in such problems.

1. Introduction

In this paper we obtain natural and sharp estimates of the curvature of some free boundaries arising in obstacle-type problems in two dimensions. We use conformal maps as an essential tool and the main result may be stated as a geometric property of the image domain under such a map: the Cauchy transform of a positive measure on the interval $(-1,1) \subset \mathbf{R}$ in the complex plane maps the exterior of the unit disc onto a domain $\Omega \subset \mathbf{C}$ which can be written as a union of discs centered on the real axis (Theorem 3.1).

An equivalent way of expressing this geometric property of Ω is to say that the inward normal rays from points on $(\partial\Omega)^+$ (the part of $\partial\Omega$ which is in the upper half-plane) never intersect in Ω^+; or that the foot point map, namely the map which takes the x-coordinate of a point on $(\partial\Omega)^+$ to (the x-coordinate of) the point where the inward normal crosses the real axis, is monotone increasing. (See Proposition 2.1 for these and other equivalent formulations.) It is in this last formulation, monotonicity of the foot point map, that our main result is proven. Writing down the statement in detail in terms of the original measure, everything comes down to proving that a certain polynomial of degree 10, in three variables and with 48 terms, is nonnegative in the unit cube.

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This result for conformal maps with Cauchy transforms is interesting in itself, but for us it is mainly a tool. Using it we show (in Section 4) that, for the obstacle problem in its simplest form (concerning a continuously differentiable function $u \geq 0$ satisfying $\Delta u = 1$ in $\{u > 0\}$), any part of the free boundary $\partial\{u > 0\}$ which can be cut off by a straight line is an envelope of circles centered on the line (Theorem 4.1). Thus we get a natural estimate, in one direction, of the curvature of the free boundary.

Some related applications concern partial balayage, quadrature domains and Hele-Shaw flow moving boundary problems, see Sections 5 and 6. For these applications the main result can be amplified to a global statement. If μ is a positive measure with compact support in $\mathbf{C}$ and K denotes the convex hull of the support, then if μ is swept out to Lebesgue measure (partial balayage), the set Ω where Lebesgue measure really is attained can be written as a union of discs centered on $\Omega \cap K$. And the inward normals from points on $\partial\Omega \setminus K$ do not intersect in $\Omega \setminus K$ (Theorem 5.5).

In particular these properties hold if Ω is a quadrature domain for subharmonic functions for μ (Corollary 5.6). Similarly, considering Hele-Shaw flow evolution $\{\Omega_t : t > 0\}$ of an initial fluid domain Ω_0, the evolution being caused by sources in Ω_0, if K denotes the closed convex hull of Ω_0, then Ω_t can be expressed as a union of discs with centers on $\Omega_t \cap K$. In addition, the inward normals from points on $\partial\Omega_t \setminus K$ do not intersect in $\Omega_t \setminus K$ (Theorem 6.1).

List of notation:

$\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$;
$\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$;
$\mathbf{D}^e = \{z \in \mathbf{C} : |z| > 1\} \cup \{\infty\}$;
$B(a,r) = \{z \in \mathbf{C} : |z-a| < r\}$ (we use $\mathbf{D} = B(0,1)$ in complex analytic contexts);
$m =$ Lebesgue measure in $\mathbf{C}$;
$p(x)$, $p(z)$: foot point map, see Proposition 2.1 and after Proposition 5.4;
N_x, N_z: normal segments, see Proposition 2.1 and after Proposition 5.4;
$\Omega^+ = \{x + iy \in \Omega : y > 0\}$ if $\Omega \subset \mathbf{C}$;
$\Omega^- = \{x + iy \in \Omega : y < 0\}$ if $\Omega \subset \mathbf{C}$;
$\Delta =$ Laplace operator $= \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} = 4\dfrac{\partial^2}{\partial z \partial \bar{z}}$;
$U^\mu(z) = \dfrac{1}{2\pi}\displaystyle\int \log \dfrac{1}{|\zeta - z|}\, d\mu(\zeta) =$ the logaritmic potential of a measure μ;
$\mathrm{Bal}\,(\mu, m) =$ partial balayage of μ onto m, see Section 5;
$\Omega(\mu)$: the saturated set for partial balayage, see (5.1);
$\mathrm{SL}^1(\Omega) =$ the set of subharmonic functions in Ω which are integrable with respect to Lebesgue measure;
$Q(\mu, \mathrm{SL}^1)$: set of quadrature domains, see Section 5.

2. Specific domains symmetric about the real axis

We formulate here in a number of different ways the statement that a domain symmetric about the real axis is a union of discs centered on the axis. The result is (to most parts) completely elementary and we state it because it will be useful to have several different points of view at hand when discussing geometry of free boundaries.

Proposition 2.1. *Let g be a positive, twice continuously differentiable function defined on an open bounded interval $I \subset \mathbf{R}$ and assume that $g(x) \to 0$ as $x \to \partial I$ (the boundary as a subset of $\mathbf{R}$). Let*

$$\begin{aligned} \Omega &= \{x + iy : x \in I,\ |y| < g(x)\}, \\ \Omega^+ &= \{x + iy : x \in I, 0 < y < g(x)\}, \\ (\partial\Omega)^+ &= \partial\Omega \cap \{y > 0\}. \end{aligned}$$

For $x \in I$, let $p(x)$ denote the foot point of the normal of $\partial\Omega$ at $z = x + ig(x) \in (\partial\Omega)^+$, i.e., the point of intersection of the normal with the real line:

$$p(x) = x + g(x)g'(x).$$

Let N_x denote the open normal segment from $x + ig(x)$ to $p(x)$:

$$N_x = \big\{t\big(x + ig(x)\big) + (1 - t)p(x) : 0 < t < 1\big\}.$$

Then the following statements are equivalent.

(i) *The function*

$$x \mapsto x^2 + g(x)^2,$$

defined on I, is convex.

(ii) *For every $c = a + ib$ with $a \in \mathbf{R}$, $b > 0$, the function*

$$x \mapsto (x - a)^2 + \big(g(x) - b\big)^2$$

is convex (alternatively: strictly convex) on each x-interval on which $g(x) > b$.

(iii) $$p'(x) \geq 0 \quad (x \in I).$$

(iv) $$\Omega = \bigcup_{x \in I} B\big(p(x), |x + ig(x) - p(x)|\big).$$

(v) *There exist radii $r = r(x) > 0$ such that*

$$\Omega = \bigcup_{x \in I} B\big(x, r(x)\big).$$

(vi) $$N_{x_1} \cap N_{x_2} = \emptyset \quad \text{for } x_1 \neq x_2.$$

(vii) *For each* $z \in (\partial\Omega)^+$ *the curvature radius of* $\partial\Omega$ *at* z *is, in case the curvature has negative sign* ($g'' < 0$), *at least as large as the distance from* z *to its foot point.*

(viii) *Every point in* Ω^+ *has a unique closest neighbour on* $(\partial\Omega)^+$.

(ix) *Every point on* $(\partial\Omega)^+$ *is a closest point on* $\partial\Omega$ *for some point on* I.

Notational remark. We shall also write $p(z)$ for $p(x)$ and N_z for N_x, where $z = x + ig(x)$. Then, e.g., statement (iv) of the proposition can be written

$$\Omega = \bigcup_{z\in(\partial\Omega)^+} B\big(p(z), |z - p(z)|\big).$$

Proof. We begin with some general considerations. For $c = a+ib$ with $a \in \mathbf{R}$, $b \geq 0$ let

$$\Phi_c(x) = \tfrac{1}{2}\big[(x-a)^2 + \big(g(x) - b\big)^2\big],$$

considered to be defined for those $x \in I$ for which $g(x) > b$. Then Φ_c is twice continuously differentiable with

$$\Phi_c'(x) = x - a + \big(g(x) - b\big)g'(x),$$
$$\Phi_c''(x) = 1 + g'(x)^2 + \big(g(x) - b\big)g''(x) = \Phi_a''(x) - bg''(x).$$

We note that (taking $b = 0$)

$$\Phi_a'(x) = p(x) - a \tag{2.1}$$

for any $a \in \mathbf{R}$. In particular

$$\Phi_{p(a)}'(a) = 0$$

for $a \in I$, i.e., the map $x \mapsto \Phi_{p(a)}(x)$ has a stationary point at $x = a$. Since $\Phi_{p(a)}(x)$ is a monotone function of the distance from $p(a)$ to $x + ig(x)$ this stationary point is a (global) minimum if and only if

$$B\big(p(a), |a + ig(a) - p(a)|\big) \subset \Omega. \tag{2.2}$$

For any point $w \in \Omega^+$ there is at least one closest point z on $\partial\Omega$, and any such z is necessarily located on $(\partial\Omega)^+$. Then $w \in N_z$ and, in particular, $w \in B\big(p(z), |z - p(z)|\big)$. Note that N_z is one of the radii in $B\big(p(z), |z - p(z)|\big)$. It follows that

$$\Omega^+ \subset \bigcup_{z\in(\partial\Omega)^+} N_z \subset \bigcup_{z\in(\partial\Omega)^+} B\big(p(z), |z - p(z)|\big). \tag{2.3}$$

Also:

$$\Omega \subset \bigcup_{z\in(\partial\Omega)^+} B\big(p(z), |z-p(z)|\big). \tag{2.4}$$

Now we turn to the statements of the proposition.

(i) $\Rightarrow$ (ii): This is seen from the formula for $\Phi_c''(x)$ above: if $c = a + ib$, $b > 0$ and $\Phi_0''(x) \geq 0$ (hence $\Phi_a''(x) \geq 0$), then $\Phi_c''(x) > 0$ follows by reading off the first expression for $\Phi_c''(x)$ in case $g''(x) \geq 0$ (recall that $g(x) > b$), the second expression in case $g''(x) < 0$.

(ii) $\Rightarrow$ (i): Just let $b \to 0$ in the second expression for $\Phi_c''(x)$.

(i) $\Leftrightarrow$ (iii): This is clear from (2.1) with $a = 0$.

(iii) $\Leftrightarrow$ (vii): The curvature of $(\partial\Omega)^+$ at $z = x + ig(x)$ is

$$\frac{g''(x)}{\big(1+g'(x)^2\big)^{3/2}}$$

and the curvature radius is one over that (taken to be $+\infty$ if $g''(x) = 0$). The center of curvature (the center of the circle which has the best fitting to $\partial\Omega$ at z) is located somewhere along the normal of $\partial\Omega$ at z (or at infinity, if $g''(x) = 0$). It may be noticed that the assertion of (vii) is exactly that this center of curvature is not located on the segment N_x.

Now, the y-coordinate of the center of curvature is easily calculated to be

$$g(x) + \frac{1+g'(x)^2}{g''(x)} = \frac{p'(x)}{g''(x)}.$$

If $g''(x) \geq 0$ then $p'(x) \geq 0$ holds automatically, and if $g''(x) < 0$ then $p'(x) \geq 0$ holds if and only if the above y-coordinate is ≤ 0. From this (iii) $\Leftrightarrow$ (vii) follows.

(i) $\Rightarrow$ (iv): By (2.4) we only need to prove (2.2) for every $a \in I$. But when Φ_0 (equivalently $\Phi_{p(a)}$) is convex then every stationary point of $x \mapsto \Phi_{p(a)}(x)$ is a global minimum, hence the desired conclusion follows from what was said in connection with (2.2).

(iv) $\Rightarrow$ (v): If the representation in (iv) holds then we get a representation as in (v) by adding small discs $B\big(x, r(x)\big) \subset \Omega$ for those $x \in I$ which are not in the range of p. (Clearly p maps I into I when (iv) holds, but it need not be onto.)

(v) $\Rightarrow$ (iv): Let $z \in (\partial\Omega)^+$. Then, if (v) holds, there exist $a_n \in I$ and $z_n \in B\big(a_n, r(a_n)\big)$ such that $z_n \to z$. The smoothness of $(\partial\Omega)^+$ and the inclusions $B\big(a_n, r(a_n)\big) \subset \Omega$ force the convergences $a_n \to p(z)$ and $r(a_n) \to |z - p(z)|$ and it follows that $B\big(p(z), |z-p(z)|\big) \subset \Omega$. Now (iv) follows from (2.4).

(iv) $\Leftrightarrow$ (ix): Note that $z \in (\partial\Omega)^+$ is a closest point of $a \in I$ if and only if $a = p(z)$ and $B\big(p(z), |p(z)-z|\big) \subset \Omega$. Thus a is determined by z and it follows immediately (in view also of (2.4)) that all $z \in (\partial\Omega)^+$ are such closest points if and only if (iv) holds.

(iv) $\Rightarrow$ (vi): Assume $w \in N_{z_1} \cap N_{z_2}$ for some $z_1, z_2 \in (\partial\Omega)^+$, $z_1 \neq z_2$. Without loss of generality $|z_1 - w| \leq |z_2 - w|$. Then $z_1 \in \overline{B(w, |z_2 - w|)}$ and it follows that $z_1 \in B\big(p(z_2), |z_2 - p(z_2)|\big)$, which contradicts (iv) since $z_1 \notin \Omega$.

(vi) $\Rightarrow$ (viii): If $w \in \Omega^+$ has two closest neighbours $z_1, z_2 \in (\partial\Omega)^+$ then $w \in N_{z_1} \cap N_{z_2}$.

(viii) $\Rightarrow$ (ii): This conclusion is somewhat analogous to [7, Theorem 2.1.30] (attributed to Motzkin) and it is slightly more tricky than the other ones.

Assume that (ii) fails and we shall produce a point $c \in \Omega^+$ with at least two closest neighbours on $(\partial\Omega)^+$. By assumption there exists $b > 0$ and an open interval $J \subset I$ on which $g(x) > b$ and such that $\Phi_{ib}(x)$ is not convex on J. We may assume that J is maximal, so that $g(x) = b$ for $x \in \partial J$.

By definition of Φ_{ib}

$$\Phi_{ib}(x) \geq \tfrac{1}{2}x^2 \quad \text{for } x \in J \cup \partial J \tag{2.5}$$

with equality for $x \in \partial J$. Let $\Psi(x)$ be the largest convex function on $J \cup \partial J$ satisfying $\Psi(x) \leq \Phi_{ib}(x)$ on $J \cup \partial J$. It follows from (2.5) and the fact that $\frac{1}{2}x^2$ is convex that

$$\tfrac{1}{2}x^2 \leq \Psi(x) \leq \Phi_{ib}(x) \quad \text{for } x \in J \cup \partial J.$$

Since $\Phi_{ib}(x)$ is not convex in J we have $\Psi(x) < \Phi_{ib}(x)$ on some subinterval $(x_1, x_2) \subset J$, which we take to be maximal with this property. In the interior (x_1, x_2) we have $\Psi''(x) = 0$, otherwise Ψ could have been made larger. At the end points $\Psi(x_j) = \Phi_{ib}(x_j)$ $(j = 1, 2)$ because (x_1, x_2) is maximal. It follows that $\Psi(x)$ is linear (affine) on $[x_1, x_2]$, say

$$\Psi(x) = ax + k \qquad (x_1 \leq x \leq x_2).$$

Then $ax + k \leq \Psi(x)$ on $J \cup \partial J$, in particular,

$$ax + k \leq \Phi_{ib}(x) \quad \text{for } x \in J \cup \partial J$$

with equality attained at least for the two points x_1 and x_2.

Now we take

$$c = a + ib.$$

Then, by the above,

$$\Phi_c(x) = \Phi_{ib}(x) - ax + \tfrac{1}{2}a^2 \geq \tfrac{1}{2}(2k + a^2) \quad \text{for } x \in J \cup \partial J \tag{2.6}$$

with equality for $x = x_1, x_2$.

We shall see that $2k + a^2 > 0$. Since $ax + k = \Psi(x) > \frac{1}{2}x^2$ in (x_1, x_2), the line $y = ax + k$ and the quadratic curve $y = \frac{1}{2}x^2$ intersect at two points (exactly).

Let $q+\frac{1}{2}iq^2$ be one of them, satisfying $q\neq 0$. Then $a=(q^2/2-k)/q=q/2-k/q$, and so

$$2k+a^2=2k+\left(\frac{q}{2}-\frac{k}{q}\right)^2=\left(\frac{q}{2}+\frac{k}{q}\right)^2\geq 0.$$

If $2k+a^2=0$, then $k=-\frac{1}{2}q^2$. This means that $y=ax+k$ has a point of tangency with $y=\frac{1}{2}x^2$ at $q+\frac{1}{2}iq^2$, contradicting $ax+k>\frac{1}{2}x^2$ in (x_1,x_2). Hence $2k+a^2>0$.

By definition of $\Phi_c(x)$, (2.6) means that $z\notin B\big(c,\sqrt{2k+a^2}\,\big)$ for all $z=x+ig(x)\in(\partial\Omega)^+$ with $x\in J\cup\partial J$ and that $z_j=x_j+ig(x_j)\in\partial B\big(c,\sqrt{2k+a^2}\,\big)$ for $j=1,2$. In particular $a\in J$ and so $c\in\Omega^+$. Since $x+ib\notin B\big(c,\sqrt{2k+a^2}\,\big)$ for $x\in\partial J$, it also follows that $z\notin B\big(c,\sqrt{2k+a^2}\,\big)$ for those $z=x+ig(x)\in(\partial\Omega)^+$ for which $x\notin J$. Thus c is a point with at least two closest neighbours on $(\partial\Omega)^+$, namely $z_j=x_j+ig(x_j)\in(\partial\Omega)^+$, as required. □

3. The main result in terms of conformal mappings

The following is our main result.

Theorem 3.1. *Let μ be a positive measure on the interval $(-1,1)$ satisfying $\int d\mu>0$ and*

$$\int_{-1}^{1}\frac{d\mu(t)}{1-t^2}<\infty, \tag{3.1}$$

and define

$$f(w)=\int_{-1}^{1}\frac{d\mu(t)}{t-w}.$$

Then f is univalent on $\mathbf{D}^e$ and maps $\mathbf{D}^e$ onto a bounded domain Ω which satisfies the assumptions (with g real analytic) and equivalent conditions in Proposition 2.1. For example, the foot point map $p(x)$ of $(\partial\Omega)^+$ is monotone increasing and Ω can be written as a union of discs centered on $\mathbf{R}$:

$$\Omega=\bigcup_{x\in I}B\big(x,r(x)\big)$$

for suitable $r(x)>0$ $(I=\Omega\cap\mathbf{R})$.

If μ is not a point mass then $p'(x)>0$ and $\overline{B(p(x),|x+ig(x)-p(x)|)}\subset\Omega\cup\{x+ig(x)\}$ for every $x\in I$. (If μ is a point mass then Ω is a disc and $p(x)$ is constant, namely equal to the center of the disc.)

Remark. Via an inversion in $\partial\mathbf{D}$ the result may equally well be stated for a function defined in the unit disc:

$$h(w)=\int_{-1}^{1}\frac{w\,d\mu(t)}{1-tw}$$

maps $\mathbf{D}$ conformally onto a domain Ω having the properties in Proposition 2.1.

Proof. An application of the argument principle shows that f is univalent in $\{|w| > 1+\varepsilon\}$ for every $\varepsilon > 0$, hence f is univalent in $\mathbf{D}^e$. Assumption (3.1) implies that $\Omega = f(\mathbf{D}^e)$ is a bounded domain, and it is clearly also symmetric about the real axis.

We shall study f on $(\partial\mathbf{D})^+$, which is mapped onto $(\partial\Omega)^+$ but with a reversion of the orientation since f maps what is outside $\partial\mathbf{D}$ to what is inside $\partial\Omega$. With $w = e^{i\theta}$, $0 < \theta < \pi$, we have

$$\operatorname{Re}\frac{d}{d\theta}f(e^{i\theta}) = \operatorname{Re}\bigl[iwf'(w)\bigr] = -\operatorname{Im}\int_{-1}^{1}\frac{e^{i\theta}\,d\mu(t)}{(t-e^{i\theta})^2} = \int_{-1}^{1}\frac{(1-t^2)\sin\theta}{|t-e^{i\theta}|^4}\,d\mu(t).$$

Since $\int d\mu > 0$ it follows that

$$\operatorname{Re}\frac{d}{d\theta}f(e^{i\theta}) > 0 \tag{3.2}$$

for $0 < \theta < \pi$.

This again implies the univalency of f. Moreover it shows that the tangent of $(\partial\Omega)^+$ is never vertical, that $(\partial\Omega)^+$ is a graph of a function g as in Proposition 2.1 and that the foot point map p exists. Since f is analytic in the entire upper half-plane, g is real analytic.

We now relate p to f and μ. We write

$$z = x+iy = f(w) \quad\text{and}\quad w = u+iv.$$

For $z \in (\partial\Omega)^+$ we have $w = e^{i\theta}$ with $0 < \theta < \pi$. We may then consider x, y, p etc. as functions of θ and we shall write x_θ, y_θ, p_θ etc. for the derivatives with respect to θ.

Our aim is to prove that the equivalent conditions in Proposition 2.1 hold by proving the third one: $p'(x) \ge 0$ $(x \in I)$. Since we saw above (3.2) that $x_\theta > 0$ this condition can be written

$$x_\theta^2 p_\theta \ge 0 \quad \text{for } 0 < \theta < \pi \tag{3.3}$$

(to be proved). It is well known that $\mathbf{D}$ can be mapped conformally onto itself by a Möbius transformation which preserves the real axis and takes any point on $(\partial\mathbf{D})^+$ onto any prescribed point on $(\partial\mathbf{D})^+$. For this reason it is enough to prove (3.3) for one single value of θ, for example for $\theta = \frac{1}{2}\pi$. This argument will be made more precise in Lemma 3.2 stated after the proof.

From

$$p(x) = x + \frac{dy}{dx}y = x + \frac{y_\theta}{x_\theta}y$$

we get

$$x_\theta^2 p_\theta = x_\theta(x_\theta^2 + y_\theta^2) + y(x_\theta y_{\theta\theta} - x_{\theta\theta}y_\theta).$$

Denoting derivatives with respect to w by prime we have $x_\theta + iy_\theta = z_\theta = z' \cdot iw$, $x_{\theta\theta} + iy_{\theta\theta} = (z' \cdot iw)' \cdot iw = -(z''w^2 + z'w)$. Using $w\overline{w} = 1$ we obtain

$$\begin{aligned} x_\theta^2 p_\theta &= \operatorname{Re}(iwz') \cdot |wz'|^2 + \operatorname{Im} z \cdot \operatorname{Im}\big[(iwz')\overline{(w^2z'' + wz')}\,\big] \\ &= \operatorname{Im}(z - wz') \cdot |z'|^2 + \operatorname{Im} z \cdot \operatorname{Re}(z'\overline{wz''}\,). \end{aligned}$$

Inserting here

$$z = \int \frac{d\mu(a)}{a-w}, \quad z' = \int \frac{d\mu(a)}{(a-w)^2}, \quad z'\overline{wz''} = \int \frac{d\mu(b)}{(b-w)^2} \cdot \int \frac{2\overline{w}\, d\mu(c)}{(c-\overline{w})^3}$$

gives

$$\begin{aligned} x_\theta^2 p_\theta &= \operatorname{Im} \int \frac{a-2w}{(a-w)^2}\, d\mu(a) \cdot \operatorname{Re}\Big(\int \frac{d\mu(b)}{(b-w)^2} \cdot \int \frac{d\mu(c)}{(c-\overline{w})^2} \Big) \\ &\quad + \operatorname{Im} \int \frac{d\mu(a)}{a-w} \cdot \operatorname{Re}\Big(\int \frac{d\mu(b)}{(b-w)^2} \cdot \int \frac{2\overline{w} d\mu(c)}{(c-\overline{w})^3} \Big) \\ &= \iiint \Big[\operatorname{Im} \frac{a-2w}{(a-w)^2} \cdot \operatorname{Re} \frac{1}{(b-w)^2(c-\overline{w})^2} \\ &\quad + \operatorname{Im} \frac{1}{a-w} \cdot \operatorname{Re} \frac{2\overline{w}}{(b-w)^2(c-\overline{w})^3} \Big]\, d\mu(a)\, d\mu(b)\, d\mu(c). \end{aligned}$$

We write the last integrand as

$$\begin{aligned} &\frac{v}{|w-a|^6|w-b|^6|w-c|^6} \cdot \Big[|w-a|^2|w-b|^2|w-c|^2(1-au) \operatorname{Re}\big((\overline{w}-b)^2(w-c)^2\big) \\ &\quad - |w-a|^4|w-b|^2 \operatorname{Re}\big(\overline{w}(\overline{w}-b)^2(w-c)^3\big) \Big] \\ &\quad = \frac{v}{|w-a|^6|w-b|^6|w-c|^6} \cdot P(u;a,b,c). \end{aligned}$$

Here $P(u;a,b,c)$ is a polynomial with real coefficients. Indeed, the expression defining P is a polynomial in w, $\overline{w}$, a, b and c which is invariant under conjugation $w \mapsto \overline{w}$ and which takes only real values. Hence, as a polynomial in u, v (and a, b, c) only even powers of v occur, and since $u^2 + v^2 = 1$ we get a polynomial in u when v is eliminated.

It is natural to symmetrize P in the variables a, b, c into

$$\begin{aligned} Q(u;a,b,c) = \tfrac{1}{6}\Big[&P(u;a,b,c) + P(u;a,c,b) \\ &+ P(u;b,a,c) + P(u;b,c,a) + P(u;c,a,b) + P(u;c,b,a) \Big]. \end{aligned}$$

This gives us

$$x_\theta^2 p_\theta = \iiint \frac{vQ(u;a,b,c)}{|w-a|^6|w-b|^6|w-c|^6}\, d\mu(a)\, d\mu(b)\, d\mu(c).$$

It is difficult to write down $Q(u;a,b,c)$ explicitly because of its size. Using e.g. Mathematica one finds that it is a polynomial in u, a, b and c of degree 5 in u and of degree 4 in each of a, b, and c, and that it has in total 232 terms.

However, as remarked above, it is enough to prove (3.3) for $\theta=\frac{1}{2}\pi$, i.e. for $w=u+iv=i$, and this reduces the complexity considerably. The polynomial $Q(0;a,b,c)$ is of total degree 10 and of degree 4 in each of a, b and c. It has 48 terms. We shall prove that it is nonnegative for $a,b,c\in[-1,1]$, which will imply (3.3).

A lengthy but straightforward calculation shows that $Q(0;a,b,c)$ can be written explicitly as

$$Q(0;a,b,c)=\tfrac{4}{3}R(a,b,c)+\tfrac{32}{3}S(a,b,c),$$

where

$$R(a,b,c)=(1-a^2)(1-b^2)(1-c^2)\big[(a-b)^2(1+c^2)+(b-c)^2(1+a^2)+(c-a)^2(1+b^2)\big]$$

and

$$\begin{aligned}S(a,b,c)&=a^2b^2+b^2c^2+c^2a^2+a^3b^3+b^3c^3+c^3a^3\\&\quad+a^2b^3c^3+a^3b^2c^3+a^3b^3c^2-a^2bc-ab^2c-abc^2\\&\quad-a^4bc-ab^4c-abc^4-a^4b^2c^2-a^2b^4c^2-a^2b^2c^4.\end{aligned}$$

We see directly that $R(a,b,c)\ge 0$, and it remains to prove that $S(a,b,c)\ge 0$. For this we may assume that $|a|\le|b|\le|c|$ and $|c|>0$, because $S(0,0,0)=0$. Set $a=Ac$ and $b=Bc$. Then $-1\le A\le 1$, $-1\le B\le 1$ and

$$S(a,b,c)=S(Ac,Bc,c)=c^4T(A,B,c^2),$$

where T is a polynomial in A, B and c^2, which is of degree 2 in c^2. Set $C=c^2$. Then

$$\begin{aligned}T(A,B,C)&=A^2-AB+B^2-A^2B-AB^2+A^2B^2\\&\quad+(A^3-AB+B^3-A^4B-AB^4+A^3B^3)C\\&\quad-A^2B^2\big((1-A)(1-B)+(A-B)^2\big)C^2.\end{aligned}$$

Since the coefficient of C^2 is nonpositive, since

$$T(A,B,-1)=(1-A)(1-B)(1-AB)(A^2+B^2)\ge 0$$

and

$$T(A,B,1)=(1+A)(1+B)(1-AB)(A-B)^2\ge 0$$

it follows that $T(A,B,C)\ge 0$ on $[-1,1]^3$. This implies that $x_\theta^2p_\theta\ge 0$ for $\theta=\frac{1}{2}\pi$ and hence proves that the equivalent statements in Proposition 2.1 hold.

Suppose now $x_\theta^2p_\theta=0$ for $\theta=\frac{1}{2}\pi$. Then $Q(0;a,b,c)=0$ a.e. with respect to $\mu\times\mu\times\mu$. But this implies $R(a,b,c)=S(a,b,c)=0$ a.e., and $R(a,b,c)=0$ holds at a point $(a,b,c)\in(-1,1)^3$ if and only if $a=b=c$. Hence $Q(0;a,b,c)=0$ a.e. implies that μ is a point measure, as required. □

The following lemma, which was used in the proof above, shows that in order to prove (3.3) for all $0<\theta<\pi$ it is enough to prove it for one single value of θ, but then for all measures μ.

Lemma 3.2. *For $l \in (-1,1)$ set*

$$L(w)=\frac{w-l}{1-lw},$$

a Möbius transformation which preserves the upper half-plane and the unit disc. Let μ and M be positive measures on $(-1,1)$ related by $dM\big(L(a)\big)=L'(a)\,d\mu(a)$. Then with the variables w and W linked by $W=L(w)$ we have the identity

$$\int\frac{dM(A)}{A-W}=\int\frac{d\mu(a)}{a-w}+C,$$

where C is a real constant.

Given any pair of points $w_0, W_0 \in (\partial\mathbf{D})^+$ the parameter l can be chosen so that $W_0=L(w_0)$, namely by taking $l=(w_0-W_0)/(1-w_0W_0)$.

Proof. A calculation shows that

$$\frac{L'(a)}{L(a)-L(w)}=\frac{1}{a-w}+\frac{l}{1-la}.$$

Integrating this with respect to μ gives

$$\int\frac{L'(a)\,d\mu(a)}{L(a)-L(w)}=\int\frac{d\mu(a)}{a-w}+\int\frac{l\,d\mu(a)}{1-la}$$

and changing the variable of integration to $A=L(a)$ in the first integral gives the desired formula. □

4. Application to the free boundary for an obstacle problem

We shall here formulate our main result as an assertion about the local geometry of the free boundary for the obstacle problem in its simplest form. Some general references for this section are [3], [8] and [10]. Regularity questions for the free boundary are treated e.g. in [1], [12] and [13].

The obstacle problem can be stated as the problem of finding the smallest superharmonic function v satisfying suitable boundary conditions and passing a given obstacle, represented by a function ψ. Assuming that ψ satisfies $\Delta\psi=-1$, at least in a small disc B under consideration, the difference $u=v-\psi$ will satisfy the following conditions within B:

$$u\in C^1(B),\qquad u\ge 0\quad\text{in } B,\qquad \Delta u=\chi_\Omega\quad\text{in } B,$$

where

$$\Omega = \{z \in B : u(z) > 0\}$$

is the subset of B where the solution goes free from the obstacle (the noncoincidence set).

By definition of Ω,

$$u = 0 \quad \text{on } B \setminus \Omega$$

and, since this is the minimum value of u,

$$\nabla u = 0 \quad \text{on } B \setminus \Omega.$$

The latter two equations mean that u, as a solution of the elliptic equation $\Delta u = 1$ in Ω, satisfies too many boundary conditions on $\partial\Omega \cap B$, which on the other hand is a free boundary (i.e., is not prescribed in advance).

For convenience we shall in the sequel take B to be centered at the origin.

Theorem 4.1. *Assume that* $\Omega^+ = \{x + iy \in \Omega : y > 0\}$ *is relatively compact in* $B = B(0, r)$, *where the obstacle satisfies* $\Delta\psi = -1$. *Let* $I = \Omega \cap \mathbf{R}$, *let* $(\Omega^+)^*$ *denote the reflection of* Ω^+ *in* $\mathbf{R}$ *and let* D *be a component of* $\Omega^+ \cup I \cup (\Omega^+)^*$.

Then D *satisfies the assumptions* (*with* g *real analytic*) *and equivalent conditions in Proposition* 2.1. *In addition,* $(\Omega^+)^* \subset \Omega$. *In particular,*

$$\Omega^+ = \bigcup_{x \in I} B\bigl(x, r(x)\bigr)^+ = \bigcup_{z \in (\partial\Omega)^+} B\bigl(p(z), |z - p(z)|\bigr)^+$$

for some radii $r(x) > 0$, *and*

$$\Omega^+ = \bigcup_{z \in (\partial\Omega)^+} N_z,$$

where the normal segments N_z *are disjoint* (*notation as in Proposition* 2.1).

Proof. We shall show that D is the conformal image of $\mathbf{D}^e$ under a map f as in Theorem 3.1. The first part of the proof essentially consists of repetitions from [5], but we need this as a background, and it will also be helpful for understanding the material in Section 5. Define

$$u^*(x + iy) = u(x - iy),$$

$$\hat{u} = u - \inf(u, u^*) = \sup(0, u - u^*)$$

in B. Since $\Delta u \le 1$, $\Delta u^* \le 1$ we have $\Delta \inf(u, u^*) \le 1$. Also, $0 \le \hat{u} \le u$. It follows that $\hat{u}$ satisfies

$$\begin{aligned} \Delta\hat{u} &\ge 0 \quad \text{in } \Omega^+, \\ \hat{u} &= 0 \quad \text{on } \partial(\Omega^+). \end{aligned}$$

Applying the maximum principle in Ω^+, using that this set is relatively compact in B, shows that $\hat{u} \le 0$ in Ω^+. This means that $u \le u^*$ in Ω^+, i.e., that $u(x+iy) \le u(x-iy)$ for $y > 0$. Hence $(\Omega^+)^* \subset \Omega$ and

$$\frac{\partial u}{\partial y} \le 0 \quad \text{on } \mathbf{R}.$$

Next we apply the maximum principle to $\partial u/\partial y$ in Ω^+. Since $\partial u/\partial y = 0$ on $(\partial\Omega)^+$ and $\Delta \partial u/\partial y = 0$ in Ω^+ this shows that $\partial u/\partial y \le 0$ in all Ω^+, in fact even that $\partial u/\partial y < 0$ in Ω^+ (having $\partial u/\partial y = 0$ in a component of Ω^+ would contradict the definition of Ω). Thus u is a decreasing function of y in Ω^+ and it follows that each component of Ω^+ is of the same form as the set Ω^+ in the assumption of Proposition 2.1, i.e., is the subgraph of a function g (for which we so far have no regularity information).

It also follows that each component D of the symmetrized domain $\Omega^+ \cup I \cup (\Omega^+)^*$ is simply connected. Let $f\colon \mathbf{D}^e \to D$ be a conformal map preserving the real axis and taking $(\mathbf{D}^e)^+$ onto D^+. Using u we can define the Schwarz function (see [2] and [14]) of $(\partial\Omega)^+$ by

$$S(z) = \overline{z} - 4\frac{\partial u}{\partial z}.$$

One immediately realizes that $S(z)$ is holomorphic in Ω^+ and equals $\overline{z}$ on $(\partial\Omega)^+$.

Thus $z \mapsto \overline{S(z)}$, for $z \in \Omega^+$, is the anticonformal reflection in $(\partial\Omega)^+$ and it can be used to extend f from $(\mathbf{D}^e)^+$ to the entire upper half-plane, namely by defining

$$f(w) = \overline{S\bigl(f(1/\overline{w})\bigr)} = f(1/\overline{w}) - 2\frac{\partial u}{\partial x}\bigl(f(1/\overline{w})\bigr) - 2i\frac{\partial u}{\partial y}\bigl(f(1/\overline{w})\bigr) \tag{4.1}$$

for $w \in \mathbf{D}^+$. It is continuous across $(\partial\mathbf{D})^+$ and hence holomorphic in $\mathbf{C}^+$. Extending f in the lower half-plane in the same way we get f defined and holomorphic in $\widehat{\mathbf{C}} \setminus [-1,1]$. We therefore can represent it as a Cauchy integral around the boundary $\partial(\widehat{\mathbf{C}} \setminus [-1,1])$, regarded as an oriented closed curve:

$$f(w) = f(\infty) + \frac{1}{2\pi i}\int_{\partial(\widehat{\mathbf{C}}\setminus[-1,1])} \frac{f(t)\,dt}{t-w} = f(\infty) + \frac{1}{2\pi i}\int_{-1}^{1} \frac{f_+(t) - f_-(t)}{t-w}\,dt.$$

Here $f_\pm(t) = \lim_{\varepsilon\searrow 0} f(t \pm i\varepsilon)$. It also follows from f being holomorphic in $\widehat{\mathbf{C}} \setminus [-1,1]$ that $(\partial D)^+$ and g are real analytic.

The jump $f_+(t) - f_-(t)$ only comes from the $\partial u/\partial y$-term in (4.1), the other terms are continuous for $w \in \mathbf{R}$, and it evaluates to

$$\begin{aligned} f_+(t) - f_-(t) &= -2i\lim_{\varepsilon\searrow 0}\biggl[\frac{\partial u}{\partial y}\Bigl(f\bigl(1/(\overline{t+i\varepsilon}\,)\bigr)\Bigr) - \frac{\partial u}{\partial y}\Bigl(f\bigl(1/(\overline{t-i\varepsilon}\,)\bigr)\Bigr)\biggr] \\ &= -4i\lim_{\varepsilon\searrow 0}\frac{\partial u}{\partial y}\bigl(f(1/t) + i\varepsilon\bigr). \end{aligned}$$

Since $\partial u/\partial y < 0$ in Ω^+ this shows that, aside for the constant $f(\infty)$, f is of the form in Theorem 3.1, namely the Cauchy transform of the measure

$$d\mu(t) = -\frac{2}{\pi} \lim_{\varepsilon \searrow 0} \frac{\partial u}{\partial y}\big(f(1/t) + i\varepsilon\big)\, dt \qquad (t \in [-1, 1]).$$

This proves the theorem. □

5. Application to partial balayage and quadrature domains

For problems of partial balayage and quadrature domains our main result has a natural global interpretation. We start with a short review of the concepts involved. More details can be found in, e.g., [11], [4] and [5].

Classical balayage is the process of sweeping a measure μ completely out to the boundary of a given domain, supposed to contain $\operatorname{supp}\mu$, in such a way that the exterior potential is left unchanged. For partial balayage there need not be any fixed domain (other than the entire plane) to start with, instead one tries to sweep the measure to have a prescribed density with respect to Lebesgue measure. The swept measure will then occupy a set which is unknown from the beginning. Thus partial balayage gives rise to a free boundary problem, which turns out to be of obstacle type.

To make the above more precise, let μ be a positive measure with compact support in $\mathbf{C}$. We shall consider partial balayage of μ onto Legesgue measure (area measure) m and use the notation $\operatorname{Bal}(\mu, m)$ for the result, a positive measure $\leq m$ having the property that its potential agrees with that of μ outside the (a priori unknown) set $\Omega = \Omega(\mu)$ on which it equals m.

In case μ has finite energy (the energy of μ being defined as $\|\mu\|^2_{\text{energy}} = \int U^\mu d\mu$) the partial balayage measure can be defined as the measure closest to μ in the energy norm among all measures which are $\leq m$. A slightly more handy and general definition is the following, which completely parallels the description of the obstacle problem given in Section 4.

Definition 5.1. Define

$$\operatorname{Bal}(\mu, m) = -\Delta V^\mu,$$

where V^μ is the largest of all locally integrable functions (or even distributions) V satisfying

$$\begin{aligned} V &\leq U^\mu \qquad \text{in } \mathbf{C}, \\ -\Delta V &\leq 1 \qquad \text{in } \mathbf{C}. \end{aligned}$$

It is easy to see, with a Perron family argument, that such a largest V^μ exists. It satisfies $0 \leq -\Delta V^\mu \leq 1$, hence has a representative which is a continuously differentiable function. It also follows that $\operatorname{Bal}(\mu, m)$ is a positive measure which is dominated by m, and it is not hard to show that it has compact support. We note that $U^{\mathrm{Bal}\,(\mu,m)} = V^\mu$.

The desired result of applying $\mathrm{Bal}\,(\mu, m)$ is usually that it shall take the form

$$\mathrm{Bal}\,(\mu, m) = \chi_\Omega m$$

for some open set Ω. This is not always achieved (namely if μ is too much spread out, e.g. has density < 1, already from beginning) but one can always define a largest good set Ω (the saturated set for $\mathrm{Bal}\,(\mu, m)$) as follows:

$$\begin{aligned}\Omega(\mu) &= \big\{\text{the largest open set in which } \mathrm{Bal}\,(\mu, m) = m\big\} \\ &= \mathbf{C} \setminus \operatorname{supp}\big(m - \mathrm{Bal}\,(\mu, m)\big).\end{aligned} \tag{5.1}$$

Then one shows that

$$U^{\mathrm{Bal}\,(\mu,m)} = U^\mu \quad \text{on } \mathbf{C} \setminus \Omega(\mu).$$

By construction of V^μ we also have

$$U^{\mathrm{Bal}\,(\mu,m)} \le U^\mu \quad \text{in all } \mathbf{C}.$$

Remark. The definition of partial balayage extends to much more general goal measures than m, e.g. to any measure of the form ϱm where the density ϱ is any locally integrable function bounded from above and below: $0 < c_1 \le \varrho \le c_2 < \infty$. In the definition above one just replaces the upper bound for $-\Delta V$ by ϱ. Also, classical (complete) balayage can be incorporated as a special case of partial balayage, see [5].

Related to partial balayage is the notion of quadrature domain for subharmonic functions. In the present paper a quadrature domain will be allowed to be disconnected. Denoting by $\mathrm{SL}^1(\Omega)$ the set of subharmonic functions in Ω which are integrable with respect to Lebesgue measure we have

Definition 5.2 [11]. Let μ be a positive measure with compact support and let Ω be a bounded open set. We say that Ω is a quadrature domain for μ for subharmonic functions, and write $\Omega \in Q(\mu, \mathrm{SL}^1)$, if

(i) $\mu(\mathbf{C} \setminus \Omega) = 0$,

(ii) $\int_\Omega \varphi \, d\mu \le \int_\Omega \varphi \, dm$ for all $\varphi \in \mathrm{SL}^1(\Omega)$.

The class $Q(\mu, \mathrm{SL}^1)$ may be empty, and if it is not empty it consists, up to nullsets, of only one element, namely $\Omega(\mu)$ (defined in (5.1)). More precisely we have the following.

Proposition 5.3. *Assume* $Q(\mu, \mathrm{SL}^1) \neq \emptyset$. *Then* $\Omega(\mu) \in Q(\mu, \mathrm{SL}^1)$ *and every element in* $Q(\mu, \mathrm{SL}^1)$ *is of the form* $\Omega(\mu) \setminus E$, *where* E *is a relatively closed subset of* $\Omega(\mu)$ *satisfying* $m(E) = 0$, $V^\mu = U^\mu$ *on* E. *Moreover,*

$$\mathrm{Bal}\,(\mu, m) = \chi_{\Omega(\mu)} m.$$

We recall that $\Omega(\mu)$ always exists, even if $Q(\mu, \mathrm{SL}^1) = \emptyset$. Our main result gives rather precise information on the geometry of $\Omega(\mu)$ outside the convex hull of $\operatorname{supp} \mu$. The following was proved in [5] (in any number of dimensions).

Proposition 5.4 [5]. *Let* μ *be a positive measure with compact support in* $\mathbf{C}$, *let* K *denote the convex hull of* $\operatorname{supp} \mu$ *and let* $\Omega = \Omega(\mu)$ *be the saturated set for* $\mathrm{Bal}\,(\mu, m)$ *as defined in* (5.1). *Then*:

(i) *Outside* K, $\mathrm{Bal}\,(\mu, m)$ *has the pure form* χ_Ω:

$$\mathrm{Bal}\,(\mu, m)|_{\mathbf{C} \setminus K} = \chi_{\Omega \setminus K} m.$$

(ii) *The part* $\partial\Omega \setminus K$ *of the boundary of* Ω *is smooth real analytic.*

(iii) *For each* $z \in \partial\Omega \setminus K$ *the inward normal ray of* $\partial\Omega$ *at* z *intersects* K. *Moreover,* $E \subset K$ *for any set* E *as in Proposition* 5.3.

Referring to the conclusions of the proposition, let $p(z)$ denote the first point of intersection of the inward normal of $\partial\Omega$ at $z \in \partial\Omega \setminus K$ with K (the foot point). Thus $p(z) \in \partial K$. Let further

$$N_z = \{tz + (1-t)p(z) : 0 < t < 1\}$$

be the open normal segment from z to $p(z)$. When stated for partial balayage our main result reads as follows.

Theorem 5.5. *With assumptions and notation as above and in Proposition* 5.4 *we have, continuing the numbering from Proposition* 5.4,

(iv)

$$\Omega = \bigcup_{z \in \partial\Omega \setminus K} B\big(p(z), |z - p(z)|\big) \cup (\Omega \cap K).$$

It follows that all of Ω *can be written as a union of discs with centers on* $\Omega \cap K$, *namely*

$$\Omega = \bigcup_{w \in \Omega \cap K} B\big(w, r(w)\big),$$

where $r(w) = |z - p(z)|$ *if* $w = p(z)$ *for some* $z \in \partial\Omega \setminus K$ *and* $r(w) = \operatorname{dist}(w, \partial\Omega)$ *if* w *is not in the range of* p.

(v) $\Omega \setminus K$ *is the disjoint union of all the normal segments* N_z:

$$\Omega \setminus K = \bigcup_{z \in \partial\Omega \setminus K} N_z, \quad N_{z_1} \cap N_{z_2} = \emptyset \quad \textit{for } z_1 \neq z_2.$$

Corollary 5.6. *With* μ *and* K *as in the theorem, assume* $\Omega \in Q(\mu, \mathrm{SL}^1)$. *Then* (ii)–(v) *in Proposition* 5.4 *and Theorem* 5.5 *hold.*

The corollary is immediate from the theorem together with Propositions 5.3 and 5.4, so we only need to prove the theorem.

The theorem will be proved by reducing it to the special case that $\operatorname{supp}\mu$ is contained in a straight line (e.g., the real axis), and for that case Theorem 3.1 applies via known properties of conformal maps onto quadrature domains. We state the special case as follows.

Lemma 5.7. *Let μ be a positive measure with compact support on $\mathbf{R}$. Then $\Omega = \Omega(\mu)$ is the unique element in $Q(\mu, \mathrm{SL}^1)$. It is symmetric about the real axis and each component of it is simply connected and satisfies the assumptions* (*with g real analytic*) *and equivalent conditions in Proposition* 2.1.

Proof. All statements except the last one (about the equivalent conditions) are well known, in fact are contained in Proposition 14.7 of [11]. To prove the last statement we may assume that Ω is connected, because each component is in itself a quadrature domain for the part of μ which it contains.

Let $f\colon \mathbf{D}^e \to \Omega$ be a conformal map which preserves the real axis and takes $(\mathbf{D}^e)^+$ onto Ω^+. It follows (essentially) from Proposition 10.19 in [11] (see also [2, p. 158]) that f is of the form

$$f(w) = f(\infty) + \int \frac{d\nu(t)}{t-w}$$

for some positive measure ν on $(-1,1)$. Indeed, f having such a form is equivalent to Ω being a quadrature domain for the smaller test class of analytic functions. The proof of the above formula for f consists of a reflection argument similar to that used in the proof of Theorem 4.1. Now the final statement in Lemma 5.7 follows from Theorem 3.1. □

Finally we show how Theorem 5.5 follows from Lemma 5.7.

Proof. We start by observing that

$$\Omega \setminus K \subset \bigcup_{z\in\partial\Omega\setminus K} N_z \subset \bigcup_{z\in\partial\Omega\setminus K} B\bigl(p(z), |z-p(z)|\bigr).$$

In fact, this is true outside H for any closed half-plane H with $K \subset H$ by the same argument as was used for (2.3) in the proof of Proposition 2.1, hence it is true as stated since K is the intersection of all such half-spaces. The fact that, in Proposition 2.1, the quantities $p(z)$ and N_z are defined with respect to what in the present context is the half-plane H, which contains K, only simplifies for the conclusion. For example, having the foot point on ∂K rather than on ∂H only makes the discs in the right member above larger.

Thus, in order to prove the theorem we only have to show that

$$B\bigl(p(z), |z-p(z)|\bigr) \subset \Omega \tag{5.2}$$

for all $z \in \partial\Omega \setminus K$ and that

$$N_{z_1} \cap N_{z_2} = \emptyset \tag{5.3}$$

for all $z_1, z_2 \in \partial\Omega \setminus K$ with $z_1 \neq z_2$.

It is known (see Theorem 4.1 in [5]) that given any closed half-plane H containing K there exists a positive measure $\nu = \nu_H$ with support on ∂H and producing the same partial balayage outside H as μ does. Precisely,

$$\mathrm{Bal}\,(\nu, m)|_{\mathbf{C}\setminus H} = \mathrm{Bal}\,(\mu, m)|_{\mathbf{C}\setminus H} \quad \text{and} \quad \mathrm{Bal}\,(\nu, m) \leq \mathrm{Bal}\,(\mu, m).$$

In particular,

$$\Omega(\nu) \setminus H = \Omega(\mu) \setminus H \quad \text{and} \quad \Omega(\nu) \subset \Omega(\mu). \tag{5.4}$$

These statements are closely related to what was obtained in the proof of Theorem 4.1 in the context of the obstacle problem. Indeed, assume $H = \{x + iy \in \mathbf{C} : y \leq 0\}$ and set $u = U^\mu - V^\mu$, $u^*(x+iy) = u(x-iy)$, $v = \inf(u, u^*)$. As in the proof of Theorem 4.1 one finds that $v = u$ (i.e., $u \leq u^*$) in $\Omega(\mu) \setminus H$ and that $\lim_{\varepsilon\searrow 0}(\partial u/\partial y)(x + i\varepsilon) \leq 0$ ($x \in \mathbf{R}$). Then the measure $\nu = \nu_H$ on $\partial H = \mathbf{R}$ is given by $\nu = (-\Delta v)|_{\mathbf{R}}$ or, what is the same, $d\nu(x) = -2\lim_{\varepsilon\searrow 0}(\partial u/\partial y)(x+i\varepsilon)\,dx$.

Now given $z \in \partial\Omega \setminus K$ choose a point $c \in N_z$. Then, since $N_z \cap K = \emptyset$ we can find a closed half-plane H such that $K \subset H$ and $c \notin H$. Since $p(z) \in H$ we have $z \notin H$. We shall apply Lemma 5.7 with ∂H identified with the real axis (and H with the lower half-plane) and with μ taken to be the $\nu = \nu_H$ defined above.

In this situation, considering $\Omega \setminus H$ in place of $\Omega \setminus K$, the inward normal segment from z will be just $N_z \setminus H$ and the foot point will be the unique point w in $\big(N_z \cup \{p(z)\}\big) \cap \partial H$. Lemma 5.7 together with (5.4) then gives that

$$N_z \setminus H \subset B(w, |z - w|) \subset \Omega(\nu_H) \subset \Omega.$$

Letting now $c \to p(z)$ (with $c \in N_z$) we have $w \to p(z)$ and it follows that

$$N_z \subset B\big(p(z), |z - p(z)|\big) \subset \Omega,$$

in particular (5.2) holds.

In order to show (5.3) we assume, to derive a contradiction, that there exists a point $c \in N_{z_1} \cap N_{z_2}$ for some $z_1, z_2 \in \partial\Omega \setminus K$ with $z_1 \neq z_2$. Again choose a closed half-plane H with $K \subset H$ and $c \notin H$. Since $p(z_1), p(z_2) \in H$ we have $z_1, z_2 \notin H$. Applying Lemma 5.7 in the same way as above gives $(N_{z_1} \setminus H) \cap (N_{z_2} \setminus H) = \emptyset$, which is the desired contradiction since c is in the intersection. Thus also (5.3) holds, so that Theorem 5.5 is now proven. □

6. Application to Hele-Shaw flow moving boundary problems

Hele-Shaw flow refers to the flow of a viscous incompressible fluid in the narrow gap between two parallel plates (see [6] and [9]). In the two-dimensional picture, averaging over the gap and looking at the plates from above, Hele-Shaw flow turns out to be a potential flow with a harmonic potential function. This makes it interesting within function theory.

We shall consider the simplest model for a Hele-Shaw blob $\Omega_t \subset \mathbf{C}$ with free boundary and which grows in time due to injection of more fluid. In case the additional fluid is injected at one single point, taken to be the origin, the standard mathematical description is the following.

An initial domain Ω_0 containing the origin is given and one seeks its evolution $\{\Omega_t\}$ in time under the rule that

$$\partial\Omega_t \text{ propagates with normal velocity } -\frac{\partial G_{\Omega_t}}{\partial n},$$

where $G_{\Omega_t}(z) = -(1/2\pi)\log|z| +$ harmonic is the Green's function of Ω_t with pole at the origin, and where $\partial/\partial n$ denotes the exterior normal derivative on $\partial\Omega_t$.

In the forward time direction the Hele-Shaw problem is well-posed and admits a unique global weak solution ("variational inequality solution"). This solution is in fact just a special case of partial balayage, namely $\{\Omega_t : 0 < t < \infty\}$ is given by

$$\mathrm{Bal}\,(t\delta + \chi_{\Omega_0} m, m) = \chi_{\Omega_t} m,$$

where δ denotes the Dirac measure at the origin. To be more precise, $\Omega_t = \Omega(t\delta + \chi_{\Omega_0} m) \in Q(t\delta + \chi_{\Omega_0} m, \mathrm{SL}^1)$. See [11] and [5]. This balayage interpretation of Hele-Shaw flow is a strengthened form of the property, discovered by S. Richardson [9], that Hele-Shaw flow preserves the complex moments of the domain.

Considering a more general source configuration than a point mass corresponds to replacing the Dirac measure above by a more general positive measure μ. In general, one may think of Hele-Shaw evolution simply as continuous partial balayage.

From Theorem 5.5 we immediately get

Theorem 6.1. *Let $\Omega_0 \subset \mathbf{C}$ be a bounded initial domain for Hele-Shaw flow and let $\mu \geq 0$ represent an arbitrary source distribution in Ω_0. Let $\{\Omega_t : 0 < t < \infty\}$ be the corresponding Hele-Shaw evolution, namely $\Omega_t = \Omega(t\mu + \chi_{\Omega_0} m)$, and let K denote the closed convex hull of Ω_0.*

Then, for any $t > 0$, Ω_t is a union of discs with centers on $\Omega_t \cap K$. Moreover, $\Omega_t \setminus K$ is the disjoint union of all the open normal segments N_z from points on $\partial\Omega_t \setminus K$ to their first points of intersection with K.

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