Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 28, 2003, 153–167

δ-STABLE FUCHSIAN GROUPS

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Abstract. We call a Fuchsian group, G, δ-stable if $\delta(G') = \dim(\Lambda(G'))$ for every quasi-Fuchsian deformation G' of G. It is well known that every finitely generated Fuchsian group has this property. We give examples of infinitely generated Fuchsian groups for which it holds and others for which it fails.

1. Introduction

Associated to a Kleinian group G are two numbers, the Hausdorff dimension of its limit set, $\dim(\Lambda)$, and the critical exponent of its Poincaré series, δ and it is natural to ask when they are equal (all definitions will be given in Section 2). The question has a simple geometric interpretation. Points of the limit set Λ naturally correspond to geodesic rays with a fixed base point $z_0 \in M = \mathbf{B}/G$. The limit set Λ can be written as the disjoint union of two special subsets; the conical limit set, Λ_c, which corresponds to geodesics which return to some compact set infinitely often (the recurrent geodesics) and the escaping limit set, Λ_e, which corresponds to geodesics which escape to infinity. Thus we always have $\dim(\Lambda) = \max\bigl(\dim(\Lambda_c), \dim(\Lambda_e)\bigr)$. It is a theorem from [16] that $\delta = \dim(\Lambda_c)$ for any non-elementary group and hence $\delta \le \dim(\Lambda)$ for all such groups with equality if and only if $\dim(\Lambda_e) \le \dim(\Lambda_c)$. Moreover, $\delta = \dim(\Lambda_b)$, where $\Lambda_b \subset \Lambda_c$ is the part of the limit set corresponding to geodesics that remain bounded in M [16].

If a Kleinian group is geometrically finite then Λ_e is at most countable (see [6], [8]) and so $\dim(\Lambda_e) = 0 \le \dim(\Lambda_c)$ clearly holds and hence $\delta = \dim(\Lambda)$. This equality is conjectured to hold for all finitely generated Kleinian groups; the best result so far says it holds if we also assume Λ has zero area (see [16]). For infinitely generated groups, the equality can fail, but it is still interesting to find conditions under which it holds.

We will restrict our attention to quasi-Fuchsian groups, i.e., groups G' which are conjugate to a Fuchsian group G of the first kind via a quasiconformal map of the plane. We shall say that a Fuchsian group G is δ-*stable* if $\delta(G') = \dim\bigl(\Lambda(G')\bigr)$ for every quasi-Fuchsian deformation G' of G. Clearly every finitely generated Fuchsian group has this property (because the deformations are geometrically

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2000 Mathematics Subject Classification: Primary 30F35.

The author is partially supported by NSF Grant DMS 0103626.

finite), but it is it not obvious whether or not any infinitely generated group does. The purpose of this note is to show both possibilities occur.

From our remarks above, we see that G is δ-stable if and only if

$$\dim\bigl(\Lambda_e(G')\bigr) \le \dim\bigl(\Lambda_c(G')\bigr) = \delta(G')$$

for every quasiconformal deformation G' of G. In other words, we want the part of the limit set corresponding to escaping geodesics to be less distorted than the part corresponding to reccurent geodesics. One way to ensure this is for the dilatation to be compactly supported (modulo G); in [15] we showed that if the deformation is compactly supported then $\dim(\Lambda_e) = 1$ (indeed, has sigma finite length) and hence $\dim(\Lambda) = \delta$. Thus every Fuchsian group with $\delta = 1$ is δ-stable with respect to compactly supported deformations (by Lemma 2.4, δ cannot decrease under a deformation in this case). A criterion which applies to certain non-compactly supported deformations is given in Lemma 5.1 of [10].

There is another way of ensuring that Λ_e is not distorted too much. It is well known that the thrice punctured sphere has a trivial deformation space, i.e., any quasiconformal conjugation gives another circle as the limit set. In [12] I give a quantified version of this fact, showing that if part of a Riemann surface "looks like" a thrice punctured sphere (i.e., consists of a union of Y-pieces, all with short boundary curves) then any deformation supported there cannot raise the dimension of the limit set very much. We will use this idea to construct δ-stable groups. To be more precise, suppose $R = \mathbf{D}/G$ is a union of Y-pieces and for any $\varepsilon > 0$, only a finite number are not ε-bounded (see Section 2 for definitions). Following the terminology in [12], we shall say such a R "approximates a thrice punctured sphere near infinity".

Theorem 1.1 *Suppose $R = \mathbf{D}/G$ approximates a thrice punctured sphere at infinity. Then G is δ-stable.*

In fact we will prove the stronger statement that for any quasi-Fuchsian deformation G' of G, $\dim\bigl(\Lambda_e(G')\bigr) = 1$. Then δ-stability follows because it is known that if $\delta(G) = 1$ then $\delta(G') \ge 1$ for any quasi-Fuchsian deformation (see Lemma 2.4). It is easy to see these examples may be taken to be either divergence type or convergence type.

In order to build an example which is not δ-stable, we will take two bordered Riemann surfaces, R_1 with $\delta = 1$ and R_2 with $\delta < 1$ and join them along a boundary component. The deformation is chosen so the dilatation has support in R_2 which is at least distance d from R_1 and we show that $\dim(\Lambda_e) > 1 + \varepsilon > 1$ independent of d. Bounded geodesics which stay in R_2 correspond to a piece of the limit set with dimension < 1, which still has dimension < 1 if the deformation has small L^∞ norm. Geodesics which stay in R_1 are far from the support of the deformation and hence correspond to a part of the limit set whose dimension is hardly changed if d is large enough. Combining these ideas gives:

Theorem 1.2. *There is a Fuchsian group G with $\delta = 1$ which is not δ-stable.*

As noted above, we require $\delta = 1$ to avoid trivial examples where $\delta(G) \neq \dim\bigl(\Lambda(G)\bigr)$. The example in Theorem 1.2 is a convergence type group and I do not know of any divergence type examples. Is every divergence type group δ-stable?

I thank the referee for carefully reading the manuscript and numerous comments and suggestions which improved it.

The rest of the paper is organized as follows. In Section 2 we recall some basic definitions and results. In Section 3 we prove that a quasiconformal map with Beltrami coefficient μ satisfies a pointwise Hölder type estimate with exponent close to 1 far from the support of μ. In Section 4 we prove Theorem 1.1. In Section 5 we prove Theorem 1.2.

2. Definitions and background

If A and B are quantities that depend on some parameter we write $A \lesssim B$ if the ratio B/A is bounded uniformly independent of the parameter. Similarly for $\gtrsim$. We write $A \simeq B$ if both $A \lesssim B$ and $A \gtrsim B$ hold and say A and B are comparable.

The terms "dist" and "diam" will always refer to Euclidean distances in this paper, except when we explicitly state otherwise (e.g., for a set on a Riemann surface, diam(E) would refer to hyperbolic diameter).

Suppose φ is an increasing continuous function from $[0,\infty)$ to itself such that $\varphi(0) = 0$. We define the *Hausdorff content* of a set $E \subset \mathbf{R}^2$ as

$$\mathscr{H}^{\varphi}_{\infty} = \inf\Bigl\{\sum \varphi(r_j) : E \subset \bigcup_j D(x_j, r_j)\Bigr\}.$$

If $\varphi(t) = t^{\alpha}$ we denote $\mathscr{H}^{\varphi}$ by $\mathscr{H}^{\alpha}$. The *Hausdorff dimension* of E is

$$\dim(E) = \inf\bigl\{\alpha : \mathscr{H}^{\alpha}_{\infty}(E) = 0\bigr\},$$

and the *Hausdorff measure* of E is

$$\mathscr{H}^{\varphi}(E) = \lim_{\delta\to 0}\Bigl[\inf\Bigl\{\sum \varphi(r_j) : E \subset \bigcup_j D(x_j, r_j), r_j \leq \delta\Bigr\}\Bigr].$$

A discrete group G of isometries of the hyperbolic metric on $\mathbf{B}^d$ is called a *Kleinian group* if $d = 3$ and *Fuchsian* if $d = 2$. A Kleinian group can also be considered as a group of linear fractional transformations on S^2. G is called *elementary* if it contains a finite index Abelian subgroup. In this paper we are only concerned with non-elementary groups. For a non-elementary G, the accumulation of any orbit in $\mathbf{B}^d \cup S^{d-1}$ is a closed set $\Lambda \subset S^{d-1}$ which is independent of the particular orbit. This is the *limit set*. The *conical limit set* Λ_c is the set of points

$x \in \Lambda$ for which there is a sequence of orbit points of 0 converging to x within a non-tangential cone in $\mathbf{B}^d$. It is easy to see that x is a conical limit point if and only if a geodesic ray in $\mathbf{B}$ ending at x projects to a geodesic in $M = \mathbf{B}^d/G$ which returns to some compact set of M infinitely often. The set $\Lambda_b \subset \Lambda_c$ denotes the subset corresponding to geodesic rays that remain bounded. We define $\Lambda_e = \Lambda \backslash \Lambda_c$ as the "escaping" part of the limit set. Points of Λ_e correspond to geodesic rays which eventually leave every compact set.

A Fuchsian group G is called *first kind* if $\Lambda = \mathbf{T}$ and otherwise it is *second kind*. It is called *cocompact* if $R = \mathbf{D}/G$ is compact and *cofinite* if R has finite hyperbolic area. A Fuchsian group is called *divergence type* if

$$\sum_{g \in G} (1 - |g(0)|) = \infty,$$

and otherwise it is called *convergence type*. The latter occurs if and only if $R = \mathbf{D}/G$ has a finite Green's function, which is given by the series

$$G_R(z,w) = \sum_{g \in G} G_{\mathbf{D}}\bigl(x, g(y)\bigr) = \sum_{g \in G} \log\left|\frac{x - g(y)}{1 - \bar{x} g(y)}\right|,$$

where x and y project to z and w respectively.

The *Poincaré exponent* (or *critical exponent*) of the group is

$$\delta(G) = \inf\left\{ s : \sum_G \exp\bigl(-s\varrho\bigl(0, g(0)\bigr)\bigr) < \infty \right\},$$

where ϱ is the hyperbolic metric in $\mathbf{B}^3$. A result from [16] says that

Theorem 2.1. *If the Fuchsian group G is a non-elementary Kleinian group then $\delta(G) = \dim\bigl(\Lambda_c(G)\bigr)$.*

For Fuchsian groups and geometrically finite Kleinian groups this was previously known, e.g., [31] and [33]. The proof given in [16] also shows

Corollary 2.2. *If G is any non-elementary, discrete Möbius group, $x \in M = \mathbf{D}/G$ and $\varepsilon > 0$ then there is a $R = R(\varepsilon, x) < \infty$ such that the set of directions (i.e., unit tangents at x) which correspond to geodesic rays starting at x which never leave the ball of radius R around x has dimension $\geq \delta(G) - \varepsilon = \dim\bigl(\Lambda_c(G)\bigr) - \varepsilon$. In particular $\dim(\Lambda_b) = \dim(\Lambda_c) = \delta(G)$.*

For Fuchsian groups this is due to Fernández and Melián in [25].

If $E \subset \mathbf{R}^d$ is compact, let $N(E, \varepsilon)$ be the minimal number of ε-balls needed to cover E. The *upper Minkowski dimension* of E is

$$\dim_M(E) = \limsup_{\varepsilon \to 0} \frac{\log N(E, \varepsilon)}{-\log \varepsilon}.$$

This is clearly an upper bound for the Hausdorff dimension of E. If $E \subset \mathbf{T}$ is compact, let $\{I_n\}$ be an enumeration of the complementary intervals, i.e., the components of $\mathbf{T} \setminus E$. The Besicovitch–Taylor index is defined as

$$\inf\left\{ s : \sum_n \operatorname{diam}(I_n)^s < \infty \right\},$$

and is well known to equal the upper Minkowski dimension of E if E has zero Lebesgue measure (e.g., [9], [35]). If G is a finitely generated Fuchsian group, it is known that the upper Minkowski and Hausdorff dimensions of the limit set agree and both agree with δ. These remarks give

Lemma 2.3. *If Λ is the limit set of a finitely generated Fuchsian group G, δ is the critical exponent and $\{I_n\}$ is an enumeration of the components of $\mathbf{T} \setminus \Lambda$ then*

$$\sum_n \operatorname{diam}(I_n)^{\delta+\varepsilon} \le C(\varepsilon, G) < \infty, \tag{1}$$

for every $\varepsilon > 0$.

The critical exponent δ also has a close relationship to λ_0, the base eigenvalue of the Laplacian on the quotient manifold M which is defined as

$$\begin{aligned} \lambda_0 &= \sup\{\lambda : \exists f \in C^\infty(M) \text{ such that } \Delta f = -\lambda f \text{ and } f > 0\} \\ &= \inf_{f \in C_0^\infty(M)} \frac{\int_M |\nabla f|^2}{|f|^2}. \end{aligned}$$

If G acts on hyperbolic n-space, then the Elstrodt–Patterson–Sullivan formula says $\lambda_0 = \delta(n-1-\delta)$ if $\delta \ge \frac{1}{2}(n-1)$ and $\lambda_0 = \frac{1}{4}(n-1)^2$ if $\delta \le \frac{1}{2}(n-1)$. See Theorem 2.17 of [34].

The base eigenvalue, in turn, can be bounded by the geometry of M using Cheeger's constant $h(M)$. This is defined as the infimum over all compact n-submanifolds N of M of $\operatorname{vol}_{n-1}\big(\partial(N)\big)/\operatorname{vol}_n(N)$. Cheeger [23] proved that $\lambda_0(M) \ge \frac{1}{4}h(M)^2$ and Buser [21] showed that $\lambda_0 \le Ch(N)$ for manifolds of bounded negative curvature (C depends on the dimension and a lower bound for the curvature). See [22] for a different proof of Buser's result. Combining these comments, we see that a Fuchsian group G has $\lambda_0 = 0$ (and hence $\delta = 1$) if the quotient surface R contains subregions S_n with $l(\partial S_n)/\operatorname{area}(S_n) \to 0$.

A homeomorphism of the plane is called K-quasiconformal if

$$\limsup_{r \to 0} \frac{\sup_{|y-x|=r} |f(y) - f(x)|}{\inf_{|y-x|=r} |f(y) - f(x)|} \le K.$$

Such maps are known (see [1]) to be differentiable almost everywhere and $\mu = f_{\bar z}/f_z$ is called the Beltrami coefficient of f and is in L^∞ with norm

$$k = (K-1)/(K+1).$$

It is also true (see [1] again) that any quasiconformal map f is Hölder continuous with an exponent $\alpha > 0$ depending only on K. In particular, we will use the fact that if f is K-quasiconformal and γ_1 and γ_2 are hyperbolic geodesics in the unit disk such that γ_1 separates 0 from γ_2 then

$$\frac{\operatorname{diam}\bigl(f(\gamma_2)\bigr)}{\operatorname{diam}\bigl(f(\gamma_1)\bigr)} \le C\left(\frac{\operatorname{diam}(\gamma_1)}{\operatorname{diam}(\gamma_2)}\right)^{\alpha}, \tag{2}$$

for some α depending only on K and C depending on $\operatorname{diam}\bigl(f(\gamma_2)\bigr)/\operatorname{diam}\bigl(f(\Omega)\bigr)$, where $\Omega \subset \mathbf{D}$ is a region separated from 0 by γ_1.

If G is a Fuchsian group and μ is a bounded measurable function on the unit disk, $\mathbf{D}$, which satisfies $\|\mu\|_\infty < 1$ and $\mu\bigl(g(z)\bigr) = \mu(z)g'(z)/\overline{g'(z)}$, for every $g \in G$, then we say μ is a G-invariant Beltrami coefficient (or complex dilatation). There is a corresponding quasiconformal mapping f_μ which is analytic outside the disk and which conjugates G to a quasi-Fuchsian group G_μ.

A conformal mapping $f\colon \mathbf{D} \to \Omega$ is called a deformation of the Fuchsian group G if for every $g \in G$, $f \circ g \circ f^{-1}$ is Möbius transformation restricted to Ω. It is called a quasiconformal deformation if f has a quasiconformal extension to the whole plane.

Bowen's theorem [18] says that if G is a cocompact Fuchsian group then for any quasi-Fuchsian deformation G' of G either $\Lambda(G')$ is a circle or $\dim\bigl(\Lambda(G')\bigr) > 1$. This was extended to all divergence type groups in [11] and is false for all convergence type groups (see [3], [4], [5]). See [17], [19] and [32] for alternate proofs of Bowen's theorem.

It is easy to see that if G' is a deformation of G then $\Lambda_c(G') = f\bigl(\Lambda_c(G)\bigr)$ and $\Lambda_e(G') = f\bigl(\Lambda_e(G)\bigr)$. A theorem of Makarov [29] says that if $E \subset \mathbf{T}$ has $\dim(E) = 1$ then $\dim\bigl(f(E)\bigr) \ge 1$ for any conformal map of the disk. Applying this to $E = \Lambda_c$ we see that

Lemma 2.4. *If G is a Fuchsian group with $\delta(G) = 1$ then $\delta(G') \ge 1$ for any deformation of G.*

An alternate proof is described in [13]. Of course, the same result holds for the escaping limit set as well. Indeed, by a result of Fernández and Melián in [26], $\dim(\Lambda_e) = 1$ for any infinitely generated Fuchsian group of the first kind. Thus

Lemma 2.5. *If G is an infinitely generated Fuchsian group of the first kind then $\dim\bigl(\Lambda_e(G')\bigr) \ge 1$ for any deformation of G.*

A theorem of Astala [2] says that if f is a K-quasiconformal map then $\dim\big(f(E)\big) \geq 2\dim(E)/\big(2K+(1-K)\dim(E)\big)$. This is a sharper version of a result of Gehring and Väisälä in [27].

A generalized Y-piece in a Riemann surface R is a region bounded by three simple closed geodesics (or punctures) which is homeomorphic to a 2-sphere minus three disks (or points). If all three boundary components have length $\leq L$ we say the Y-piece is L-bounded (punctures count as zero length). We say that R has a L-bounded Y-piece decomposition if it can be written as a union of L-bounded Y-pieces with disjoint interiors. Let Γ be the union of all simple closed geodesics which occur as boundary arcs in the Y-piece decomposition and let $\Gamma^\varepsilon \subset \Gamma$ denote the union of all those with lengths $\geq \varepsilon$. By the collar lemma (e.g., [28], [30]) there is a $C > 0$ (depending only on L) so that the hyperbolic C-neighborhoods of elements of Γ are pairwise disjoint.

The following is the result from [7] which we will use to prove Theorem 1.1.

Theorem 2.6 *Given $L, K < \infty$ and $\eta > 0$ there are $\varepsilon > 0$ and $r < \infty$ so that the following holds. Suppose $R = \mathbf{D}/G$ is a Riemann surface which has a decomposition into L-bounded Y-pieces. Suppose $F\colon R \to S$ is a K-quasiconformal map with Beltrami coefficient μ and $\operatorname{dist}\big(\operatorname{supp}(\mu), \Gamma^\varepsilon\big) > r$ (here "dist" denotes hyperbolic distance on R). Then the corresponding quasi-Fuchsian deformation of G has limit set of dimension $\leq 1+\eta$.*

3. A Hölder type estimate

It is well known that quasiconformal maps satisfy a Hölder condition with exponent that depends only on the quasiconformal constant (e.g. p. 47 of [1]). In this section we wish to prove that they satisfy a pointwise Hölder type estimate with exponent close to 1 if the support of the Beltrami coefficient μ is sufficiently "thin" near the point.

Theorem 3.1. *Suppose Ω is a K-quasidisk and μ is Beltrami coefficient supported on Ω (hence zero outside Ω) with $\|\mu\|_\infty \leq k < 1$. Let f_μ be the corresponding quasiconformal map of the plane fixing 0, 1 and ∞. Given $\varepsilon > 0$ there is a $r = r(K, k, \varepsilon) < \infty$ and a $C = C(K, k, \varepsilon) < \infty$ so that the following holds. Suppose $x \in \partial\Omega$, $z \in \Omega$, $s = \operatorname{dist}(z, \partial\Omega)$ and $\gamma \subset \Omega$ is a hyperbolic geodesic connecting z to x. Suppose the hyperbolic distance (in Ω) from γ to the support of μ is at least r. Then for any $0 < t < s$,*

$$\frac{1}{C}\left(\frac{t}{s}\right)^{1+\varepsilon} \leq \frac{\operatorname{diam}\big(f\big(B(x,t)\big)\big)}{\operatorname{diam}\big(f\big(B(x,s)\big)\big)} \leq C\left(\frac{t}{s}\right)^{1-\varepsilon}.$$

Before giving the proof, we will recall a few facts which we will need. If $0 < a < b < \infty$ we let $R(a,b) = \{x \in \mathbf{R}^n : a < |x| < b\}$. Let L_f denote the inner dilatation

$$L_f(x) = \frac{J_f(x)}{l\big(f'(x)\big)^n},$$

where $l\big(f'(x)\big) = \inf_{|h|=1} |f'(x)h|$. For $n = 2$ (the only case we will use here) this agrees with the usual dilatation of f. Let

$$M_f(r) = \max_{|x|=r} |f(x)|, \qquad m_f(r) = \min_{|x|=r} |f(x)|.$$

Also let ω_{n-1} be the $n-1$ measure of the unit sphere in $\mathbf{R}^n$. The following combines Corollaries 2.21 and 2.34 of [14].

Lemma 3.2. *Suppose $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is quasiconformal, $f(0) = 0$ and $n \geq 2$. Then*

$$\log \frac{b}{a} - \log \frac{M_f(b)}{m_f(a)} \leq \frac{1}{\omega_{n-1}} \int_{R(a,b)} \frac{L_f(x) - 1}{|x|^n}\, dx$$

and

$$\log \frac{m_f(b)}{M_f(a)} - \log \frac{b}{a} \leq \frac{1}{\omega_{n-1}} \int_{R(a,b)} \frac{L_f(x) - 1}{|x|^n}\, dx.$$

The following is an easy consequence of this.

Lemma 3.3. *Given $K < \infty$ and $\varepsilon > 0$ there is a $\delta > 0$ and $C < \infty$ so that the following holds. Suppose $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ is a K-quasiconformal map with Beltrami coefficient supported on a set E. Suppose that for $0 < r \leq 1$, E satisfies $\operatorname{area}\big(E \cap B(x,r)\big) \leq \delta r^2$. Then*

$$\frac{1}{C} r^{1+\varepsilon} \leq \frac{\operatorname{diam}\big(f_\mu\big(B(x,r)\big)\big)}{\operatorname{diam}\big(f_\mu\big(B(x,1)\big)\big)} \leq C r^{1-\varepsilon}.$$

Proof. Let N be the smallest integer such that $2^{-N} \leq r$. Then

$$\begin{aligned}
\int_{R(r,1)} \frac{L_f(x) - 1}{|x|^2}\, dx &\leq \sum_{n=0}^{N} \int_{E \cap R(2^{-n-1}, 2^{-n})} (K-1) 2^{2n+2}\, dx \\
&\leq \sum_{n=0}^{N} (\delta 2^{-2n})(K-1) 2^{2n+2} \\
&\leq 4\delta(K-1) \sum_{n=0}^{N} 1 \leq 4\delta(K-1)\left(\log_2 \frac{1}{r} + 1\right).
\end{aligned}$$

Thus by Lemma 3.2,

$$\frac{M_f(1)}{m_f(r)} \geq \left(\frac{1}{r}\right)^{1-O(\delta)}, \qquad \frac{m_f(1)}{M_f(r)} \leq \left(\frac{1}{r}\right)^{1+O(\delta)}.$$

For quasiconformal maps $M_f(r) \simeq m_f(r) \simeq \operatorname{diam}\big(f\big(B(0,r)\big)\big)$, so this proves the lemma. ◻

We also need a few easy facts about quasidisks. The proofs are included for completeness, but the results are well known.

Lemma 3.4. *Suppose Ω is a K-quasidisk and $z_0 \in B(x,r)\cap\Omega$ is a point which satisfies* $\mathrm{dist}(z_0,\partial\Omega) \ge Mr$. *Then there is a $C<\infty$ and an $a>0$ so that*

$$\{w\in\Omega\cap B(x,r) : \varrho(w,z_0)\ge s\} \subset \{w\in\Omega\cap B(x,r) : \mathrm{dist}(w,\partial\Omega)\le Cr\exp(-as)\}.$$

Proof. By rescaling we may assume $r=1$. Any point w_0 in $B(x,1)\cap\Omega$ which is distance d from the boundary can be joined to z_0 be a path of length at most C which satisfies $\mathrm{dist}(z,\partial\Omega)\ge c|z-w_0|$. The quasi-hyperbolic length of such a path is at most $C\log(1/d)$ and hence the same is true of its hyperbolic length. Thus $\varrho(z_0,w_0)\le C\log(1/d)$ or $d\le \exp\bigl(-\varrho(z_0,w_0)/C\bigr)$, as desired. □

Lemma 3.5. *If Γ is a K-quasiarc then there is a $\alpha>0$ and $C<\infty$ (both depending only on K) so that the ε-neighborhood of Γ has area $\le C\varepsilon^\alpha \mathrm{diam}(\Gamma)^2$.*

Proof. This follows from the fact that quasicircles are porous, i.e., there is an N (depending on K), such that if we divide a square Q into N^2 disjoint subsquares, then Γ misses at least one of them (this follows from the Ahlfors 3-point condition, e.g., [1]). After dropping down k times, the squares of size $\varepsilon_k = N^{-k}$ γ hits have area at most $(1-N^{-2})^k=\varepsilon_k^\alpha$ where $\alpha=-\log(1-N^{-2})/\log N$. Tripling each square covers an ε-neighborhood of Γ and only increases the area by a factor of 9. □

The proof of Theorem 3.1 is immediate from the previous lemmas. We will now deduce some consequences that we will need later. In what follows Ω will be a bounded K-quasicircle with a fixed base point z_0 which satisfies $\mathrm{dist}(z_0,\partial\Omega)\simeq \mathrm{diam}(\Omega)$. We will let $W\subset\Omega$ be a subdomain bounded by hyperbolic geodesics. Moreover we assume there is an $a>0$ so that any two geodesics in ∂W are at least hyperbolic distance a apart. If W does not contain z_0 then the component of ∂W which separates z_0 from W will be called the "top edge" of W and the other components will be called "bottom edges". The following are easy consequences of the results above.

Corollary 3.6. *With notation as above, let γ denote the top edge of W and let $\{\gamma_k\}$ be an enumeration of the bottom edges. Let $E=\partial W\cap\partial\Omega$. Suppose μ is a Beltrami coefficient supported on Ω with $\|\mu\|_\infty\le k<1$. Then given $\eta>0$ there is an $r=r(K,k,\eta)$ so that if the support of μ is at least hyperbolic distance r from W then*

$$\sum_k \left(\frac{\mathrm{diam}\bigl(f(\gamma_k)\bigr)}{\mathrm{diam}\bigl(f(\gamma)\bigr)}\right)^{1+\eta} \le C\sum_k \frac{\mathrm{diam}(\gamma_k)}{\mathrm{diam}(\gamma)}, \tag{3}$$

and

$$\frac{\dim(E)}{1+\eta}\le \dim\bigl(f(E)\bigr)\le(1+\eta)\dim(E). \tag{4}$$

Corollary 3.7. *Suppose μ and ν are two Beltrami coefficients on $\mathbf{D}$ with disjoint supports and $E \subset \mathbf{T}$ is such that for every $x \in E$,*

$$\liminf_{r \to 1} \varrho\big(rx, \operatorname{supp}(\nu)\big) = \infty.$$

Then $\dim\big(f_\mu(E)\big) = \dim\big(f_{\mu+\nu}(E)\big)$.

4. Proof of Theorem 1.1

Suppose $R = \mathbf{D}/G$ is a Riemann surface which approximates the thrice punctured sphere near infinity and suppose f is a K-quasiconformal deformation of G to a quasi-Fuchsian group G'. By Lemma 2.5, $\dim\big(\Lambda_e(G')\big) = \dim\big(f\big(\Lambda_e(G)\big)\big) \geq 1$, so we only have to prove the opposite inequality.

Fix some $\eta > 0$ and use Theorem 2.6 to choose ε and r. Let μ be the Beltrami coefficient of f and write $\mu = \mu_1 + \mu_2$ where each μ_i is G invariant, they have disjoint supports, μ_2 is supported on a finite union of Y-pieces and $\varrho\big(\operatorname{supp}(\mu_1), \Gamma^\varepsilon\big) > r$. Let f_1 be the deformation of G to a quasi-Fuchsian group G_1 corresponding to μ_1. By Theorem 2.6, $\dim\big(f_1(\mathbf{T})\big) \leq 1 + \eta$.

Let E denote $\Lambda_e(G)$, minus the (at most countably many) parabolic fixed points of G. Then each geodesic corresponding to a point of E moves arbitrarily far from $\operatorname{supp}(\mu_2)$. Thus by Corollary 3.7,

$$\dim\big(\Lambda_e(G)\big) = \dim\big(f(E)\big) = \dim\big(f_{\mu_1}(E)\big) \leq 1 + \eta.$$

Since this holds for every $\eta > 0$, we deduce $\dim\big(\Lambda_e(G')\big) = 1$, as desired.

5. Proof of Theorem 1.2

Let Y_0 be a Y-piece that has three equal length boundary components. Construct an X-piece, X_0, by identifying two copies of Y_0, which we will denote Y_1 and Y_2, along one boundary component of each. We will consider several Riemann surfaces which are unions of copies of X_0 with various boundary identifications. The first is the compact, genus two Riemann surface R_0 we obtain by identifying the two remaining boundary components of Y_1 and the two remaining boundary components of Y_2 (there are many ways to do this, but any choice will be sufficient for our purposes). Let G_0 be a Fuchsian group such that $R_0 = \mathbf{D}/G_0$. We will think of R_0 as being labeled by a multi-graph Γ_0 with one vertex and two edges. See Figure 1. Given any multigraph Γ which covers Γ_0 we can define an associated Riemann surface $R = \mathbf{D}/G$ which covers R_0 and hence G is a subgroup of G_0.

One such covering graph is Γ_2, the infinite regular, degree four tree. It is easy to check that Cheeger's constant for the corresponding surface $R_2 = \mathbf{D}/G_2$ is positive and hence the corresponding critical exponent satisfies $\delta(G_2) < 1$. See

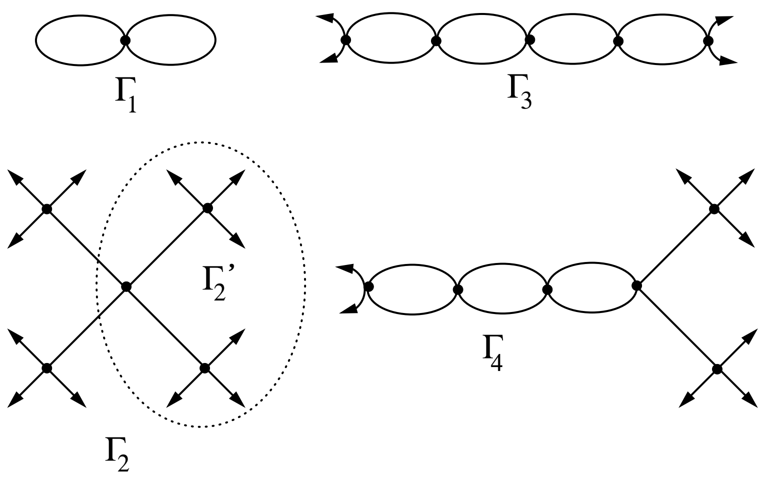

Figure 1. The graphs Γ_1, Γ_2, Γ_2' and Γ_3.

also [20] and [24]. For future reference we will denote this number as $\delta_2 = \delta(G_2)$. Since R_2 has a finite upper bound for its injectivity radius, the limit set of G_2 is the whole circle and hence G_2 is a Fuchsian group for which $\delta < \dim(\Lambda)$.

The example in Theorem 1.2 is obtained by modifying R_2 in order to make $\delta = 1$. Choose a vertex $z_0 \in \Gamma_2$ to be the root and let Γ_2' be the component containing z_0 when two of the four edges adjacent of z_0 is removed (thus Γ_2' is a union of two of the four "branches" which meet at z_0). Let Γ_3 be the multigraph with vertex set $\mathbf{N} = \{1, 2, 3, \dots\}$ and such that vertex n is connected to $n+1$ by exactly two edges. Define Γ_1 to be the union of Γ_2' and Γ_3 with z_0 and $\{1\}$ joined by two edges.

Clearly Γ_1 covers Γ_0 and is covered by Γ_2. Thus the associated Riemann surface R_1 covers R_0 and is covered by R_2 and the corresponding Fuchsian groups satisfy $G_2 \subset G_1 \subset G_0$. We claim that G_1 has the properties claimed in Theorem 1.2.

To prove this, we need to show $\delta_1 = \delta(G_1) = 1$ and to construct a G_1 invariant Beltrami coefficient μ so that the corresponding quasiconformal deformation f satisfies $\dim\bigl(f\bigl(\Lambda_c(G_1)\bigr)\bigr) < \dim\bigl(f\bigl(\Lambda_e(G_1)\bigr)\bigr)$.

The first part is easy. Considering the part of R_2 corresponding to $\Gamma_3 \subset \Gamma_1$, one easily shows that the the Cheeger constant for R_2 is zero. Thus $\delta_1 = 1$ as desired by the Elstrodt–Patterson–Sullivan formula.

To prove the second part, we will actually construct a sequence of G_1-invariant coefficients $\{\mu_n\}$ and show that the corresponding deformations $\{f_n\}$ satisfy

$$\dim\bigl(f_n(\Lambda_e)\bigr) \geq 1 + \varepsilon \tag{5}$$

for some ε independent of n and

$$\dim\bigl(f_n(\Lambda_c)\bigr) \to 1 \tag{6}$$

as $n \to \infty$. Together, these clearly imply the desired result if n is large enough.

We now define μ_n. Choose η_1 so that $\delta_2 < \eta_1 < 1$. By taking $k > 0$ small enough, we may assume that any map with Beltrami coefficient bounded by k is Hölder of order η_1. Taking k smaller, if necessary, we may also assume $k \le 1 - \frac{1}{2}\delta_2$.

Since R_0 is a compact surface, Bowen's theorem says every deformation of G_0 gives a quasi-Fuchsian group whose limit set is either a circle or has critical exponent $\delta > 1$. Thus we can choose a non-trivial deformation G_0' of G_0 so that $\|\mu\|_\infty \le k$ and an $\varepsilon > 0$ so that $\delta(G_0') > 1 + \varepsilon$. Let f denote the corresponding deformation of G_0. By Astala's theorem and the fact that $k < 1 - \frac{1}{2}\delta_2$,

$$\dim(E) \le \delta_2 \quad \Rightarrow \quad \dim\bigl(f(E)\bigr) < 1. \tag{7}$$

The G_0-invariant coefficient μ is also G_2-invariant (since G_2 is a subgroup of G_0). Let G_2' be the corresponding quasi-Fuchsian deformation of G_2. Note that

$$\delta(G_2') = \dim\bigl(\Lambda_c(G_2')\bigr) = \dim\bigl(f(\Lambda_c(G_2)\bigr) < 1$$

by (7). Thus

$$\dim\bigl(\Lambda_e(G_2')\bigr) = \dim\bigl(\Lambda(G_2')\bigr) = \dim\bigl(f(\mathbf{T})\bigr) = 1 + \varepsilon.$$

Moreover, $\Lambda_e(G_2)$ breaks into four pieces depending on which branch (i.e., component of $R_2 \setminus X_0$ the corresponding geodesic ray eventually stays in). We claim that the f-image of each of the four sets has dimension equal to $\dim\bigl(f(\Lambda_e)\bigr)$. Given one such piece E, there is clearly an element $g \in G_0$ so that $\bigcup_n g^n(E)$ is all of $\Lambda_e(G_2)$ except for one point (the attracting fixed point of g). Since g is conjugated to a Möbius transformation by f (since it is a deformation of G_0), this says that $f(\Lambda_e)$ is the union of one point and a countable number of Möbius images of $f(E)$. Thus $\dim\bigl(f(E)\bigr) = \dim\bigl(f(\Lambda_e)\bigr)$, as desired.

The coefficient μ is also G_1 invariant (since $G_1 \subset G_0$) and the sequence $\{\mu_n\}$ will be defined by restricting μ to certain subregions of the disk.

Label the vertices of Γ_2' by their distance to the root z_0. This gives a labeling of the corresponding X-pieces in R_2 and we will let $S_n^+ \subset R_2$ be the union of all X-pieces with labels $\ge n$. Let $\Omega_n^+ \subset \mathbf{D}$ be the preimage of S_n under the quotient map. Similarly, let $S_n^- = R_2 \setminus S_n^+$ and let Ω_n^- be the lift of S_n^-. Let $\Gamma_n = \partial\Omega_n \cap \mathbf{D}$. Note that Γ is a union of infinite geodesics (each a lift of a boundary geodesic of an X-piece) and that any two components are a uniform hyperbolic distance apart. Note for future use that the hyperbolic distance from S_1^- to S_n^+ is $\ge cn$ for some fixed $c > 0$.

Let μ_n be the restriction of the Beltrami coefficient μ to Ω_{2n}^+ and let f_n be the corresponding quasiconformal map. We claim that (5) and (6) hold for these maps.

First we prove (5). Consider a component W of Ω_0^+ and let $E = \Lambda_e(G_1) \cap \partial W$. Then f_n can be written as a composition of f with a quasiconformal map whose dilatation is supported on $f(\mathbf{D} \setminus \Omega_{2n})$. Applying Corollary 3.7, we see that $\dim\bigl(f_n(E)\bigr) = \dim\bigl(f(E)\bigr)$. Since $\dim\bigl(f(E)\bigr) = \dim\bigl(\Lambda_e(G_2')\bigr) = 1 + \varepsilon$ and $f_n(E) \subset \Lambda_e(G_1')$ we clearly have $\dim\bigl(\Lambda_e(G_1')\bigr) \geq 1 + \varepsilon$, as desired.

Next we prove (6). For any $m < \infty$, let $\Lambda_b^m \subset \Lambda(G_1)$ denote the subset corresponding to geodesic rays which never enter Ω_m^+. Then $\Lambda_b(G_1) \subset \bigcup_{m=1}^\infty \Lambda_b^m$ and so $\delta(G_1) \leq \sup_m \dim(\Lambda_b^m)$ by Theorem 2.1. Similarly, $\delta(G_1') = \sup_m \dim\bigl(f_n(\Lambda_b^m)\bigr)$. Thus it suffices to show that given $\eta > 0$ there is a n_0 (independent of m) so that $\dim\bigl(f_n(\Lambda_b^m)\bigr) \leq 1 + \eta$ for all $n \geq n_0$.

Suppose $x \in \Lambda_b^m$ is a point corresponding to a geodesic γ. Then one of the following must be hold for γ:

(1) γ eventually never leaves S_1^+,

(2) γ eventually never leaves S_n^- or

(3) γ alternately leaves S_1^+ and S_n^- infinitely often.

Let E_1, E_2 and E_3 denote the subsets of Λ_b^m which correspond to each of these possibilities. Note that $E_1 \subset \partial\Omega_{n,m} = \partial(\Omega_n^+ \cap \Omega_m^-)$. Moreover, if Ω is a component of $\Omega_{n,m}$ then $\partial\Omega \cap \mathbf{T}$ is the limit set of a finitely generated Fuchsian group $\widetilde{G}$ corresponding to the Riemann surface obtained by attaching funnels to boundary components of $S_{n,m} = S_n^+ \cap S_m^-$. Since $\delta(S_{n,m}) \leq \delta(R_1) = \delta_2 < 1$ we see that $\dim(\partial\Omega) \leq \delta_2$. Since E_1 is contained in a countable union of such sets, its dimension is also $\leq \delta_2$. Hence $\dim\bigl(f_n(E_1)\bigr) \leq 1$ by (7), independent of n and m.

Next consider the set $E_2 = \partial\Omega_n^- \cap \mathbf{T}$. Since the hyperbolic distance from Ω_n^- to the support of μ_n is at least cn (since μ_n is supported in Ω_{2n}), (4) of Corollary 3.6 says that $\dim\bigl(f(E_2)\bigr) < 1 + \frac{1}{2}\varepsilon$ if n is large enough.

Finally, consider the set E_3. It is separated from 0 by infinitely many alternating curves from Γ_0 and Γ_n. Given a component $\gamma \subset \Gamma_n$ the maximal components of Γ_0 separated from 0 by γ will be denoted $\{\gamma_k\}$. Since $\eta_1 > \delta_2$, by Lemma 2.3, they satisfy

$$\sum_k \left(\frac{\operatorname{diam}(\gamma_k)}{\operatorname{diam}(\gamma)}\right)^{\eta_1} \leq C.$$

By our choice of μ, f_n is η_1-Hölder, so we get for any $\eta > 0$

$$\begin{aligned}
\sum_k \left(\frac{\operatorname{diam}\bigl(f_n(\gamma_k)\bigr)}{\operatorname{diam}\bigl(f_n(\gamma)\bigr)}\right)^{1+\eta} &\leq \sum_k \left(\frac{\operatorname{diam}(\gamma_k)}{\operatorname{diam}(\gamma)}\right)^{\eta_1(1+\eta)} \\
&\leq Ce^{-\eta_1\eta cn} \sum_k \left(\frac{\operatorname{diam}(\gamma_k)}{\operatorname{diam}(\gamma)}\right)^{\eta_1} \\
&\leq Ce^{-\eta_1\eta cn}.
\end{aligned}$$

This is ≤ 1 if n is large enough.

On the other hand, if $\gamma \subset \Gamma_0$ and $\{\gamma_j\}$ are the maximal components of Γ_n separated from 0 by γ then trivially

$$\sum_j \frac{\operatorname{diam}(\gamma_j)}{\operatorname{diam}(\gamma)} \leq 1.$$

If we apply Theorem 3.1, with ε so small that $(1-\varepsilon)(1+\eta) \geq 1 + \frac{1}{2}\eta$, then we see that if n is large enough

$$\begin{aligned}\sum_j \left[\frac{\operatorname{diam}\bigl(f_n(\gamma_j)\bigr)}{\operatorname{diam}\bigl(f_n(\gamma)\bigr)}\right]^{1+\eta} &\leq C\biggl(\sum_j \frac{\operatorname{diam}(\gamma_j)}{\operatorname{diam}(\gamma)}\biggr)^{1+\eta/2} \\ &\leq Ce^{-\eta cn/2}\biggl(\sum_j \frac{\operatorname{diam}(\gamma_j)}{\operatorname{diam}(\gamma)}\biggr) \\ &\leq Ce^{-\eta cn},\end{aligned}$$

which is ≤ 1 if n is large enough. Thus if n is large (depending on η), there is a cover of $f_n(E_3)$ with $(1+\eta)$-Hausdorff sum bounded by 1. Taking η small enough completes the proof that $\dim\bigl(f_n(\Lambda_b^m)\bigr) \leq 1 + \frac{1}{2}\varepsilon$ if n is large enough (independent of m) and hence finishes the proof of Theorem 1.2.

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Received 21 May 2002