Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 28, 2003, 303–314

INFINITELY MANY ASYMPTOTIC VALUES OF LOCALLY UNIVALENT MEROMORPHIC FUNCTIONS

Karl F. Barth and Philip J. Rippon

Syracuse University, Department of Mathematics
Syracuse, NY 13244, U.S.A.; kfbarth@mailbox.syr.edu

The Open University, Department of Pure Mathematics
Walton Hall, Milton Keynes MK7 6AA, United Kingdom; p.j.rippon@open.ac.uk

Abstract. McMillan and Pommerenke showed that a locally univalent meromorphic function f in the disc, without Koebe arcs, has at least three asymptotic values in each boundary arc. The modular function shows that the number three is best possible. We show that if f satisfies certain further conditions, each of which narrowly excludes the modular function, then the number of asymptotic values in each boundary arc must be infinite.

1. Introduction

Let D denote the unit disc $\{z : |z| < 1\}$, C the unit circle $\{z : |z| = 1\}$, and $\widehat{\mathscr{C}}$ the extended complex plane. Let the function f be meromorphic in D. We say that f is *locally univalent* if it has at most simple poles and $f' \neq 0$, and that f has *Koebe arcs* if there are curves $J_n \subset D$ such that, for some $\alpha < \beta < \alpha + 2\pi$ and some $a \in \widehat{\mathscr{C}}$,

(a) J_n meets the radii $\arg z = \alpha$ and $\arg z = \beta$, for $n = 1, 2, \dots$;
(b) $|z| \to 1$, for $z \in J_n$, $n \to \infty$;
(c) $f(z) \to a$, for $z \in J_n$, $n \to \infty$.

A curve $\gamma : z(t)$, $0 \le t < 1$, in D is a *boundary path* if $|z(t)| \to 1$ as $t \to 1$. The set $\overline{\gamma} \cap C$ is called the *end* of γ. We say that f has the *asymptotic value* $a \in \widehat{\mathscr{C}}$ if there is a boundary path $\gamma : z(t)$, $0 \le t < 1$, such that

$$f\bigl(z(t)\bigr) \to a \qquad \text{as } t \to 1.$$

If f maps γ one-to-one onto a line segment, then the asymptotic value a is said to be *linearly accessible* along γ. Whenever the end of γ is contained in a subset E of C, we say that f has the asymptotic value a *in* E; if the end of γ is a singleton $\{\zeta\}$, then we say that f has the (*point*) asymptotic value a *at* ζ. It is clear that if f has no Koebe arcs, then all the asymptotic values of f are point asymptotic values.

In [17], McMillan and Pommerenke proved the following elegant result.

2000 Mathematics Subject Classification: Primary 30D40, 32A18.

Theorem A. *If f is meromorphic and locally univalent in D, and has no Koebe arcs, then f has at least three distinct point asymptotic values in each non-trivial arc of C.*

The modular function is locally univalent and its only asymptotic values are 0, 1 and ∞; see [14, p. 56] for a summary of facts about the modular function. Thus the number three is best possible in Theorem A. In this paper we show that if f satisfies certain further conditions, each of which narrowly excludes the modular function, then $\Gamma_P(f,\gamma)$, which denotes the set of point asymptotic values of f in any non-trivial arc γ of C, is infinite.

First we give a result for locally univalent analytic functions for which the maximum modulus $M(r,f)=\max\{|f(z)|:|z|=r\}$ does not grow too quickly.

Theorem 1. *Let f be analytic and locally univalent in D, with*

$$(1-r)\log^+ M(r,f)=o(1) \qquad \text{as } r\to 1. \tag{1.1}$$

Then for each non-trivial arc γ in C, the set $\Gamma_P(f,\gamma)$ is infinite.

The modular function satisfies $(1-r)\log^+ M(r,f)=O(1)$, because it is normal; see below. Thus the condition $o(1)$ in (1.1) cannot be replaced by $O(1)$. Also, the assumption that f is locally univalent cannot be dropped. Indeed, there are unbounded analytic functions in D which grow 'arbitrarily slowly' but have just one asymptotic value, namely ∞; see [13] and the references therein.

We give an example in Section 4 to show that the conclusion of Theorem 1 cannot be strengthened to 'uncountable' no matter what restriction is placed on the growth of an unbounded $M(r,f)$. We remark that Bagemihl [3] gave several conditions on unbounded analytic functions f in D under which the set $\Gamma_P(f,\gamma)$ is always of positive linear measure.

Next recall that a meromorphic function f is said to be *normal* if the functions

$$f\bigl(\phi(z)\bigr), \qquad \text{where } \phi(z)=e^{i\theta}\left(\frac{z+a}{1+\bar{a}z}\right),\ |a|<1,\ \theta\in\mathscr{R},$$

form a normal family or, equivalently, if

$$\sup_{z\in D}(1-|z|^2)f^{\#}(z)<\infty, \qquad \text{where } f^{\#}(z)=\frac{|f'(z)|}{1+|f(z)|^2}; \tag{1.2}$$

see [10]. For example, the modular function is normal because it omits the three values 0, 1 and ∞. By a theorem of Bagemihl and Seidel [4], normal meromorphic functions do not have Koebe arcs, so all their asymptotic values are point asymptotic values, and even angular limits by a theorem of Lehto and Virtanen [12]. Thus Theorem A applies to normal meromorphic functions that are locally univalent, and the modular function again shows that the number three is best possible.

Note that there exist normal meromorphic functions in D with no asymptotic values. See [12] for an example based on a modification of the modular function; alternatively, certain other Schwarz triangle functions with multiple points give an example directly (those described in [18, p. 294, case 3]).

The class $\mathscr{N}_0$ consists of functions meromorphic in D such that

$$(1.3) \qquad (1-|z|^2)f^{\#}(z) \to 0 \qquad \text{as } |z| \to 1.$$

Such 'little normal' functions have been characterised in various ways; see [2], and also [7] where they were called 'strongly normal'. As noted in [1, p. 31], the hypothesis (1.3) means that the spherical radius of the largest schlicht disc around $f(z)$ on the Riemann image surface of f tends to 0 as $|z| \to 1$. In particular, every univalent function is in $\mathscr{N}_0$. The 'little Bloch' class $\mathscr{B}_0$ consists of analytic functions in D which satisfy

$$(1-|z|^2)|f'(z)| \to 0 \qquad \text{as } |z| \to 1,$$

and these evidently lie in $\mathscr{N}_0$. There are functions in $\mathscr{B}_0$ which have finite angular limits almost nowhere on C, but all such functions must have finite angular limits on a set of Hausdorff dimension 1, by a result of Makarov; see [20, Chapters 8 and 11].

For locally univalent functions in $\mathscr{N}_0$, the conclusion of Theorem A can be greatly strengthened.

Theorem 2. *Let f be meromorphic in D, locally univalent and in $\mathscr{N}_0$. Then for each non-trivial arc γ in C, the set $\Gamma_P(f,\gamma)$ is of positive linear measure.*

We remark that the set of asymptotic values of a meromorphic function is known to be analytic and hence linearly measurable; see [16]. The conclusion of Theorem 2 is best possible, and the modular function shows that the hypothesis in Theorem 2 that $f \in \mathscr{N}_0$ cannot be replaced by the hypothesis that f is normal.

Finally, recall that a function f analytic in D is said to be in the MacLane class $\mathscr{A}$ if f is non-constant and has point asymptotic values at a dense set of points in C; see [14]. It is known that if

$$(1.4) \qquad \int_0^1 \log^+ \log^+ M(r,f)\, dr < \infty,$$

then $f \in \mathscr{A}$; see [11]. In particular, normal analytic functions are in $\mathscr{A}$ because they satisfy $(1-r)\log^+ M(r,f) = O(1)$ as $r \to 1$; see [10, p. 165].

The class $\mathscr{A}_m$, introduced in [5], is the extension of the class $\mathscr{A}$ to meromorphic functions. Theorem A shows that locally univalent normal meromorphic functions belong to $\mathscr{A}_m$. Theorem 2 shows that a stronger result holds if 'normal' is replaced by $\mathscr{N}_0$. We say that a set is *uncountably dense* in C if the set meets each non-trivial arc of C in an uncountable set.

Corollary 1. *Let f be meromorphic in D, locally univalent and in $\mathscr{N}_0$. Then f has angular limits at an uncountably dense set of points of C.*

It is natural to ask about the asymptotic values of a meromorphic function f in $\mathscr{N}_0$ without the hypothesis of local univalence. If

$$(1-|z|^2)f^{\#}(z) = O(1)(1-|z|)^{\varepsilon} \qquad \text{as } |z| \to 1,$$

where $\varepsilon > 0$, then it follows from a result of Carleson [6, p. 61] that f has angular limits at all points of C apart from a set of (an appropriate) capacity zero. It is plausible that if f is meromorphic in D and in $\mathscr{N}_0$, then f belongs to $\mathscr{A}_m$; we intend to return to this matter in another paper.

2. Proof of Theorem 1

It is sufficient to consider the case when the function f is unbounded near each interior point of the arc γ. Indeed, if f is bounded near some interior point ζ of γ, then $\Gamma_P(f,\gamma)$ has positive linear measure by [8, p. 120].

We assume that the set $\Gamma_P(f,\gamma)\setminus\{\infty\}$ of finite asymptotic values of f at points of γ is finite, and obtain a contradiction. It follows from (1.1), (1.4) and Cauchy's estimate that f and f' belong to the class $\mathscr{A}$. Since $f' \neq 0$, we deduce from [15, Theorem 9] that f has at least three distinct point asymptotic values in each non-trivial arc of C. Hence there is a cross-cut γ' of D ending at distinct points of γ on which f is bounded. Let $D(\gamma)$ be the component of $D\setminus\gamma'$ such that $\partial D(\gamma)\cap C \subset \gamma$. We can choose $R > 0$ so that

$$\Gamma_P(f,\gamma)\setminus\{\infty\} \subset \{|w| \le R\}, \tag{2.1}$$

and

$$|f(z)| \le R, \qquad \text{for } z \in \gamma'. \tag{2.2}$$

Since f is unbounded near each interior point of γ, there is at least one component, Ω say, of $\{z : |f(z)| > R+1\}$ lying entirely in $D(\gamma)$, by (2.2). Pick $z_0 \in \Omega$, let $w_0 = f(z_0)$, and let g denote the branch of f^{-1} such that $g(w_0) = z_0$. By (2.1), (2.2) and the local univalence of f, the branch g can be analytically continued along any path from w_0 in $\{|w| > R\}$ without meeting any singularity. Indeed, since the Riemann surface of f has no branch points, such a singularity would give rise to an asymptotic value α of f, with $|\alpha| > R$, at a point of γ.

Thus if $H = \{t : \mathrm{Re}(t) > \log R\}$ and t_0 satisfies $e^{t_0} = w_0$, then $h(t) = g(e^t)$ can be analytically continued from t_0 along any path in H to give a single-valued analytic function, by the monodromy theorem. Two cases can then arise [18, p. 283], as follows.

(a) The function h is univalent in H.
(b) The function h is periodic with period $2\pi i m$, where m is a minimal positive integer.

In the present situation case (b) cannot occur. Indeed, if case (b) holds, then the function $G(w) = g(w^m)$ must be univalent in $\Delta_0 = \{|w| > R^{1/m}\}$, so it has a Laurent expansion there of the form

$$G(w) = a_1 w + a_0 + a_{-1} w^{-1} + \cdots .$$

Since g takes values in D, we have $a_1 = 0$. Hence G has a univalent extension to $\Delta_0 \cup \{\infty\}$ and so $a_{-1} \neq 0$. Then $G(\infty) = a_0$ is a pole of f in D, which is impossible.

Thus only case (a) can occur. Since the values taken by the function h in H all lie in the component of $\{z : |f(z)| > R\}$ which contains Ω, we deduce that Ω is conformally equivalent to $\{t : \mathrm{Re}(t) > \log(R+1)\}$ and that $\partial\Omega$ consists of a simple analytic curve in $D(\gamma)$.

Now we use the fact that f belongs to the MacLane class $\mathscr{A}$. Then by [14, Theorem 1], both ends of the simple curve $\partial\Omega$ must approach points of γ. Moreover, by the Carathéodory boundary correspondence theorem [20, p. 24], both ends of $\partial\Omega$ must approach the same point, ζ_0 say, of γ. It follows that the function

$$u(z) = \begin{cases} \log\bigl(|f(z)|/(R+1)\bigr), & \text{for } z \in \Omega, \\ 0, & \text{for } z \in D \setminus \Omega, \end{cases}$$

is subharmonic in D and tends to zero at each point of $C \setminus \{\zeta_0\}$. The hypothesis (1.1) implies that $\max\{u(z) : |z| = r\} = o(1)/(1-r)$ as $r \to 1$ so, by a special case of Dahlberg's radial maximum theorem [9, Theorem 2], u must be identically zero, a contradiction. This completes the proof of Theorem 1.

Remark. By a result of MacLane [15, Theorem 8], it follows that under the hypotheses of Theorem 1 the set of linearly accessible finite asymptotic values of f is unbounded and so infinite.

3. Proof of Theorem 2

First we consider the case when there exists $\alpha \in \widehat{\mathscr{C}}$ such that f is bounded away from α, in the spherical metric, near some interior point ζ of γ. Then f or $g(z) = 1/\bigl(f(z) - \alpha\bigr)$ is bounded near ζ, and it follows from [8, p. 120] that $\Gamma_P(f, \gamma)$ is of positive linear measure.

There remains the case when, for each $\alpha \in \widehat{\mathscr{C}}$, the function f takes values arbitrarily close to α, in the spherical metric, in any neighbourhood of any interior point of γ. To deal with this case, we suppose that the set $\Gamma_P(f, \gamma)$ is of linear measure zero, and obtain a contradiction.

Since normal functions do not have Koebe arcs [4], we deduce from Theorem A that there is a cross-cut γ' of D ending at distinct points of γ on which f is bounded, say

$$|f(z)| \le R, \qquad \text{for } z \in \gamma', \tag{3.1}$$

for some $R>3$. Let $D(\gamma)$ be the component of $D\setminus\gamma'$ such that $\partial D(\gamma)\cap C\subset\gamma$. Since f takes values arbitrarily close to any $\alpha\in\widehat{\mathscr{C}}$ near each point of γ, there is some $z_0\in D(\gamma)$ such that

$$R+2<|f(z_0)|<2R, \tag{3.2}$$

and, by (1.3),

$$(1-|z_0|^2)f^{\#}(z_0)<\frac{1}{1+4R^2}. \tag{3.3}$$

Moreover, since f is an open mapping and $\Gamma_P(f,\gamma)$ has linear measure zero, we can assume, by varying z_0 slightly if necessary, that

$$w_0=f(z_0)\text{ does not belong to }\Gamma_P(f,\gamma). \tag{3.4}$$

From (3.2) and (3.3), we deduce that

$$(1-|z_0|^2)|f'(z_0)|<1. \tag{3.5}$$

For $z\in D$, let $d(z)$ denote the radius of the maximal schlicht disc in the image surface $\mathscr{F}$ of f centred at the point of $\mathscr{F}$ corresponding to $f(z)$. Since f is locally univalent the surface $\mathscr{F}$ has no branch points, so this maximal schlicht disc has a (linearly accessible) asymptotic value of f on its boundary corresponding to an inverse function singularity of f. It follows from Schwarz's lemma (see [21]) that if f is analytic at z, then

$$d(z)\le(1-|z|^2)|f'(z)|. \tag{3.6}$$

We claim that there is a path γ_0 in D with one endpoint at z_0 such that

$$\text{there exist distinct }\zeta_n\in\gamma_0\text{ such that }f(\zeta_n)=w_0,\ \text{ for }n=1,2,\dots, \tag{3.7}$$

and

$$|f(z)-w_0|\le 2d(z_0),\qquad\text{for }z\in\gamma_0. \tag{3.8}$$

To prove this claim, let Σ_0 denote the maximal schlicht disc in $\mathscr{F}$ centred at the point of $\mathscr{F}$ corresponding to $w_0=f(z_0)$ and let w_0' be a singularity of f^{-1} on the boundary of Σ_0. Then $|w_0'-w_0|=d(z_0)<1$ by (3.5) and (3.6). Also, let Γ_0' denote the radius of Σ_0 from w_0 to w_0'.

Next let g denote the branch of f^{-1} such that $g(w_0)=z_0$, so g is univalent in Σ_0. Let Σ_0' denote the open disc in $\mathscr{C}$ with diameter along the line segment Γ_0'.

Then Σ_0' has radius $\frac{1}{2}d(z_0)$ and centre $\frac{1}{2}(w_0 + w_0') = w_0^*$, say. We consider the analytic continuation of g from near w_0 along paths in $\mathscr{C} \setminus \overline{\Sigma_0'}$.

First suppose that such analytic continuation of g meets no singularities of f^{-1}. Choose t_0 such that

$$w_0^* + \tfrac{1}{2}d(z_0)e^{t_0} = w_0. \tag{3.9}$$

Then the function

$$h(t) = g\big(w_0^* + \tfrac{1}{2}d(z_0)e^{t}\big),$$

can be analytically continued along any path in $H = \{\mathrm{Re}(t) > 0\}$, starting near t_0. Thus h is a single-valued analytic function in H, by the monodromy theorem, and $h(t_0) = z_0$. Two cases can then arise [18, p. 283], as follows.

(a) The function h is univalent in H.

(b) The function h is periodic with period $2\pi i m$, where m is a minimal positive integer.

Once again, case (b) cannot occur in the present situation. Indeed, if case (b) holds, then the function $G(w) = g\big(w_0^* + (w - w_0^*)^m\big)$ must be univalent in $\Delta_0 = \big\{|w - w_0^*| > \big(\frac{1}{2}d(z_0)\big)^{1/m}\big\}$, so it has a Laurent expansion there of the form

$$G(w) = a_1(w - w_0^*) + a_0 + a_{-1}(w - w_0^*)^{-1} + \cdots.$$

Since g takes values in D, we have $a_1 = 0$. Hence G has a univalent extension to $\Delta_0 \cup \{\infty\}$ and so $a_{-1} \neq 0$. Then $G(\infty) = a_0$ is a pole of f which must be simple. Writing $z = G(w)$, we obtain $f(z) = w_0^* + (w - w_0^*)^m$ and

$$z - a_0 = \frac{a_{-1}}{w - w_0^*}\big(1 + o(1)\big) \qquad \text{as } w \to \infty,$$

so

$$f(z) = w_0^* + \frac{(a_{-1})^m}{(z - a_0)^m}\big(1 + o(1)\big) \qquad \text{as } z \to a_0.$$

Thus $m = 1$, so g is univalent in $\Delta_0 \cup \{\infty\}$. Because $\partial\Delta_0 \setminus \{w_0'\} \subset \Sigma_0$, it follows that g is analytic in $\Sigma_0 \cup (\Delta_0 \cup \{\infty\}) = \widehat{\mathscr{C}} \setminus \{w_0'\}$. Since g takes values in D, this contradicts Liouville's theorem.

Thus case (a) occurs, so h is univalent in H. Moreover, we can find open discs D_n with centres $t_n = t_0 + 2\pi i n$, $n \in \mathscr{Z}$, such that h is univalent in the larger set

$$H' = H \cup \bigcup_{n \in \mathscr{Z}} D_n.$$

This follows from (3.9) and the fact that the analytic continuation of g along any path in $\mathscr{C} \setminus \overline{\Sigma_0'}$ which ends at w_0 can be extended to a neighbourhood of w_0; for otherwise $w_0 \in \Gamma_P(f, \gamma)$ contrary to (3.4).

We now take a simple path β_0 in $H' \cap \{\mathrm{Re}(t) < 1\}$ with one endpoint at t_0, passing through every $t_n, n = 0, 1, 2, \ldots$, and tending to ∞. Then $\gamma_0 = h(\beta_0)$ is a simple path in D with one endpoint at z_0. Since $f(h(t)) = w_0^* + \frac{1}{2}d(z_0)e^t$, we deduce that (3.7) holds, with $\zeta_n = h(t_n)$, and also (3.8) holds, as required.

Suppose, on the other hand, that the analytic continuation of g from w_0 along some path Γ_0'' in $\mathscr{C} \setminus \overline{\Sigma_0'}$ *does* encounter a singularity of f^{-1}, at w_0'' say. Choose $r_0 > 0$ such that $r_0 < \min\{d(z_0), \mathrm{dist}\,(w_0', \Gamma_0'')\}$. Then let Γ_0 be a Jordan curve in $\mathscr{C}$ which surrounds w_0' and consists of two chords through w_0 of the disc $\overline{\Sigma_0'}$ together with an arc of a circle $\{|w - w_0'| = r\}$, where $0 < r < r_0$. Since $\Gamma_P(f, \gamma)$ has linear measure zero, it fails to meet almost all such chords and almost all such circular arcs, so we can assume by (3.2) that $\Gamma_0 \subset \{|w| > R\} \setminus \Gamma_P(f, \gamma)$. In particular, the branch g can be analytically continued from w_0 around Γ_0 arbitrarily far (in either direction). Note that by construction Γ_0, Γ_0', Γ_0'' meet only at w_0.

Let γ_0, γ_0', γ_0'' denote the paths in D obtained by analytic continuation of g along Γ_0, Γ_0', Γ_0'', respectively. Then γ_0' and γ_0'' are asymptotic paths meeting only at z_0, so the path $\gamma_0' \cup \gamma_0''$ forms a cross-cut of D. The path γ_0 can meet $\gamma_0' \cup \gamma_0''$ only at z_0. Thus, by the local univalence of f, the path γ_0 must pass through z_0 exactly once, as it crosses $\gamma_0' \cup \gamma_0''$. Also, if the part of γ_0 on one side of the cross-cut were to intersect itself, then at the first point of intersection there would again be a contradiction to local univalence. It follows that γ_0 is a simple path passing through z_0 such that (3.7) and (3.8) hold. Thus we have established the claim.

It follows from (3.7) that $|\zeta_n| \to 1$ as $n \to \infty$. Hence $d(\zeta_n) \to 0$ as $n \to \infty$ by (1.3) and (3.6), because f is bounded on γ_0 by (3.8). Thus we can truncate γ_0 at a point, z_1 say, chosen from the ζ_n, such that

$$f(z_1) = w_0, \quad 1 - |z_1| < \tfrac{1}{2} \quad \text{and} \quad d(z_1) \le \tfrac{1}{2}d(z_0) < \tfrac{1}{2}.$$

Now we repeat the above argument with z_0 replaced by z_1 to obtain a path γ_1 in D with one endpoint at z_1 such that

$$\text{there exist distinct } \zeta_n \in \gamma_1 \text{ such that } f(\zeta_n) = w_0, \text{ for } n = 1, 2, \ldots,$$

and

$$|f(z) - w_0| \le 2d(z_1), \quad \text{for } z \in \gamma_1.$$

We then truncate γ_1 at a point z_2 such that

$$f(z_2) = w_0, \quad 1 - |z_2| < \tfrac{1}{4} \quad \text{and} \quad d(z_2) \le \tfrac{1}{2}d(z_1) < \tfrac{1}{4}.$$

By repeating this process, we obtain a sequence of such truncated paths γ_n, $n = 0, 1, 2, \ldots$, such that $f(z)$ tends to w_0 along the path

$$\gamma_\infty = \gamma_0 \cup \gamma_1 \cup \gamma_2 \cup \cdots,$$

and, by (3.2), $|f(z)| > R$ for $z \in \gamma_\infty$. By (3.1) the end of γ_∞ must lie in γ, so γ_∞ must tend to a point of γ because f has no Koebe arcs. Hence $w_0 \in \Gamma_P(f, \gamma)$, which contradicts (3.4). The proof of Theorem 2 is complete.

Remarks. 1. The proof can be adapted to show that if, in any neighbourhood of any point of γ, the locally univalent function f in $\mathscr{N}_0$ takes values arbitrarily close to each $\alpha \in \widehat{\mathscr{C}}$, in the spherical metric, then $\Gamma_P(f,\gamma)$ has positive linear measure in any neighbourhood of each point of $\widehat{\mathscr{C}}$. This holds near ∞ by taking R arbitrarily large in (3.1) and, for other points α, we consider the function in $\mathscr{N}_0$ obtained by composing f with a rotation of the Riemann sphere taking α to ∞. It seems possible that under these circumstances we must actually have $\Gamma_P(f,\gamma) = \widehat{\mathscr{C}}$.

2. An earlier version of this proof, which was more like the proof of Theorem 1, examined the components of $\{z : |f(z)| > R\}$ in $D(\gamma)$. This raised the question of whether the level sets $L(R) = \{z : |f(z)| = R\}$, $R > 0$, end at points of C, in the sense of MacLane [14, p. 8]. In fact, the proof of Theorem A in [17] shows that the 'three asymptotic values' obtained in each arc are linearly accessible. It follows that if f satisfies the hypotheses of Theorem A, then every level set $L(R)$, $R > 0$, of f must end at points.

4. An example

We show here that the conclusion in Theorem 1 cannot be strengthened to 'uncountable'.

Example 1. *Let μ be an increasing function on $[0,1)$ such that $\mu(0) = 1$ and $\mu(r) \to \infty$ as $r \to 1$. Then there exists a function f analytic and locally univalent in D, such that*

$$M(r,f) \le \mu(r), \qquad \text{for } 0 \le r < 1, \tag{4.1}$$

and the set of asymptotic values of f is countable.

Our approach is to follow the geometric construction of MacLane [13] for an analytic function in D of 'arbitrarily slow growth' with level curves C_n, $n = 1,2,\ldots$, which are nested Jordan curves that expand towards C, on which $|f| = \varrho_n$, $n = 1,2,\ldots$, where $\varrho_n \to \infty$ as $n \to \infty$. The method is to construct the image Riemann surface of the function by a process of successive enlargements, and the main difference in our construction is that we replace MacLane's adjoined two-sheeted surfaces by adjoined logarithmic spirals.

We first fix a positive strictly increasing sequence ϱ_n, with $\varrho_n \to \infty$ as $n \to \infty$, and put $\alpha_n = n\sqrt{2}$, $n = 1,2,\ldots$. Then define

$$a_{n,k} = \varrho_n \exp(i\theta_{n,k}), \qquad \text{where } \theta_{n,k} = 2\pi\alpha_n + \frac{2\pi k}{\nu_n},\ k = 1,2,\ldots,\nu_n. \tag{4.2}$$

For $n = 1,2,\ldots$, the points $a_{n,k}$, $k = 1,2,\ldots,\nu_n$, are equally spaced on the circle $\{|w| = \varrho_n\}$ and no two $a_{n,k}$, $k = 1,2,\ldots,\nu_n$, $n = 1,2,\ldots$, are in the same direction from 0. The sequences ϱ_n and α_n are fixed throughout, but ν_n will be varied to determine the required Riemann surface.

Let

$$L(n,k) = \{|w| \geq \varrho_n,\ \arg w = \theta_{n,k}\}, \qquad k = 1,2,\dots,\nu_n,\ n = 1,2,\dots,$$

so the rays $L(n,k)$, $k = 1,2,\dots,\nu_n$, $n = 1,2,\dots$, are mutually disjoint. The required surface $\mathscr{S}$ is built in stages. We begin with the surface $\mathscr{S}_1$ consisting of the w-plane slit along all rays $L(1,k)$, $k = 1,2,\dots,\nu_1$. Then the surface $\mathscr{S}_2$ is constructed by first adjoining a half-logarithmic spiral to each edge of each slit in $\mathscr{S}_1$, and then making a slit over every ray $L(2,k)$, $k = 1,2,\dots,\nu_2$, in all the resulting sheets. The surface $\mathscr{S}_3$ is constructed in an analogous manner by adjoining half-logarithmic spirals to edges of existing slits in $\mathscr{S}_2$ wherever possible and then making new slits over the rays $L(3,k)$, $k = 1,2,\dots,\nu_3$, wherever possible.

Repeating this process indefinitely, we obtain a surface $\mathscr{S}$ without algebraic branch points. The surface $\mathscr{S}$ is simply connected because at each stage the addition of each pair of half logarithmic spirals preserves the property of being simply connected, and $\mathscr{S} = \bigcup_{n=1}^{\infty} \mathscr{S}_n$. Now let $\mathscr{T}_n$ denote that component of $\mathscr{S}_n$ over $\{|w| < \varrho_n\}$ which contains the origin in $\mathscr{S}_1$. If $\nu_1, \nu_2, \dots, \nu_{n-1}$ are fixed and $\nu = \nu_n$ is variable, then as $\nu \to \infty$ the sequence of surfaces $\mathscr{T}_{n+1} = \mathscr{T}_{n+1}(\nu)$ converges to its Carathéodory kernel, the unique maximal Riemann surface all of whose compact subsets K can be embedded in $\mathscr{T}_{n+1}(\nu)$, for $\nu \geq \nu(K)$. This kernel is $\mathscr{T}_n$. We now need the following generalisation of the Carathéodory kernel theorem, due essentially to L. I. Volkovyskii; see [13].

Theorem B. *Let R_n, $n = 1,2,\dots$, be a sequence of positive numbers such that $\lim_{n\to\infty} R_n = R$, $0 < R < \infty$. Let $\mathscr{F}_n = \{|z| < R_n\}$ and $\mathscr{F} = \{|z| < R\}$. Let $[\mathscr{G}_n; Q_n]$ be a sequence of Riemann surfaces over the w-plane, each containing a schlicht disc of radius $s_0 > 0$ about Q_n, where Q_n is a point of $\mathscr{G}_n$ over $w = 0$. Let $w = F_n(z)$ be holomorphic in $\mathscr{F}_n$ and map $\mathscr{F}_n$ one-one onto $\mathscr{G}_n$ with $F_n(0) = Q_n$ and $F_n'(0) = 1$. Let Φ_n be the inverse of F_n. Then the following are true.*

(a) *If $F_n \to F$ locally uniformly on $\mathscr{F}$, then $[\mathscr{G}_n; Q_n]$ converges to its kernel $[\mathscr{G}; Q]$ and F maps $\mathscr{F}$ one-one onto $\mathscr{G}$, with $F(0) = Q$ and $F'(0) = 1$. Also, $\Phi_n \to \Phi$ locally uniformly on $\mathscr{G}$ and Φ is the inverse of F.*

(b) *If $\{F_n\}$ is a normal family on $\mathscr{F}$ and if $[\mathscr{G}_n; Q_n]$ converges to its kernel $[\mathscr{G}; Q]$, then $F_n \to F$ locally uniformly on $\mathscr{F}$.*

Now let f_n map $\{|z| < r_n\}$ onto $\mathscr{T}_n$, with $f_n(0) = 0 \in \mathscr{S}$ and $f_n'(0) = 1$. Since $g_n = f_{n+1}^{-1} \circ f_n$ maps $\{|z| < r_n\}$ into $\{|z| < r_{n+1}\}$ with $g_n(0) = 0$ and $g_n'(0) = 1$, we deduce that

$$r_n < r_{n+1}, \qquad \text{for } n = 1,2,\dots. \tag{4.3}$$

Let δ_n be any positive sequence such that

$$\sum_{n=1}^{\infty} \delta_n < \infty. \tag{4.4}$$

We shall show that the sequence ν_n, $n = 1, 2, \dots$, can be chosen so that

$$r_{n+1} < r_n + \delta_n, \qquad \text{for } n = 1, 2, \dots . \tag{4.5}$$

To do this, suppose that $\nu_1, \nu_2, \dots, \nu_{n-1}$ have already been chosen. As above, we indicate dependence on $\nu = \nu_n$ by writing $\mathscr{T}_{n+1} = \mathscr{T}_{n+1}(\nu)$, $f_{n+1}(z) = f_{n+1}(z;\nu)$ and $r_{n+1} = r_{n+1}(\nu)$. Now choose a subsequence of values of ν so that

$$r_{n+1}(\nu) \to r_{n+1}(\infty) \geq r_n.$$

We claim that $r_{n+1}(\infty) = r_n$. Indeed, the subsequence $f_{n+1}(z;\nu)$ is defined eventually on each compact subset of $\{|z| < r_{n+1}(\infty)\}$, is uniformly bounded by ϱ_{n+1} and hence forms a normal family. Also, the sequence $\mathscr{T}_{n+1}(\nu)$ converges to its kernel $\mathscr{T}_n$. Thus, by Theorem B, part (b),

$$f_{n+1}(z;\nu) \to f_{n+1}(z;\infty) \qquad \text{as } \nu \to \infty,$$

locally uniformly and, by Theorem B, part (a), $f_{n+1}(z;\infty)$ maps $\{|z| < r_{n+1}(\infty)\}$ onto $\mathscr{T}_n$ with the same normalization as f_n. Hence $r_{n+1}(\infty) = r_n$. Thus we can indeed choose ν_n so that (4.5) holds.

Now we apply Theorem B to the sequences f_n and $\mathscr{T}_n$. By (4.4), $r_n \to R < \infty$ and by construction $\mathscr{T}_n$ converges to its kernel $\mathscr{S}$. To show that the sequence f_n is normal, let f^* map $\{|z| < R^*\} = D^*$ onto $\mathscr{S}$ with $f^*(0) = 0 \in \mathscr{S}$ and $(f^*)'(0) = 1$; here $0 < R^* \leq \infty$. Since $\mathscr{T}_n \subset \mathscr{S}$, the functions $h_n = (f^*)^{-1} \circ f_n$ are conformal and normalised in the usual way. Thus the family $\{h_n\}$ is normal and it follows that the family $f_n = f^* \circ h_n$ is normal in $\{|z| < R\}$. Hence, by Theorem B, part (b), $f_n \to f$ locally uniformly in $\{|z| < R\}$, where f maps $\{|z| < R\}$ onto $\mathscr{S}$, the kernel of the sequence $\mathscr{T}_n$, with $f(0) = 0 \in \mathscr{S}_1$ and $f'(0) = 1$.

The proof that the sequence ν_n can be chosen in such a way that the condition (4.1) holds, is based on (4.3) and (4.4), as in [13, Section 3]. Finally, it is clear from the construction of $\mathscr{S}$ that the function f mapping $\{|z| < R\}$ onto $\mathscr{S}$ is locally univalent and has countably many finite asymptotic values, namely $a_{n,k}$, $k = 1, 2, \dots, \nu_n$, $n = 1, 2, \dots$. This completes the proof of Example 1.

References

[1] Anderson, J. M., J. Clunie, and Ch. Pommerenke: On Bloch functions and normal functions. - J. Reine Angew. Math. 270, 1974, 12–37.

[2] Aulaskari, R., and R. Zhao: Some characterizations of normal and little normal functions. - Complex Variables Theory Appl. 28, 1995, 135–148.

[3] Bagemihl, F.: Sets of asymptotic values of positive linear measure. - Ann. Acad. Sci. Fenn. Ser. A I Math. 373, 1965, 3–7.

[4] Bagemihl, F., and W. Seidel: Koebe arcs and Fatou points of normal functions. - Comment. Math. Helv. 36, 1961, 9–18.

[5] Barth, K. F.: Asymptotic values of meromorphic functions. - Michigan Math. J. 13, 1966, 321–340.

[6] Carleson, L.: Selected Problems on Exceptional Sets. - Van Nostrand Mathematical Studies 13, 1967.

[7] Chen, H., and P. Gauthier: On strongly normal functions. - Canad. Math. Bull. 39, 1996, 408–419.

[8] Collingwood, E. F., and A. J. Lohwater: The Theory of Cluster Sets. - Cambridge University Press, 1966.

[9] Dahlberg, B. E. J.: On the radial boundary values of subharmonic functions. - Math. Scand. 40, 1977, 301–317.

[10] Hayman, W. K.: Meromorphic Functions. - Oxford University Press, 1964.

[11] Hornblower, R. J. M.: A growth condition for the MacLane class $\mathscr{A}$. - Proc. London Math. Soc. (3) 23, 1971, 371–384.

[12] Lehto, O., and K. I. Virtanen: Boundary behaviour of normal meromorphic functions. - Acta Math. 97, 1957, 47–65.

[13] MacLane, G. R.: The geometry of functions holomorphic in the unit circle, of arbitrarily slow growth, which tend to infinity on a sequence of curves approaching the circumference. - Duke Math. J. 29, 1962, 191–197.

[14] MacLane, G. R.: Asymptotic values of holomorphic functions. - Rice Univ. Studies 49, 1963.

[15] MacLane, G. R.: Exceptional values of $f^{(n)}(z)$, asymptotic values of $f(z)$ and linearly accessible asymptotic values. - In: Mathematical Essays Dedicated to A. J. MacIntyre, Ohio University Press, 1970, 271–288.

[16] McMillan, J. E.: On local asymptotic properties, the asymptotic value sets, and ambiguous properties of functions meromorphic in the open unit disc. - Ann. Acad. Sci. Fenn. Ser. A I Math. 384, 1965, 1–12.

[17] McMillan, J. E., and Ch. Pommerenke: On the asymptotic values of locally univalent meromorphic functions. - J. Reine Angew. Math. 249, 1969, 31–33.

[18] Nevanlinna, R.: Analytic Functions. - Springer-Verlag, 1970.

[19] Pommerenke, Ch.: Normal functions. - In: Proceedings of the NRL Conference on Classical Function Theory, Washington DC, 1970, 76–93.

[20] Pommerenke, Ch.: Boundary Behaviour of Conformal Maps. - Springer-Verlag, 1992.

[21] Seidel, W., and J. L. Walsh: On the derivative of functions analytic in the unit circle and their radius of univalence and p-valence. - Trans. Amer. Math. Soc. 52, 1942, 128–216.

Received 14 November 2002