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ON A DISTANCE DEFINED BY THE LENGTH SPECTRUM ON TEICHMÜLLER SPACE

Hiroshige Shiga

Tokyo Institute of Technology, Department of Mathematics
Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan; shiga@math.titech.ac.jp

Dedicated to the memory of Professor Nobuyuki Suita

Abstract. We consider a distance d_L on the Teichmüller space $T(S_0)$ of a hyperbolic Riemann surface S_0. The distance is defined by the length spectrum of Riemann surfaces in $T(S_0)$ and we call it the length spectrum metric on $T(S_0)$. It is known that the distance d_L determines the same topology as that of the Teichmüller metric if S_0 is a topologically finite Riemann surface.

In this paper we show that there exists a Riemann surface S_0 of infinite type such that the length spectrum distance d_L on $T(S_0)$ does not define the same topology as that of the Teichmüller distance. Also, we give a sufficient condition for these distances to have the same topology on $T(S_0)$.

1. Introduction and results

On the Teichmüller space $T(S_0)$ of a hyperbolic Riemann surface S_0, we have the Teichmüller distance $d_T(\cdot\,,\cdot)$, which is a complete distance on $T(S_0)$. In this paper, we study another distance $d_L(\cdot\,,\cdot)$ which is defined by the length spectrum on Riemann surfaces in $T(S_0)$. Li [4] discussed the distance $d_L(\cdot\,,\cdot)$ on the Teichmüller space of a compact Riemann surface of genus $g \geq 2$ and showed that the distance d_L defines the same topology as that of the Teichmüller distance. Recently, Liu [5] showed that the same statement is true even if S_0 is a Riemann surface of topologically finite type, and he asked whether the statement holds for Riemann surface of infinite type. Our first result gives a negative answer to this question.

Theorem 1.1. *There exist a Riemann surface S_0 of infinite type and a sequence $\{p_n\}_{n=0}^{\infty}$ in $T(S_0)$ such that*

$$d_L(p_n, p_0) \to 0, \quad n \to \infty,$$

while

$$d_T(p_n, p_0) \to \infty, \quad n \to \infty.$$

From the proof of this theorem, we show the incompleteness of the length spectrum distance.

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Corollary 1.1. *There exists a Riemann surface of infinite type such that the length spectrum distance d_L is incomplete in the Teichmüller space.*

Next, we give a sufficient condition for the length distance to define the same topology as that of the Teichmüller distance as follows.

Theorem 1.2. *Let S_0 be a Riemann surface. Assume that there exists a pants decomposition $S_0 = \bigcup_{k=1}^{\infty} P_k$ of S_0 satisfying the following conditions.*

(1) *Each connected component of ∂P_k is either a puncture or a simple closed geodesic of S_0, $k = 1, 2, \dots$.*

(2) *There exists a constant $M > 0$ such that if α is a boundary curve of some P_k then*

$$0 < M^{-1} < l_{S_0}(\alpha) < M$$

holds.

Then d_L defines the same topology as that of d_T on the Teichmüller space $T(S_0)$ of S_0.

2. Preliminaries

Let S_0 be a hyperbolic Riemann surface. We consider a pair (S, f) of a Riemann surface S and a quasiconformal homeomorphism f of S_0 onto S. Two such pairs (S_j, f_j), $j = 1, 2$, are called *equivalent* if there exists a conformal mapping $h\colon S_1 \to S_2$ which is homotopic to $f_2 \circ f_1^{-1}$, and we denote the equivalence class of (S, f) by $[S, f]$. The set of all equivalence classes $[S, f]$ is called *the Teichmüller space* of S_0: we denote it by $T(S_0)$.

The Teichmüller space $T(S_0)$ has a complete distance d_T called *the Teichmüller distance* which is defined by

$$d_T([S_1, f_1], [S_2, f_2]) = \inf_f \log K[f],$$

where the infimum is taken over all quasiconformal mappings $f\colon S_1 \to S_2$ homotopic to $f_2 \circ f_1^{-1}$ and $K[f]$ is the maximal dilatation of f.

We define another distance on $T(S_0)$ by length spectrum of Riemann surfaces. Let $\Sigma(S)$ be the set of closed geodesics on a hyperbolic Riemann surface S. For any two points $[S_j, f_j]$, $j = 1, 2$, in $T(S_0)$, we set

$$\varrho([S_1, f_1], [S_2, f_2]) = \sup_{c \in \Sigma(S_1)} \max\left\{ \frac{l_{S_1}(c)}{l_{S_2}\bigl(f_2 \circ f_1^{-1}(c)\bigr)}, \frac{l_{S_2}\bigl(f_2 \circ f_1^{-1}(c)\bigr)}{l_{S_1}(c)} \right\},$$

where $l_S(\alpha)$ is the hyperbolic length of a closed geodesic on S freely homotopic to a closed curve α. For two points $[S_j, f_j] \in T(S_0)$, $j = 1, 2$, we define a distance d_L called *the length spectrum distance* by

$$d_L([S_1, f_1], [S_2, f_2]) = \log \varrho\bigl([S_1, f_1], [S_2, f_2]\bigr).$$

Wolpert ([7]) shows that $l_{S_2}\bigl(f(c)\bigr) \le K[f] l_{S_1}(c)$ holds for every quasiconformal mapping $f\colon S_1 \to S_2$ and for every $c \in \Sigma(S_1)$. Thus, we have immediately:

Lemma 2.1. *An inequality*

$$d_L(p,q) \le d_T(p,q)$$

holds for every $p,q \in T(S_0)$.

3. Proofs of Theorem 1.1 and Corollary 1.1

3.1. Proof of Theorem 1.1. First, we take monotone divergent sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ of positive numbers so that $a_{n+1} = b_n$ and $b_n/a_n > n$, $n = 1, 2, \dots$. For each n, we take a right-angled hexagon H_n so that the lengths of three edges are a_n, b_n, b_n as Figure 1.

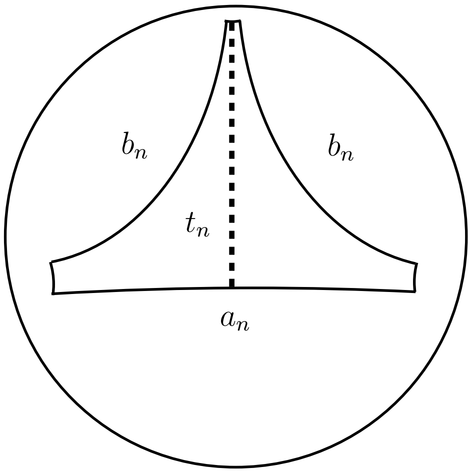

Figure 1.

We take b_n so large that

$$t_n > na_n \tag{1}$$

holds, where t_n is the height of H_n, the distance from the bottom edge of length a_n to the top edge. Gluing two copies of H_n, we obtain a pair of pants P_n as in Figure 2.

The hyperbolic lengths of the three boundary curves of P_n are $2a_n, 2b_n, 2b_n$. Note that the lengths of boundaries of P_n are long but the distances between any two boundaries are short. From these pairs of pants $\{P_n\}_{n=1}^{\infty}$, we construct a Riemann surface S_0 as follows.

Step 1. For each $k \in \mathbf{N}$, glue two copies of P_k along the boundaries of length $2a_k$. Then we obtain a Riemann surface of type $(0,4)$, say $S_{k,1}$. Let γ_k denote the "core" curve in $S_{k,1}$ with length $2a_k$ (see Figure 3).

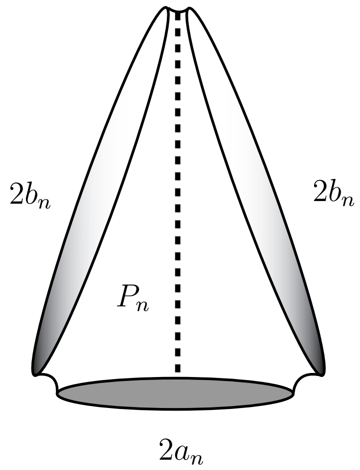

Figure 2.

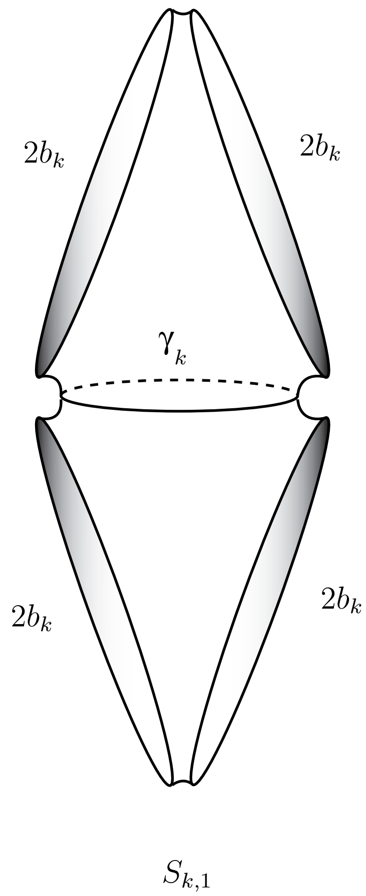

Figure 3.

Step 2. Make four copies of P_{k+1} and glue each copy with $S_{k,1}$ along their boundary curves of length $2b_k = 2a_{k+1}$. The resulting Riemann surface $S_{k,2}$ is of type $(0,8)$.

Step 3. Continue the above construction inductively. Namely, make 2^{n+1} copies of P_{k+n} and glue each copy with $S_{k,n-1}$ along their boundary curves of length $2b_{k+n-1} = 2a_{k+n}$. Then we obtain a Riemann surface $S_{k,n}$ of type $(0, 2^{n+1})$.

Step 4. Take $n(k) \in \mathbf{N}$ so large that the distance between γ_k and $\partial S_{k,n(k)}$ is greater than ka_k. Put $S^{(k)} = S_{k,n(k)}$.

From the construction, we see the following:

Observation. For any points $p, q \in \gamma_k$, let γ be a geodesic arc from p to q. Then the hyperbolic length $l_{S_0}(\gamma)$ of γ satisfies

$$l_{S_0}(\gamma) > ka_k, \tag{2}$$

if $\gamma \not\subset \gamma_k$.

Step 5. Make another copy $S^{(-k)}$ of $S^{(k)}$ for each $k \in \mathbf{N}$ and construct a Riemann surface S_0 of infinite type from $\{S^{(k)}\}_{k=-\infty}^{-1} \cup \{S^{(k)}\}_{k=1}^{\infty}$ and another pair of pants with geodesic boundaries such that each $S^{(k)}$ is isometrically embedded in S_0.

Let f_n be the positive Dehn twist for γ_n, $n \in \mathbf{N}$. Here, the "positive" Dehn twist means the Dehn twist with left turning. Set $p_n = [S_0, f_n]$ and $p_0 = [S_0, \mathrm{id}]$. The following lemma shows that $d_T(p_n, p_0) \to \infty$.

Lemma 3.1. *Let K_n be the maximal dilatation of the extremal quasiconformal mapping which is homotopic to f_n. Then, $\lim_{n\to\infty} K_n = \infty$. Thus, $\lim_{n\to\infty} d_T(p_n, p_0) = \infty$ as $n \to \infty$.*

Proof. We consider a neighbourhood U_n of γ_n which is defined by

$$U_n = \{z \in S_0 \mid d_{S_0}(z, \gamma_n) < \varepsilon\}$$

for some $\varepsilon > 0$, where $d_{S_0}(\,\cdot\,,\cdot\,)$ is the hyperbolic distance on S_0. We take $\varepsilon > 0$ small enough that U_n is conformally equivalent to an annulus. We define the Dehn twist f_n such that $f_n \mid U_n$ is the standard Dehn twist on the annulus and the identity on $S_0 \setminus U_n$. That is, $f_n \mid U_n$ is defined in terms of the polar coordinates in the annulus by

$$r \exp(i\theta) \mapsto r \exp\left\{i\left(\theta + 2\pi \frac{r-1}{R-1}\right)\right\},$$

if U_n is equivalent to $\{1 < |z| < R\}$.

Let α_n be a simple closed geodesic in $S_{n,1} \subset S_0$ perpendicular to γ_n. We consider connected components of $\pi^{-1}(U_n)$, $\pi^{-1}(\gamma_n)$ and $\pi^{-1}(\alpha_n)$ on $\mathbf{H}$, where $\pi\colon \mathbf{H} \to S_0$ is a universal covering map. We may assume that the connected component of $\pi^{-1}(\gamma_n)$ is the imaginary axis and that of $\pi^{-1}(\alpha_n)$ is $\delta := \{|z| = 1\} \cap \mathbf{H}$, the unit circle in $\mathbf{H}$. Let $\widetilde{U}_n$ denote the connected component of $\pi^{-1}(U_n)$ containing the imaginary axis.

Let $F_n\colon \mathbf{H} \to \mathbf{H}$ be a lift of an extremal quasiconformal mapping which is homotopic to f_n. We may take F_n so that $F_n(0) = 0$, $F_n(i) = i$ and $F_n(\infty) = \infty$. It is well known that F_n can be extended to a homeomorphism of $\overline{\mathbf{H}}$ and the boundary mapping $F_n \mid \mathbf{R}$ depends only on the homotopy class of f_n up to $\mathrm{Aut}(\mathbf{H})$.

Let z_1, z_2, $\mathrm{Re}\, z_1 < 0 < \mathrm{Re}\, z_2$, be the points of $\delta \cap \partial \widetilde{U}_n$. Since f_n is the positive Dehn twist, we see that $F_n(z_1) = z_1$ and $F_n(z_2) = e^{2a_n} z_2$. Hence, $F_n(\delta \cap \widetilde{U}_n)$ is an arc connecting z_1 and $e^{2a_n} z_2$ in $\widetilde{U}_n$. Applying the similar argument to a subarc of δ in each component of $\pi^{-1}(U_n)$, we see that

$$-1 < F_n(-1) < e^{2a_n} < F_n(1).$$

In particular, $\lim_{n\to\infty} F_n(1) = \infty$. Therefore, for the cross ratio

$$[a, b, c, d] = (a - b)(c - d)(a - d)^{-1}(c - b)^{-1}$$

we have

$$[-1, 0, 1, \infty] = -1$$

and

$$\lambda_n = [F_n(-1), F_n(0), F_n(1), F_n(\infty)] = [F_n(-1), 0, F_n(1), \infty] = \frac{F_n(-1)}{F_n(1)}.$$

Thus, we have $\lim_{n\to\infty} \lambda_n = 0$. Therefore, the conformal modulus of a quadrilateral $\mathbf{H}$ with vertices $F_n(-1), F_n(0), F_n(1)$ and $F_n(\infty)$ degenerates as $n \to \infty$. Since the quasiconformal mapping F_n maps a quadrilateral $\mathbf{H}$ with vertices $-1, 0, 1$ and ∞ onto the quadrilateral $\mathbf{H}$ with vertices $F_n(-1), F_n(0), F_n(1)$ and $F_n(\infty)$, we have $K_n = K(F_n) \to +\infty$. □

Next, we shall show that $d_L(p_n, p_0) \to 0$.

Let α be a closed geodesic on S_0. If $\alpha \cap \gamma_n = \emptyset$, then it is obvious that $l_{S_0}(\alpha) = l_{S_0}\big(f_n(\alpha)\big)$.

Suppose that $^{\#}(\alpha \cap \gamma_n) = m > 0$. Since each point of $\alpha \cap \gamma_n$ makes a Dehn twist, we have

$$l_{S_0}\big(f_n(\alpha)\big) \le l_{S_0}(\alpha) + m l_{S_0}(\gamma_n) = l_{S_0}(\alpha) + 2ma_n. \tag{3}$$

On the other hand, from (2) we have

$$l_{S_0}(\alpha) > mna_n. \tag{4}$$

Combining (3) and (4), we have

$$\frac{l_{S_0}\big(f_n(\alpha)\big)}{l_{S_0}(\alpha)} < 1 + \frac{1}{2n} \to 0,$$

as $n \to \infty$. Since f_n^{-1} is also a Dehn twist, from the same argument as above, we have

$$\frac{l_{S_0}(\alpha)}{l_{S_0}\big(f_n(\alpha)\big)} < 1 + \frac{1}{2n}.$$

Therefore, we note that $\lim_{n\to\infty} d_L(p_n, p_0) = 0$ and complete the proof of Theorem 1.1.

3.2. Proof of Corollary 1.1. We use the same Riemann surface S_0 and the same quasiconformal mappings f_n as in the proof of Theorem 1.1. Set $F_n = f_1 \circ f_2 \circ \cdots \circ f_n$ and $q_n = [S_0, F_n]$. Then, we see that $\lim_{m,n\to\infty} d_L(q_m, q_n) = 0$ and $\{q_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in $T(S_0)$ with respect to the length spectrum distance d_L. However, it does not converge to any point in $T(S_0)$ because $F_\infty = \prod_{n=1}^{\infty} f_n$ is not homotopic to a quasiconformal mapping. Indeed, by using the same argument as in the proof of Lemma 3.1, we see that the maximal dilatation of any homeomorphism homotopic to F_∞ is not finite. Hence, the distance d_L is not complete in $T(S_0)$.

4. Proof of Theorem 1.2

It follows from Lemma 2.1 that $d_L(p_n, p_0) \to 0$ when $d_T(p_n, p_0) \to 0$. Thus, it suffices to show that $d_T(p_n, p_0) \to 0$ as $d_L(p_n, p_0) \to 0$.

By using Lemma 2.1 again, we see that the condition of Theorem 1.2 is quasiconformal invariant, that is, any Riemann surface which is quasiconformally equivalent to S_0 satisfies the condition of Theorem 1.2 for some constant. Hence we may assume that $p_0 = [S_0, \mathrm{id}]$.

Put $p_n = [S_n, f_n] \in T(S_0)$ and assume that $\lim_{n\to\infty} d_L(p_n, p_0) = 0$. Let $\mathscr{C}_P$ be the set of closed geodesics which are boundaries of some P_k in S_0. For each $\alpha \in \mathscr{C}_P$, there exist a closed geodesic in S_n homotopic to $f_n(\alpha)$. We denote the closed geodesic by $[f_n(\alpha)]$. The set $\{[f_n(\alpha)]\}_{\alpha\in\mathscr{C}_P}$ together with punctures of S_n gives a pants decomposition of S_n. Let $P_k^{(n)}$ denote a pair of pants in the pants decomposition of S_n such that each boundary component is a closed geodesic homotopic to a component of $f_n(P_k)$ or a puncture of $\partial f_n(P_k)$.

From the definition of d_L, we have

$$\big(d_L(p_n, p_0)\big)^{-1} l_{S_0}(\alpha) \le l_{S_n}\big([f_n(\alpha)]\big) \le d_L(p_n, p_0) l_{S_0}(\alpha)$$

for any $\alpha \in \mathscr{C}_P$. As $M^{-1} \le l_{S_0}(\alpha) \le M$ for $\alpha \in \mathscr{C}_P$, we see that $\big\{l_{S_n}\big([f_n(\alpha)]\big)\big\}_{n=1}^{\infty}$ converges to $l_{S_0}(\alpha)$ uniformly on $\mathscr{C}_P$, that is, for any $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$

such that if $n \geq n_0$, then

$$\bigl|l_{S_n}\bigl([f_n(\alpha)]\bigr) - l_{S_0}(\alpha)\bigr| < \varepsilon \tag{5}$$

holds for any $\alpha \in \mathscr{C}_P$.

It is known that the lengths of boundaries determine the moduli of the pair of pants. Hence, if n is sufficiently large, then from (5) we verify that there exists a quasiconformal mapping $g_{k,n}$ of P_k onto $P_k^{(n)}$ with small dilatation. However, we need to find quasiconformal mappings g_n on the *whole surface* S_0 such that $\lim_{n\to\infty} K[g_n] = 1$. To obtain such mappings, we consider the "Fenchel–Nielsen coordinates" of the infinite-dimensional Teichmüller space $T(S_0)$. The classical Fenchel–Nielsen coordinates are defined in the Teichmüller space of Riemann surfaces of finite type. We define the coordinates in $T(S_0)$ by using an exhaustion of S_0.

We take a subregion $S_0^{(m)}$, $m = 1, 2, \dots$, of S_0 satisfying the following conditions (after rearrangement of the numbers of $\{P_k\}_{k=1}^{\infty}$).

(1) $S_0^{(m)} = \mathrm{Int}\bigl(\bigcup_{k=1}^{k(m)} \overline{P}_k\bigr)$ for some $k(m) \in \mathbf{N}$,
(2) $S_0^{(1)} \subset S_0^{(2)} \subset \cdots \subset S_0^{(m)} \subset S_0^{(m+1)} \subset \cdots$, and
(3) $S_0 = \bigcup_{m=1}^{\infty} S_0^{(m)}$.

Similarly, we take an exhaustion $\{S_n^{(m)}\}_{m=1}^{\infty}$ of S_n from $\{P_k^{(n)}\}_{k=1}^{\infty}$.

Let $\widehat{S}_0^{(m)}$, respectively $\widehat{S}_n^{(m)}$ be the Nielsen extension of $S_0^{(m)}$, respectively $S_n^{(m)}$. Since $\bigl(f_n\bigr)_*(\pi_1(S_0^{(m)})\bigr) = \pi_1(S_n^{(m)})$, we see that there exists a quasiconformal mapping $f_{n,m}\colon \widehat{S}_0^{(m)} \to \widehat{S}_n^{(m)}$ such that

$$(f_{n,m})_* \circ (\iota_0^{(m)})_* = (\iota_n^{(m)})_* \circ (f_n)_* \mid \pi_1(S_0^{(m)})$$

on $\pi_1(S_0^{(m)})$, where $\iota_0^{(m)}$ and $\iota_n^{(m)}$ are the natural inclusion maps from $S_0^{(m)}$ to $\widehat{S}_0^{(m)}$ and from $S_n^{(m)}$ to $\widehat{S}_n^{(m)}$, respectively. A pair $(\widehat{S}_n^{(m)}, f_{n,m})$ gives a point in $T(\widehat{S}_0^{(m)})$. Obviously, if $m' > m$, then

$$(f_{n,m'})_* \mid \pi_1(S_0^{(m)}) = (f_{n,m})_*. \tag{6}$$

Now, we consider the Fenchel–Nielsen coordinates (cf. [3]) on $T(\widehat{S}_0^{(m)})$ with respect to the pants decomposition given by $\{P_k\}_{k=1}^{k(m)}$. The Fenchel–Nielsen coordinates consist of length coordinates and twist coordinates. The length coordinates are the collection of lengths of boundaries of the pants decomposition and the twist coordinates are the collection of twist angles along the boundaries of the pants decomposition. From (6) or the construction of $\widehat{S}_n^{(m)}$, we see that if $\alpha \in S_0^{(m)}$ and $m' > m$, then

$$l_{\hat{S}_n^{(m)}}\bigl([f_{n,m}(\alpha)]\bigr) = l_{\hat{S}_n^{(m')}}\bigl([f_{n,m'}(\alpha)]\bigr) = l_{S_n}\bigl([f_n(\alpha)]\bigr).$$

Thus, it follows from (5) that the length coordinates of $\hat{p}_n^{(m)} = [\widehat{S}_n^{(m)}, f_{n,m}]$ converge to those of $\hat{p}_0^{(m)} = [\widehat{S}_0^{(m)}, \mathrm{id}]$ as $n \to \infty$.

Similarly, it is also seen that the twist parameter along $[f_{m,n}(\alpha)]$ does not depend on m if $[f_n(\alpha)] \subset S_n^{(m)}$. Thus, we may denote the twist parameter along $[f_n(\alpha)]$, $\alpha \in \mathscr{C}_P$, by $\theta_n(\alpha)$.

Lemma 4.1. *The sequence $\{\theta_n(\alpha)\}_{n\in\mathbf{N}}$ converges to $\theta_0(\alpha)$, the twist parameter along α of $p_0 \in T(S_0)$, uniformly on $\mathscr{C}_P$. Namely, for any $\varepsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that if $n \geq n_0$, then*

$$|\theta_n(\alpha) - \theta_0(\alpha)| < \varepsilon$$

holds for any $\alpha \in \mathscr{C}_P$.

Proof. For any closed geodesic $\alpha \in \mathscr{C}_P$, the following two cases occur (Figure 4):

(A) α is a non-dividing curve and is contained in a subregion $T(\alpha)$ of S_0 of genus one with one geodesic boundary curve in $\mathscr{C}_P$.

(B) α is a dividing curve and is contained in a planar subregion of S_0 bounded by four geodesic curves in $\mathscr{C}_P$.

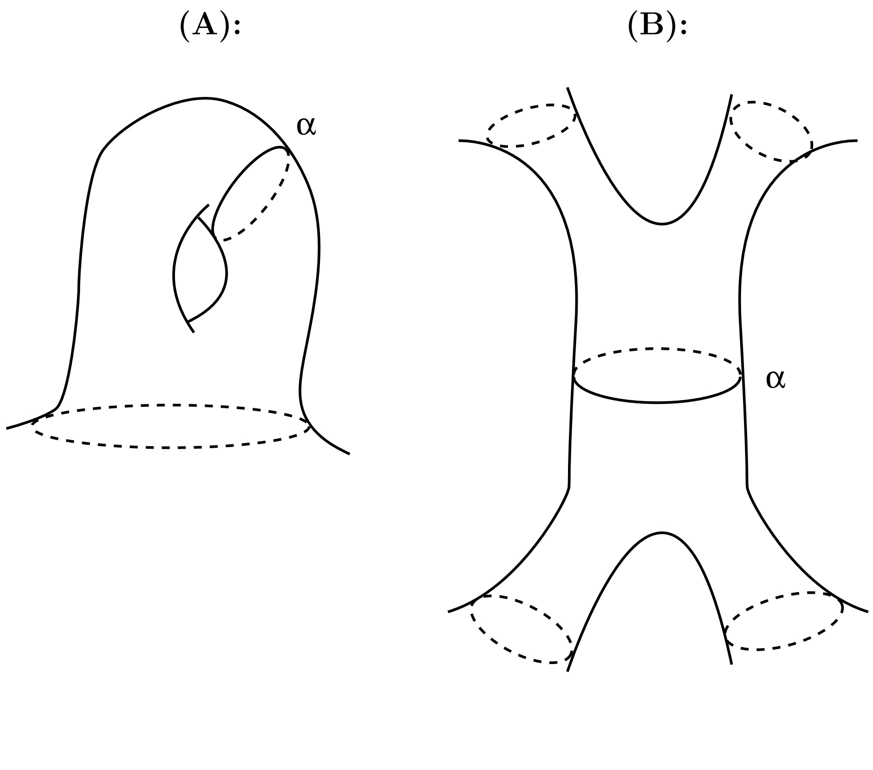

Figure 4.

We shall show the statement of the lemma in Case (A) since the proof is similar in Case (B).

Let α^* be the shortest simple closed geodesic in all closed curves intersecting α (Figure 5). From the second condition in Theorem 1.2 we verify that there exists a constant $C_1 = C_1(M) > 0$ depending only on M such that

$$0 < C_1^{-1} < l_{S_0}(\alpha^*) < C_1. \tag{7}$$

Indeed, since $l_{S_0}(\alpha) < M$, the collar theorem (cf. [2, Chapter 4]) guarantees that the geodesic $\alpha \subset S_0$ has a collar with some width depending only on M. Hence the existence of a positive lower bound of $l_{S_0}(\alpha^*)$ is obvious and the lower bound depends only on M. On the other hand, since the hyperbolic length of any boundary curve of any P_k is less than M, the hyperbolic distance between any two boundary curves of P_k is less than some constant C_1 depending only on M. Therefore, it is easy to see $l_{S_0}(\alpha^*) < C_1$.

Next, we show that there exists a constant Θ depending only on M such that

$$|\theta_n(\alpha)| < \Theta \tag{8}$$

holds for any $\alpha \in \mathscr{C}_P$ and for any $n \in \mathbf{N}$.

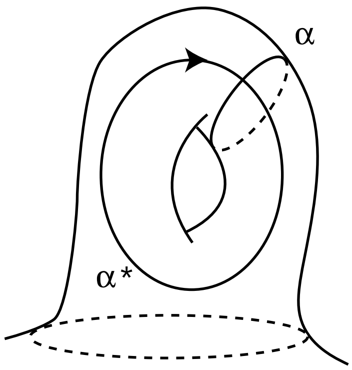

Figure 5.

Indeed, if such a constant does not exist, then there exist sequences $\{n_t\}_{t\in\mathbf{N}}$ in $\mathbf{N}$ and $\{\alpha_{n_t}\}_{t\in\mathbf{N}}$ in $\mathscr{C}_P$ such that

$$\lim_{t\to\infty} |\theta_{n_t}(\alpha_{n_t})| = +\infty.$$

Considering that $l_{S_{n_t}}(\alpha_{n_t}) > M^{-1} > 0$, we have that

$$\lim_{t\to\infty} l_{S_{n_t}}\bigl(f_{n_t}(\alpha^*_{n_t})\bigr) = +\infty.$$

Therefore, from (7) we see that

$$d_L(p_{n_t}, p_0) \geq \log \frac{l_{S_{n_t}}\big(f_{n_t}(\alpha^*_{n_t})\big)}{l_{S_0}(\alpha^*_{n_t})} \to +\infty$$

as $t \to \infty$, and we have a contradiction.

Now, we prove the lemma. Assume that $\{\theta_n(\alpha)\}_{n\in\mathbf{N}}$ does not uniformly converge to $\theta_0(\alpha)$. Then, there exist a constant $\varepsilon_0 > 0$, a sequence $\{n_t\}_{t\in\mathbf{N}}$ in $\mathbf{N}$ and $\{\alpha_{n_t}\}_{t\in\mathbf{N}}$ in $\mathscr{C}_P$ such that

$$|\theta_{n_t}(\alpha_{n_t}) - \theta_0(\alpha_{n_t})| \geq \varepsilon_0 \tag{9}$$

holds for any n_t. From this inequality and (8), we can find a simple closed geodesic β in S_0 such that

$$\big|l_{S_{n_t}}\big(f_{n_t}(\beta)\big) - l_{S_0}(\beta)\big| > \delta$$

hold for some $\delta > 0$ and for all n_t (cf. [8]). Hence, we have

$$\left|\frac{l_{S_{n_t}}\big(f_{n_t}(\beta)\big)}{l_{S_0}(\beta)} - 1\right| > \frac{\delta}{l_{S_0}(\beta)} > 0.$$

It contradicts $\lim_{t\to\infty} d_L(p_{n_t}, p_0) = 0$ and thus we complete the proof of the lemma. □

Now, the proof of Theorem 1.2 is immediate. From the boundedness of the length and the angle parameters and from the uniform convergence of them, we can construct quasiconformal mappings from $\widehat{S}_0^{(m)}$ onto $\widehat{S}_n^{(m)}$ with small dilatations. More precisely, for any $\varepsilon > 0$ there exist $n_0 \in \mathbf{N}$ and quasiconformal mappings $g_{n,m}$ from $\widehat{S}_0^{(m)}$ onto $\widehat{S}_n^{(m)}$, $m = 1, 2, \dots$, such that

(1) $g_{n,m}$ is homotopic to $f_{n,m}$.

(2) If $n \geq n_0$, then the maximal dilatation $K[g_{n,m}]$ of $g_{n,m}$ is less than $(1+\varepsilon)$.

By taking the limit of $\{g_{n,m}\}_{m\in\mathbf{N}}$ as $m \to \infty$, we have a quasiconformal mapping g_n from S_0 onto S_n with $K[g_n] \leq 1 + \varepsilon$. From the construction, $[S_n, g_n] = [S_n, f_n] = p_n$ and we conclude that $\lim_{n\to\infty} d_T(p_n, p_0) = 0$ as we desired.

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