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EXISTENCE OF BIG PIECES OF GRAPHS FOR PARABOLIC PROBLEMS

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Abstract. In this paper we define parabolic chord arc domains and generalize, to a parabolic setting, a theorem of Semmes concerning quantitative approximation by graphs with small Lipschitz constants.

1. Background and notation

In this paper, motivated by applications to the analysis of parabolic partial differential equations (see [HLN]) and parabolic singular integrals on domains not locally given by graphs, we define parabolic chord arc domains and generalize, to a parabolic setting, a theorem of Semmes concerning quantitative approximation by graphs with small Lipschitz constants.

If $X = (x_0, x)$, we note for $n > 1$ that [LM], [HL] considered graph domains of the form $\widetilde{\Omega} = \{(X,t) \in \mathbf{R}^{n+1} : x_0 > \psi(x,t)\}$, where $\psi = \psi(x,t) : \mathbf{R}^n \to \mathbf{R}$ had compact support and satisfied

$$(1.1)\quad |\psi(x,t) - \psi(y,s)| \le b_1(|x-y| + |s-t|^{1/2}) < \infty \qquad \text{for all } (x,t),(y,s) \in \mathbf{R}^n,$$

and

$$(1.2)\qquad \|D^t_{1/2}\psi\|_* \le b_2 < \infty.$$

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Here $D^t_{1/2}\psi(x,t)$ denotes the $\frac{1}{2}$ derivative in t of $\psi(x,\cdot)$, x fixed. This half derivative in time can be defined by way of the Fourier transform or by

$$D^t_{1/2}\psi(x,t) \equiv \hat{c} \int_{\mathbf{R}} \frac{\psi(x,s)-\psi(x,t)}{|s-t|^{3/2}}\, ds$$

for properly chosen $\hat{c}$. Also $\|\cdot\|_*$ denotes the bounded mean oscillation norm on $\mathbf{R}^n$ taken with respect to rectangles of side length r in the space direction and r^2 in the time direction (see the display following (2.37) for a definition). If $n=1$ the above authors considered graph domains Ω for which (1.1), (1.2) held with $\psi=\psi(t)$. Through the work of [H], [LM], [HL] it has become clear that this is the right smoothness condition to impose on the domain, from the perspective of mutual absolute continuity of caloric measure with respect to a certain projective surface measure as well as from the point of view of parabolic singular integrals. In this paper we consider sets not necessarily locally given by graphs but impose conditions on the set in such a manner that the conditions allow us to extract (in a scale invariant fashion) big and very big pieces of time-varying graphs from the set, i.e., graphs given by a functions ψ satisfying the type of smoothness conditions stated above. The results of this paper are applied in [HLN] where we obtain parabolic analogues of results by Kenig–Toro in [KT], [KT1] concerning regularity and free boundary regularity on Reifenberg flat chord arc domains.

To formulate our results we need some definitions and notation. Let (X,t), $X=(x_0,\ldots,x_{n-1})$, $t\in\mathbf{R}$ denote a point in $\mathbf{R}^{n+1}$ and for given $r>0$ set

$$C_r(X,t)=\big\{(Y,s): |Y-X|<r,\ |t-s|<r^2\big\}.$$

Given a Borel set $F\subset\mathbf{R}^{n+1}$ let $\overline{F}$, ∂F denote the closure, boundary of F, respectively, and put $\sigma(F)=\int_F d\sigma_t\, dt$ where $d\sigma_t$ is $(n-1)$-dimensional Hausdorff measure on the time slice $F\cap\big(\mathbf{R}^n\times\{t\}\big)$. Let Ω be a connected open set in $\mathbf{R}^{n+1}$. We say that $\partial\Omega$ separates $\mathbf{R}^{n+1}$ and is δ_0 Reifenberg flat, $0<\delta_0\leq 1/10$, if given any $(X,t)\in\partial\Omega$, $R>0$, there exists an n-dimensional plane $\widehat{P}=\widehat{P}(X,t,R)$ containing (X,t) and a line parallel to the t axis with unit normal $\hat{n}=\hat{n}(X,t,R)$ such that

$$\begin{aligned} &\big\{(Y,s)+r\hat{n}\in C_R(X,t):(Y,s)\in\widehat{P},\ r>\delta_0R\big\}\subset\Omega,\\ &\big\{(Y,s)-r\hat{n}\in C_R(X,t):(Y,s)\in\widehat{P},\ r>\delta_0R\big\}\subset\mathbf{R}^{n+1}\setminus\Omega. \end{aligned} \tag{1.3}$$

For short we shall just say $\partial\Omega$ separates $\mathbf{R}^{n+1}$ when (1.3) holds for some δ_0. Note that if $\partial\Omega$ separates $\mathbf{R}^{n+1}$, then a line segment drawn parallel to $\hat{n}$ and with endpoints in each of the sets in (1.3), also intersects $\partial\Omega$. Below we will, for $(X,t)\in\partial\Omega$ and $r>0$, by $\Delta(X,t,r)$ denote a surface cube defined as $\Delta(X,t,r)=\partial\Omega\cap C_r(X,t)$.

We say that $\partial\Omega$ satisfies an (M,R) Ahlfors condition, $M \geq 4$, if for all $(X,t) \in \partial\Omega$ and $0 < r \leq R$,

$$\sigma\big(\partial\Omega \cap C_r(X,t)\big) \leq M r^{n+1}. \tag{1.4}$$

Using the above note, (1.3), (1.4), the fact that Hausdorff measure does not increase under a projection we deduce for $0 < r \leq R$, $(X,t) \in \partial\Omega$,

$$\big(\tfrac{1}{2}r\big)^{n+1} \leq \sigma\big(\partial\Omega \cap C_r(X,t)\big) \leq M r^{n+1}, \tag{1.5}$$

whenever $\partial\Omega$ separates $\mathbf{R}^{n+1}$ and satisfies an (M,R) Ahlfors condition.

Next let

$$d(F_1,F_2) = \inf\big\{|X-Y| + |s-t|^{1/2} : (X,t) \in F_1,\ (Y,s) \in F_2\big\}, \tag{1.6}$$

denote the parabolic distance between the sets F_1, F_2 and for Ω as in (1.5) set

$$\gamma(Z,\tau,r) = \inf_P \left[r^{-n-3} \int_{\partial\Omega \cap C_r(Z,\tau)} d\big(\{(Y,s)\}, P\big)^2 \, d\sigma(Y,s) \right] \tag{1.7}$$

where the infimum is taken over all n-dimensional planes P containing a line parallel to the t axis. Let

$$d\nu(Z,\tau,r) = \gamma(Z,\tau,r)\, d\sigma(Z,\tau) r^{-1}\, dr. \tag{1.8}$$

We say that ν is a Carleson measure on $\big[\partial\Omega \cap C_R(Y,s)\big] \times (0,R)$ if there exists $M_1 < \infty$ such that whenever $(X,t) \in \partial\Omega$ and $C_\varrho(X,t) \subset C_R(Y,s)$, we have

$$\nu\big(\big[C_\varrho(X,t) \cap \partial\Omega\big] \times (0,\varrho)\big) \leq M_1 \varrho^{n+1}. \tag{1.9}$$

The least such M_1 in (1.9) is called the Carleson norm of $\big[\partial\Omega \cap C_R(Y,s)\big] \times (0,R)$. We write $\|\nu\|_+$ for the Carleson norm of ν when (1.9) holds for all $\varrho > 0$. As in [DS], [DS1] we say that $\partial\Omega$ is uniformly rectifiable (in the parabolic sense) if $\|\nu\|_+ < \infty$ and (1.5) holds all $R > 0$. We remark that the work in [DS], [DS1] was in turn motivated by earlier work of [Jo], [Jo1], where L^∞ versions of (1.7) were introduced. If $\partial\Omega$ separates $\mathbf{R}^{n+1}$ and is uniformly rectifiable, we shall call Ω a parabolic regular domain. Finally in analogy with [KT] we say that Ω is a chord arc domain with vanishing constant if Ω is a parabolic regular domain and

$$\sup_{\substack{(X,t)\in\partial\Omega \\ 0<\varrho\leq r}} \big[\varrho^{-(n+1)} \nu\big(\big[C_\varrho(X,t) \cap \partial\Omega\big] \times (0,\varrho)\big)\big] \to 0 \quad \text{as } r \to 0. \tag{1.10}$$

In the following we are interested in showing the existence of big pieces of graphs, in particular we prove for parabolic regular domains that if δ_0 in (1.3) is sufficiently small, then $\partial\Omega$ contains big pieces of time varying graphs in the following sense. Given $(X,t) \in \partial\Omega$ and $R > 0$, there exists after a possible rotation in the space variable, a function ψ as in (1.1), (1.2) and $\widetilde{\Omega}$ as above such that for some $M_2 = M_2(M, \|\nu\|_+, \delta_0)$

$$M_2 \sigma\big(\partial\widetilde{\Omega} \cap \Delta(X,t,R)\big) \geq R^{n+1}. \tag{1.11}$$

We will also obtain a parabolic analogue of Semmes decomposition, [S], concerning very big pieces of time varying graph domains when $\|\nu\|_+$ is sufficiently small.

Given Ω as in (1.3) let $(X,t) \in \partial\Omega$ and let $\|\nu\|$ be the Carleson norm of ν on $\Delta(X,t,800R) \times (0,800R)$ (see (1.9)). We first prove

Theorem 1. *If Ω satisfies (1.3), (1.5), and $\|\nu\| < \infty$, then there exists $c(M)$ such that if $0 < \delta_0 \leq c(M)^{-1}$ in (1.3), then $\partial\Omega$ contains big pieces of time-varying graphs in the sense of (1.11) with $b_1 + b_2 \leq c(M)(1 + \|\nu\|)^{1/2}$ in (1.1), (1.2). Moreover for $(X,t), R$ as in (1.11), we can make the construction so that either $\widetilde{\Omega} \cap C_{2R}(X,t) \subset \Omega$ or $\Omega \cap C_{2R}(X,t) \subset \widetilde{\Omega}$.*

Concerning very big pieces we obtain the following parabolic version of the theorem of Semmes for chord arc domains with small constants (see [S], [S1], [KT]).

Theorem 2. *Let $\Omega, \delta_0, (X,t)$ be as in Theorem 1 and suppose for some $\hat{\delta}$, $0 < \hat{\delta} \leq \delta_0$, that $\|\nu\| = \delta^{\kappa}$ with $\delta \leq \hat{\delta}$ and $\kappa = 24(n+3)\max\{n-1, 1\}$. If $\hat{\delta} = \hat{\delta}(M)$ is small enough, then $\Delta(X,t,R)$ contains very big pieces of time-varying graphs with small constants in the following sense. There exists after a possible rotation in the space variable, $\psi, \widetilde{\Omega}, c_2 = c_2(M)$ satisfying (1.1), (1.2) and*

(a) $b_1 + b_2 \leq c_2\delta$.

(b) $\sigma\big(\Delta(X,t,R) \setminus \partial\widetilde{\Omega}\big) + \sigma\big(\partial\widetilde{\Omega} \cap C_R(X,t) \setminus \partial\Omega\big) \leq e^{-1/(c_2\delta)} R^{n+1}$.

(c) $\Delta(X,t,2R) = G \cup B$, *where* $G \subset \partial\widetilde{\Omega}$ *and* $\sigma(B) \leq e^{-1/(c_2\delta)} R^{n+1}$.

(d) *If p denotes the orthogonal projection of $\mathbf{R}^{n+1}$ on $\{0\} \times \mathbf{R}^n$, $d(\cdot,\cdot)$ is as in (1.6), and $(Y,s) \in B$, then $|y_0 - \psi(y,s)| \leq c_2\delta d\big(\{p(Y,s)\}, p(G)\big)$,*

(e) *We can make the construction so that*

$$\text{either} \quad \widetilde{\Omega} \cap C_R(X,t) \subset \Omega \quad \text{or} \quad \Omega \cap C_R(X,t) \subset \widetilde{\Omega}.$$

The rest of the paper is devoted to the proof of Theorem 1 and 2. The two sections of the paper are organized as follows. In Section 2 we start by jointly treating Theorem 1 and 2. Our proof scheme is motivated by the work in [DS]. Because of the refined analysis needed in the proof of Theorem 2 just the proof of Theorem 1 is completed in that section. In Section 3 we finish the proof of Theorem 2.

Finally in a remark following the proof of Theorem 2, we point out the differences between chord arc domains with small constants in the sense of [KT] and our definition (using smallness of the Carleson norm in (1.9)). In the first case it turns out that there are several equivalent definitions, including one which is an elliptic analogue of our definition. However, in the parabolic situation our definition is the only correct one, as we show by examples.

2. Proof of the theorems

In the sequel c denotes a positive constant ≥ 1, depending only on n. In general $c(a_1, \dots, a_m)$ denotes a positive constant ≥ 1, depending only on $n, a_1, \dots, a_m$, not necessarily the same at each occurrence. Given Ω as in (1.3) let $(X,t) \in \partial\Omega$ and let $\|\nu\|$ be the Carleson norm of ν on $\Delta(X,t,800R) \times (0,800R)$ (see (1.9)).

Proof. If $n=1$ then (d) of Theorem 2, $\psi(y,s)$ should be replaced by $\psi(s)$. To simplify matters we shall assume that $n>1$ in the proof of Theorems 1 and 2. At the end of the proofs, we briefly treat the case $n=1$. We shall also just make the construction so that $\widetilde{\Omega} \cap C_R(X,t,R) \subset \Omega$.

We begin the proof of both theorems together since their proofs involve many of the same steps. We introduce for $(Z,\tau) \in \partial\Omega$, $r>0$,

$$\gamma_\infty(Z,\tau,r) = \inf_P \sup_{(Y,s)\in\Delta(Z,\tau,r)} \frac{d(Y,s,P)}{r}, \tag{2.1}$$

where the infimum is taken over all n planes P containing a line parallel to the t axis. Given $(Z,\tau), r$ as above we claim that

$$\gamma_\infty(Z,\tau,r)^{n+3} \leq 16^{n+3}\gamma(Z,\tau,2r). \tag{2.2}$$

To prove (2.2) we note that the infimum in the definition of γ (see (1.7)) is actually obtained for some plane say $\widetilde{P}$. Clearly there exists $(Z_1,\tau_1) \in \bar{\Delta}(Z,\tau,r)$ with

$$\varrho = r\gamma_\infty(Z,\tau,r) \leq d\big(\{(Z_1,\tau_1)\}, \widetilde{P}\big).$$

From (1.5) we have $\sigma\big(\Delta(Z_1,\tau_1,\varrho/8)\big) \geq (\varrho/16)^{n+1}$. Moreover every point in $\Delta(Z_1,\tau_1,\varrho/8)$ is contained in $\Delta(Z,\tau,2r)$ and lies at least $\frac{1}{2}\varrho$ from $\widetilde{P}$. Thus, (2.2) is valid. Integrating (2.2) with respect to $d\sigma(z,\tau)r^{-1}\,dr$ over $\Delta(Y,s,4\varrho)\times(0,4\varrho)$ and making simple estimates we conclude that

$$\gamma_\infty(Y,s,\varrho)^{n+3} \leq c\|\nu\| \tag{2.3}$$

whenever $(Y,s) \in \Delta(X,t,100R)$ and $0<\varrho\leq 100R$. From (2.3) we see in the situation of Theorem 2 that there exists $c_3 = c_3(n)$ with

$$\gamma_\infty(Y,s,\varrho) \leq c_3\delta^{\kappa/(n+3)} \quad \text{for all } (Y,s)\in\Delta(X,t,100R),\ 0<\varrho\leq 100R. \tag{2.4}$$

Let $\delta_1 = \delta_0$ in the proof of Theorem 1 and $\delta_1 = 2c_3\delta^{\kappa/(n+3)}$ in the proof of Theorem 2. From the definition of γ_∞ we see when $\delta_1 = 2c_3\delta^{\kappa/(n+3)} < 1/10$ that (1.3) still holds with $(X,t), R, \delta_0$ replaced by $(Y,s), \varrho, \delta_1$ provided $\widehat{P}(Y,s,\varrho)$ is replaced by a plane through (Y,s) which is parallel to the plane that minimizes $\gamma_\infty(Y,s,\varrho)$. From now on we denote this minimizing plane (for ease of notation) by $\widehat{P}(Y,s,\varrho)$ whenever $(Y,s) \in \Delta(X,t,100R)$ and $0 < \varrho \leq 100R$. In the proof of Theorems 1 and 2 we assume, as we may, thanks to (1.3) that $\widehat{P}(X,t,R) = \big\{(Y,s) \in \mathbf{R}^{n+1} : y_0 = -10\delta_1 R\big\}$ and

$$\begin{aligned}\big\{(-10\delta_1 R + r, y, s) \in C_R(X,t),\ r > \delta_1 R\big\} \subset \Omega,\\ \hat{n}(X,t,R) = e_0 = (1,0,\ldots,0).\end{aligned}$$

As in Theorem 2 let $p(Y,s) = (0,y,s)$ be the orthogonal projection of $\mathbf{R}^{n+1}$ onto $\{0\} \times \mathbf{R}^n$ and observe from (1.3) as in (1.5) that

$$\big(\tfrac{1}{2}R\big)^{n+1} \leq H^n\big(p\big[\Delta(X,t,R)\big]\big), \tag{2.5}$$

where H^n denotes Hausdorff n measure on $\{0\} \times \mathbf{R}^n$.

Let E be the set of all points $(Y,s) \in \Delta(X,t,R)$ for which there exists r, $0 < r \leq R$, with

$$H^n\big(p\big[\Delta(Y,s,r)\big]\big) \leq \theta r^{n+1}, \tag{2.6}$$

where $\theta = 1000^{-(n+1)} M^{-1}$. Thus $E \subset \Delta(X,t,R)$ consists of those points (Y,s) such that $\Delta(Y,s,r)$ has a small projection on $\{0\} \times \mathbf{R}^n$ for some $r > 0$. Using a well-known covering argument we see there exists a covering of E by sets $\big\{\Delta(Y_i,s_i,r_i)\big\}$ with $\big\{\Delta\big(Y_i,s_i,\tfrac{1}{5}r_i\big)\big\}$ pairwise disjoint and $(Y_i,s_i) \in E$. Moreover (2.6) holds with Y,s,r replaced by Y_i,s_i,r_i for each i and $0 < r_i \leq R$. Using (1.5), (2.5) we get

$$\begin{aligned}H^n\big(p(E)\big) &\leq \sum_i H^n\big(p\big[\Delta(Y_i,s_i,r_i)\big]\big)\\ &\leq \theta \sum_i r_i^{n+1} \leq 10^{n+1}\theta \sum_i \sigma\big(\Delta\big(Y_i,s_i,\tfrac{1}{5}r_i\big)\big)\\ &\leq 10^{n+1}\theta\sigma\big(\Delta(X,t,2R)\big) \leq 2^{-(n+2)}R^{n+1}\\ &\leq \tfrac{1}{2}H^n\big(p\big[\Delta(X,t,R)\big]\big).\end{aligned}$$

Thus if $F = \Delta(X,t,R) \setminus E$, then

$$H^n\big(p(F)\big) \geq \tfrac{1}{2}H^n\big(p\big[\Delta(X,t,R)\big]\big) \geq 2^{-(n+2)}R^{n+1}. \tag{2.7}$$

Next suppose $(Y,s) \in F$, $0 < r \le \frac{1}{2}R$, $\hat{n}$ is the normal in (1.3) defined relative to $\widehat{P}(Y,s,r)$ and $\eta = |\langle \hat{n}, e_0\rangle|$. We consider two cases. First if $\eta \le 1/10$ observe that $\widehat{P}(Y,s,r)$ and $p(\mathbf{R}^{n+1})$ intersect in an $(n-1)$-dimensional plane containing a line parallel to the t axis on which p is the identity mapping. Also $e' = e_0 - \langle \hat{n}, e_0\rangle \hat{n}$ lies in a plane through 0 that is parallel to $\widehat{P}(Y,s,r), |e'| \ge \frac{1}{2}$, and $|p(e')| \le \eta$. Hence p maps $K = C_r(Y,s) \cap \widehat{P}(Y,s,r)$ onto a set with H^n measure $k\eta r^{n+1}$, where $c^{-1} \le k \le c$. Since each point of $\Delta(Y,s,r)$ lies at most $\delta_1 r$ from $\widehat{P}$ and $p(K)$ is convex we conclude from the above analysis that

$$c^{-1}(\eta + \delta_1) r^{n+1} \le H^n\big(p\big[\Delta(Y,s,r)\big]\big) \le c(\eta + \delta_1) r^{n+1}.$$

From this inequality we deduce for $1/10 \ge \eta$, that there exists $c_+ = c_+(n) \ge 1$, such that if (2.6) is false, then

$$(c_+ M)^{-1} \le \eta \tag{2.8}$$

On the other hand, if $\eta \ge 1/10$, then (2.8) clearly holds so (2.8) is true in both cases. We note that if $(Y,s) \in F$, (the set with big projections), then (2.8) is true for $0 < r \le R$. Next we use (2.8) to show that if $(Y,s),(Z,\tau) \in F$, then

$$|Z - Y| \le cM(|z-y| + |s-t|^{1/2}). \tag{2.9}$$

In fact if $r = 2(|Z-Y| + |s-\tau|^{1/2})$ and $r \ge R$, then (2.9) follows from (1.3) for $\delta_1, \widehat{P}(X,t,R)$. Otherwise we write $Z = Z' + Z''$, $Y = Y' + Y''$ where (Y', s), (Z', τ) are the orthogonal projections of (Y,s), (Z,τ) onto $\widehat{P}(Y,s,r)$ and

$$|Y'' - Z''| \le \delta_1 r. \tag{2.10}$$

Moreover using the same argument and notation as in the proof of (2.8) we have

$$|z' - y'| \ge \eta |Z' - Y'|. \tag{2.11}$$

Using (2.8), (2.10), (2.11), the triangle inequality and $\delta_1 \le \eta/1000$, we get

$$|Z - Y| \le \delta_1 r + (1/\eta)|z' - y'| \le r/100 + (1/\eta)|z-y| \le r/100 + c_+ M|z-y|.$$

Clearly this inequality implies (2.9). From (2.9) we see that p is invertible on $F \cap (\mathbf{R}^{n-1} \times \{s\})$ for each $s \in \mathbf{R}$. Thus there exists $\psi^*\colon p(F) \to \mathbf{R}$ with

$$|\psi^*(y,s) - \psi^*(z,\tau)| \le cM(|y-z| + |s-\tau|^{1/2}) \tag{2.12}$$

and $\big(\psi^*(y,s), y, s\big) \in F$ for all $(y,s) \in p(F)$.

We now use our Carleson measure assumption and argue as in [DS, Chapter 13]. Let

$$f(Z,\tau)=\int_0^{100R}\gamma(Z,\tau,r)r^{-1}\,dr,\qquad (Z,\tau)\in\Delta(X,t,R),$$

and observe from (1.8), (1.9) that

$$\int_{\Delta(X,t,100R)} f(Z,\tau)\,d\sigma(Z,\tau)\le\|\nu\|(100R)^{n+1}.$$

From this inequality and 'Tchebychev's inequality' we find for $A\ge 1000$,

$$\sigma\big(\big\{(Z,\tau)\in\Delta(X,t,100R):f(Z,\tau)\ge A^{n+1}\|\nu\|\big\}\big)\le(100R/A)^{n+1}\le(R/10)^{n+1}.$$

Using the above inequality, (2.7), the fact that Hausdorff measure does not increase under a projection we deduce the existence of a closed set $F_1=F_1(A)$ with $F_1\subset F$,

$$f(Z,\tau)\le A^{n+1}\|\nu\|,\quad (Z,\tau)\in F_1, \tag{2.13}$$

and

$$H^n\big(p(F_1)\big)\ge 2^{-(n+3)}R^{n+1}. \tag{2.14}$$

We remark that we shall later obtain better estimates for the σ measure of a set F_1 as above, when A is small (depending on δ). To extend ψ^* off of $p(F_1)$ we identify $\mathbf{R}^n$ with $\{0\}\times\mathbf{R}^n$ and put

$$Q_r(z,\tau)=\big\{(y,s)\in\mathbf{R}^n:|y_i-z_i|<r,\ i=1,\dots,n-1,\ |s-\tau|<r^2\big\}$$

for $(z,\tau)\in\mathbf{R}^n$, $r>0$. Let $\big\{\bar{Q}_i=\bar{Q}_{r_i}(\hat{x}_i,\hat{t}_i)\big\}$ be a Whitney decomposition of $\mathbf{R}^n\setminus p(F_1)$ into rectangles. That is, $Q_i\cap Q_j=\emptyset$, $i\neq j$, and

$$10^{-10n}d\big(Q_i,p(F_1)\big)\le r_i\le 10^{-8n}d\big(Q_i,p(F_1)\big). \tag{2.15}$$

Let $\{v_i\}$ be a partition of unity adapted to $\{Q_i\}$. Thus,

(2.16)

(a) $\sum_i v_i\equiv 1$ on $\mathbf{R}^n\setminus p(F_1)$,

(b) $v_i\equiv 1$ on Q_i and $v_i\equiv 0$ in $\mathbf{R}^n\setminus\bar{Q}_{2r_i}(\hat{x}_i,\hat{t}_i)$ for all i,

(c) v_i is infinitely differentiable on $\mathbf{R}^n$ with

$$r_i^{-l}\left|\frac{\partial^l}{\partial x^l}v_i\right|+r_i^{-2l}\left|\frac{\partial^l}{\partial t^l}v_i\right|\le c(l,n)\text{ for } l=1,2,\dots.$$

In (c), $\partial^l/\partial x^l$ denotes an arbitrary lth partial in the space variable x. Set

$$C_\varrho(z,\tau) = \big\{(y,s) \in \mathbf{R}^n : |z-y| < \varrho,\ |\tau - s| < \varrho^2\big\} \quad \text{whenever } (y,s) \in \mathbf{R}^n,\ \varrho > 0.$$

Next for each i choose $(x_i', t_i') \in p(F_1)$ with

$$\varrho_i = d\big(\{(x_i',t_i')\}, Q_i\big) = d\big(p(F_1), Q_i\big)$$

and set $\Lambda = \big\{i : \bar{Q}_i \cap \bar{C}_{2R}(x,t) \neq \emptyset\big\}$. We consider two cases. Under the assumptions of Theorem 1 we put

$$\tag{2.17} \psi(y,s) = \begin{cases} \psi^*(y,s), (y,s) \in p(F_1), \\ \displaystyle\sum_{i\in\Lambda} \big(\psi^*(x_i',t_i') + c^* M \varrho_i\big) v_i(y,s),\ (y,s) \in \mathbf{R}^n \setminus p(F_1). \end{cases}$$

With ψ now defined on $\mathbf{R}^n$ we can use (2.12), (2.15)–(2.17), and the usual Whitney type argument to get that (1.1) holds with b_1 replaced by cM. One needs to be slightly careful when (y,s) is in the closure of two cubes say Q_i, Q_j with $i \in \Lambda$, $j \notin \Lambda$. However this case follows easily from the fact that $|\psi^*| \le cMR$ and $|\partial v_k/\partial y_l|(y,s) \le c/R$ for $1 \le l \le n-1$, $k = i,j$. Also $\psi \equiv 0$ in $\mathbf{R}^n \setminus C_{4R}(X,t)$ and $\widetilde{\Omega} \cap C_{2R}(X,t) \subset \Omega$ for c^* large enough since otherwise (2.8) would be false. The proof of Theorem 1 is now complete except for (1.2). We continue under the scenario of Theorem 2. Let

$$h(K_1,K_2) = \sup_{(Y,s)\in K_1} d\big(\{(Y,s)\}, K_2\big) + \sup_{(Y,s)\in K_2} d\big(\{(Y,s)\}, K_1\big)$$

be the parabolic Hausdorff distance between K_1 and K_2. From (1.3) it is easily deduced that for $(Z,\tau), (Y,s) \in \Delta(X,t,100R)$ and $0 < \frac{1}{4} r \le \varrho \le 4r \le 100R$, we have

$$\tag{2.18} \begin{aligned} &(\alpha) \quad |\hat{n}(Y,s,r) - \hat{n}(Z,\tau,\varrho)| \le 10^5 \delta_1 \text{ whenever } |Y-Z| + |s-\tau|^{1/2} \le 4r, \\ &(\beta) \quad h\big(\widehat{P}(Y,s,r) \cap C_r(Y,s), \widehat{P}(Z,\tau,\varrho) \cap C_r(Y,s)\big) \le 10^5 \delta_1 r,\ (Z,\tau), \\ &\qquad\ (Y,s),\ r,\ \varrho \text{ as in } (\alpha). \end{aligned}$$

We now extend the definition of ψ^* from $p(F_1)$ to $\mathbf{R}^n$. Let

$$\begin{aligned} (X_i', t_i') &= (\psi^*(x_i',t_i'), x_i', t_i'), \\ \widehat{P}(X_i', t_i', \varrho_i) &= \big\{\big(\varphi_i(y,s), y, s\big) : (y,s) \in \mathbf{R}^n\big\} \text{ for } i \in \Lambda \end{aligned}$$

which defines φ_i: $\mathbf{R}^n \to \mathbf{R}$ and as in [DS] put

$$\tag{2.19} \hat{\psi}(y,s) = \begin{cases} \psi^*(y,s), & \text{for } (y,s) \in p(F_1), \\ \displaystyle\sum_{i\in\Lambda} \varphi_i(y,s) v_i(y,s) & \text{when } (y,s) \in \mathbf{R}^n \setminus p(F_1). \end{cases}$$

We note from (2.15) that

$$r_i/r_j \le \varrho_i/r_j \le c \quad \text{whenever } \overline{Q}_i \cap \overline{Q}_j \ne \emptyset. \tag{2.20}$$

We claim for $(y,s),(z,\tau) \in \mathbf{R}^n$ that

$$\begin{aligned} &(+) \quad |\varphi_i(y,s) - \varphi_i(z,\tau)| \le cM(|y-z| + |s-\tau|^{1/2}) \text{ for all } i,\\ &(++) \quad |\varphi_i(y,s) - \varphi_j(y,s)| \le cM^2\delta_1 \min(\varrho_i,\varrho_j) \text{ when } (y,s) \in \overline{Q}_i\\ &\qquad\quad \text{and } \overline{Q}_i \cap \overline{Q}_j \ne \emptyset. \end{aligned} \tag{2.21}$$

(2.21)(+) is a direct consequence of (2.8). Also from (2.18), (2.8) and elementary geometry, we have for $(y,s) \in \overline{Q}_i$ and $\overline{Q}_i \cap \overline{Q}_j \ne \emptyset$,

$$|\varphi_i(y,s) - \varphi_j(y,s)| \le cMd\big(\{(\varphi_i(y,s),y,s)\}, \widehat{P}(X'_j,t'_j,\varrho_j)\big) \le cM^2\delta_1 \min(\varrho_i,\varrho_j)$$

which is (2.21)(++).

From (2.18)–(2.21) and (2.16) it is easily deduced for δ_1 small enough that

$$\begin{aligned} &(*) \quad |\hat\psi(y,s) - \hat\psi(z,\tau)| \le cM(|y-z| + |s-\tau|^{1/2}),\\ &(**) \quad \left|\frac{\partial^2}{\partial y_l \partial y_k}\hat\psi\right|(y,s) + \left|\frac{\partial}{\partial s}\hat\psi\right|(y,s) \le cM^2\delta_1\varrho_i^{-1} \text{ for } (y,s) \in \overline{Q}_i,\\ &\qquad 1 \le l,k \le n-1. \end{aligned} \tag{2.22}$$

For example, if $(y,s) \in \overline{Q}_i$ and $\lambda = \{j : \overline{Q}_j \cap \overline{Q}_i \ne \emptyset\} \subset \Lambda$, then $\partial^2\varphi_j/\partial y_l \partial y_k \equiv 0$ since ϕ_j is linear so

$$\frac{\partial^2 \hat\psi}{\partial y_l \partial y_k} = \sum_{j\in\lambda}\left[\frac{\partial \varphi_j}{\partial y_l}\frac{\partial v_j}{\partial y_k} + \frac{\partial \varphi_j}{\partial y_k}\frac{\partial v_j}{\partial y_l} + \varphi_j \frac{\partial^2}{\partial y_l \partial y_k} v_j\right] = T_1 + T_2 + T_3.$$

From (2.21)(++) we deduce first that $|\nabla\varphi_i - \nabla\varphi_j| \le cM^2\delta_1$ when $\overline{Q}_i \cap \overline{Q}_j \ne \emptyset$ and second at $(y,s) \in \overline{Q}_i$ that

$$|T_1| + |T_2| \le \sum_{j\in\lambda}\left[\left|\frac{\partial\varphi_i}{\partial y_l} - \frac{\partial\varphi_j}{\partial y_l}\right|\left|\frac{\partial v_j}{\partial y_k}\right| + \left|\frac{\partial\varphi_i}{\partial y_k} - \frac{\partial\varphi_j}{\partial y_k}\right|\left|\frac{\partial v_j}{\partial y_l}\right|\right] \le cM^2\delta_1\varrho_i^{-1}.$$

Also at (y,s),

$$|T_3| \le \sum_{j\in\lambda}|\varphi_i - \varphi_j|\left|\frac{\partial^2}{\partial y_l \partial y_k}v_j\right| \le cM^2\delta_1\varrho_i^{-1}.$$

Combining these inequalities we get (2.22)$(**)$ in this case.

Let $\Gamma = \{(\hat\psi(y,s),y,s) : (y,s) \in \mathbf{R}^n\}$ and suppose that γ', ν' are the measures in (1.7)–(1.9) corresponding to Γ.

We shall prove that $\|\nu'\|_+ \le c(M)(\delta_1^2 + \|\nu\|)$. That is, we shall show that we can control the relevant Carleson norm with respect to Γ using δ_1 and the local Carleson norm, $\|\nu\|$, defined relative to $\Delta(X,t,800R)$. To begin we claim that if $(Y,s) \in \Gamma \cap \overline{C}_{100R}(X,t)$, then

$$d(\{(Y,s)\},\partial\Omega) \le \tilde{c}M^2\delta_1 d(\{(y,s)\},p(F_1)). \tag{2.23}$$

To prove this inequality note that if $(Y,s) \in F_1$, then this inequality is trivally true so assume $(Y,s) = (\hat\psi(y,s),y,s)$ with $(y,s) \in \overline{Q}_i$. If $\overline{Q}_i \cap \overline{Q}_j \ne \emptyset$ for some $j \notin \Lambda$, then from (2.15) we have $c\varrho_i \ge R$ which implies (2.23) in view of (1.3). Otherwise, from (2.21) (++), (1.3) we have

$$\begin{aligned} d(\{(Y,s)\},\partial\Omega) &\le d(\{(Y,s)\},\widehat{P}(X_i',t_i',\varrho_i)) + c\delta_1\varrho_i \\ &\le |\hat\psi(y,s) - \phi_i(y,s)| + c\delta_1\varrho_i \le cM^2\delta_1\varrho_i. \end{aligned}$$

Thus (2.23) is true. We continue under the assumption that δ_1 is so small that $\tilde{c}M^2\delta_1 \le 10^{-6}$.

Let $\Gamma_i = \{(Y,s) \in \Gamma : (y,s) \in Q_i\}$ and put $\hat{c} = 2\tilde{c}M^2$. Then from (2.23) we see there exists a covering of $\overline{\Gamma}_i \subset C_{100R}(X,t)$ by cylinders $\{C_{\hat{c}\delta_1\varrho_i}(Z_j,\tau_j)\}$ with centers, $(Z_j,\tau_j) \in \partial\Omega$ and $\{C_{\hat{c}\delta_1\varrho_i/5}(Z_j,\tau_j)\}$ pairwise disjoint. We note that each point in $\cup C_{\hat{c}\delta_1\varrho_i}(Z_j,\tau_j)$ lies in at most $c(n)$ cylinders in the covering as follows from disjointness of the smaller cylinders and the fact that if $\overline{C}_{\hat{c}\delta_1\varrho_i}(Z_j,\tau_j) \cap \overline{C}_{\hat{c}\delta_1\varrho_i}(Z_l,\tau_l) \ne \emptyset$ then $\overline{C}_{\hat{c}\delta_1\varrho_i}(Z_j,\tau_j) \subset C_{4\hat{c}\delta_1\varrho_i}(Z_l,\tau_l)$. Let P be a plane containing a line parallel to the t axis. For $(Z,\tau) \in \overline{\Gamma}_i \cap \overline{C}_{\hat{c}\delta_1\varrho_i}(Z_j,\tau_j)$ we have

$$d(\{(Z,\tau)\},P) \le cM^2\delta_1\varrho_i + \min_{(Y,s)\in\bar{\Delta}(Z_j,\tau_j,\hat{c}\delta_1\varrho_i)} d(\{(Y,s)\},P).$$

Let $(\widehat{X},\hat{t}\,) \in F_1, \overline{C}_r(\widehat{X},\hat{t}\,) \subset C_{80R}(X,t), \Gamma_{i,j} = \Gamma_i \cap C_{\hat{c}\delta_1\varrho_i}(Z_j,\tau_j)$ and put

$$\partial\Omega_i = \{(Y,s) \in \partial\Omega : d(\{(Y,s)\},\Gamma_i) \le 2\hat{c}\delta_1\varrho_i\}.$$

We note that if $\overline{\Gamma}_{i,j} \cap C_r(\widehat{X},\hat{t}\,) \ne \emptyset$, then $C_{\hat{c}\delta_1\varrho_i}(Z_j,\tau_j) \subset C_r(\widehat{X},\hat{t},\tfrac{5}{4}r)$ as follows from (2.15) and the fact that $\varrho_i \le 2r$. Also, $\sigma(\overline{\Gamma}_i) \le cM\varrho_i^{n+1}$ since from (2.22) (*)

$$d\sigma(Y,s) = \sqrt{1 + |\nabla\hat\psi(y,s)|^2}\,dy\,ds \le 2M\,dH^n(y,s) \qquad \text{for } (Y,s) \in \Gamma. \tag{2.24}$$

Here $\nabla\hat\psi$ denotes the gradient of $\hat\psi$ in the space variable only. Using these notes, the above inequality, (1.5), we get

$$\int_{\Gamma_i\cap C_r(\hat{X},\hat{t}\,)} d(\{(Y,s)\},P)^2\,d\sigma(Y,s) \le \sum_j \int_{\Gamma_{i,j}\cap C_r(\hat{X},\hat{t}\,)} d(\{(Y,s)\},P)^2\,d\sigma(Y,s)$$

$$\begin{aligned}&\le c(M^2\delta_1\varrho_i)^2\sigma\big(\Gamma_i\cap C_r(\widehat X,\hat t\,)\big)\\
(2.25)\quad &+cM\sum_j\int_{C_{5r/4}(\hat X,\hat t\,)\cap\Delta(Z_j,\tau_j,\hat c\delta_1\varrho_i)} d\big(\{(Y,s)\},P\big)^2\,d\sigma(Y,s)\\
&\le c(M)\bigg[\delta_1^2\varrho_i^{n+3}+\int_{C_{5r/4}(\hat X,\hat t\,)\cap\partial\,\Omega_i} d\big(\{(Y,s)\},P\big)^2\,d\sigma(Y,s)\bigg].\end{aligned}$$

Finally we observe for fixed l that $\partial\,\Omega_l\cap\partial\,\Omega_k\neq\emptyset$ for at most $c(n)$ integers k as we find from (2.15). We conclude from this observation upon summing (2.25) that if

$$\xi=\xi(\widehat X,\hat t,r)=\big\{i:\Gamma_i\cap C_r(\widehat X,\hat t\,)\neq\emptyset\big\},$$

then

$$\begin{aligned}&\int_{\Gamma\cap C_r(\hat X,\hat t\,)} d\big(\{(Y,s)\},P\big)^2\,d\sigma(Y,s)\\
(2.26)\qquad &\le c(M)\bigg[\delta_1^2\sum_{i\in\xi}\varrho_i^{n+3}+\int_{\partial\,\Omega\cap C_{5r/4}(\hat X,\hat t\,)} d\big(\{(Y,s)\},P\big)^2\,d\sigma(Y,s)\bigg].\end{aligned}$$

Let γ', ν' be the measures in (1.7)–(1.9) corresponding to Γ. Using (2.22), (2.26) we shall show that

$$(2.27)\qquad \|\nu'\|_+\le c(M)(\delta_1^2+\|\nu\|),$$

where $\|\nu'\|_+$ denotes the Carleson norm of ν' relative to $\Gamma\times(0,\infty)$ as defined after (1.9). To do this first suppose $(\widehat X,\hat t\,)\in F_1$ and $C_r(\widehat X,\hat t\,)\subset C_{80R}(X,t)$. Then from (2.26) and the fact that $F_1\subset\partial\,\Omega\cap\Gamma$ we get

$$(2.28)\qquad \gamma'(\widehat X,\hat t,r)\le c(M)\bigg[\delta_1^2\sum_{i\in\xi}(\varrho_i/r)^{n+3}+\gamma\big(\widehat X,\hat t,\tfrac54 r\big)\bigg].$$

For given $\hat\varrho>0,(\widehat Z,\hat\tau)\in\Gamma$ with $C_{\hat\varrho}(\widehat Z,\hat\tau)\subset C_{20R}(X,t)$, we integrate (2.28) over $F_1\cap C_{\hat\varrho}(\widehat Z,\hat\tau)$. If $F_1\cap C_{\hat\varrho}(\widehat Z,\hat\tau)=\emptyset$ the following inequality is trivially true. Otherwise, summing and interchanging the order of integration we obtain with $r_i'(\widehat X,\hat t\,)=d\big(\{(\widehat X,\hat t\,)\},\Gamma_i\big)+\varrho_i$,

$$\begin{aligned}\nu'\big[F_1\cap C_{\hat\varrho}(\widehat Z,\hat\tau)\times(0,\hat\varrho)\big]&=\int_0^{\hat\varrho}\int_{F_1\cap C_{\hat\varrho}(\hat Z,\hat\tau)}\gamma'(\widehat X,\hat t,r)\,d\sigma(\widehat X,\hat t\,)r^{-1}\,dr\\
&\le c(M)\delta_1^2\int_{F_1\cap C_{\hat\varrho}(\hat Z,\hat\tau)}\bigg(\sum_{i\in\xi(\hat X,\hat t,\hat\varrho)}\int_{r_i'(\hat X,\hat t\,)}^{\hat\varrho}(\varrho_i/r)^{n+3}r^{-1}\,dr\bigg)\,d\sigma(\widehat X,\hat t\,)\end{aligned}$$

$$
\begin{aligned}
(2.29)\quad & + c(M)\nu\big[F_1 \cap C_{5\hat\varrho/4}(\widehat{Z},\hat\tau) \times (0, 2\hat\varrho)\big] \\
&\le c(M)\delta_1^2 \sum_{i\in\xi(\hat Z,\hat\tau,\hat\varrho)} \int_{F_1\cap C_{\hat\varrho}(\hat Z,\hat\tau)} \left(\frac{\varrho_i}{r_i'(\widehat{X},\hat t\,)}\right)^{n+3} d\sigma(\widehat{X},\hat t\,) + c(M)\|\nu\|\hat\varrho^{n+1} \\
&\le c(M)\Bigg[\delta_1^2 \sum_{i\in\xi(\hat Z,\hat\tau,\hat\varrho)} \varrho_i^{n+1} + \|\nu\|\hat\varrho^{n+1}\Bigg] \\
&\le c(M)(\delta_1^2 + \|\nu\|)\hat\varrho^{n+1},
\end{aligned}
$$

where we have used (1.5) to estimate the last integral. Next for $r>0$, $(\hat x,\hat t\,)\in\mathbf{R}^n$ let

$$
\kappa(\hat x,\hat t,r) = r^{-(n+3)} \inf_L \int_{C_r(\hat x,\hat t\,)} |\hat\psi(y,s) - L(y)|^2\, dy\, ds,
$$

where the infimum is over all linear functions of y (only). We note that

$$
\big(c(M)\big)^{-1}\kappa(\hat x,\hat t,r) \le \gamma'(\widehat{X},\hat t,r) \le c(M)\kappa(\hat x,\hat t,r) \tag{2.30}
$$

as we see from (2.24), (2.22)$\,(*)$, and an argument similar to the one used in (2.8). Using (2.30), (2.22)$\,(**)$ and Taylor's theorem we deduce for $0<r\le\frac{2}{3}\varrho_i$ and $(\hat x,\hat t\,)\in\overline{Q}_i$ that

$$
\gamma'(\hat X,\hat t,r) \le c(M)\kappa(\hat x,\hat t,r) \le c(M)\delta_1^2 r^2\varrho_i^{-2}.
$$

If $\frac{2}{3}\varrho_i < r \le 8\varrho_i$, then this inequality is also valid as we see from (1.3) with $(X,t),R$ replaced by $(X_i',t_i'),8\varrho_i$. Thus

$$
\nu'\big[\,\overline{\Gamma}_i \times (0,8\varrho_i)\big] \le c(M)\delta_1^2\varrho_i^{n+1}. \tag{2.31}
$$

If $8\varrho_i<r$, then from (2.15), (2.30) we see for $(\hat x,\hat t\,)\in\overline{Q}_i$ that

$$
\gamma'(\widehat{X},\hat t,r) \le c(M)\gamma'\big(X_i',t_i',\tfrac{3}{2}r\big).
$$

Integrating this inequality, using (2.28) as well as (2.13) we find for $(\widehat{X},\hat t\,)\in\overline{\Gamma}_i$, $\overline{\Gamma}_i\subset C_{40R}(X,t)$ and $C_{\hat\varrho}(\widehat{X},\hat t\,)\subset C_{40R}(X,t)$ that

$$
\begin{aligned}
\int_{8\varrho_i}^{\hat\varrho} \gamma'(\widehat{X},\hat t,r) r^{-1}\, dr &\le c(M)f(X_i',t_i') + c(M)\delta_1^2 \int_{2\varrho_i}^{\hat\varrho} \sum_{j\in\xi(X_i',t_i',2r)} (\varrho_j/r)^{n+3} r^{-1}\, dr \\
&\le c(M)\|\nu\| + c(M)\delta_1^2 \sum_{j\in\xi(X_i',t_i',2\hat\varrho)} \left(\frac{\varrho_j}{\varrho_i+\varrho_j+d(Q_i,Q_j)}\right)^{n+3}.
\end{aligned}
$$

Integrating the above inequality over $\overline{\Gamma}_i$ with respect to σ and summing for $i \in \xi(\widehat{Z}, \hat{\tau}, 2\hat{\varrho})$ we obtain when $C_{\hat{\varrho}}(\widehat{Z}, \hat{\tau}) \subset C_{20R}(X,t)$ is as in (2.29)

$$\begin{aligned}
&\sum_{i\in\xi(\hat{Z},\hat{\tau},2\hat{\varrho})} \int_{\overline{\Gamma}_i} \int_{8\varrho_i}^{\hat{\varrho}} \gamma'(\widehat{X}, \hat{t}, r) r^{-1}\, dr\, d\sigma(\widehat{X}, \hat{t}\,) \leq c(M)\|\nu\|\hat{\varrho}^{n+1} \\
&\qquad + c(M)\delta_1^2 \sum_{j\in\xi(\hat{Z},\hat{\tau},2\hat{\varrho})} (\varrho_j)^{n+3} \int_{\mathbf{R}^n} \big(\varrho_j + d\big(\{(y,s)\}, \{(x_j', t_j')\}\big)\big)^{-(n+3)}\, dy\, ds \\
&\quad \leq c(M)\Bigg[\|\nu\|\hat{\varrho}^{n+1} + \delta_1^2 \sum_{j\in\xi(\hat{Z},\hat{\tau},2\hat{\varrho})} \varrho_j^{n+1}\Bigg] \\
&\quad \leq c(M)(\delta_1^2 + \|\nu\|)\hat{\varrho}^{n+1}.
\end{aligned}$$

Combining this inequality with (2.31) we deduce first that

$$\nu'\big[(\Gamma \setminus F_1) \cap C_{\hat{\varrho}}(\widehat{X}, \hat{t}\,) \times (0, \hat{\varrho})\big] \leq c(M)(\delta_1^2 + \|\nu\|)\hat{\varrho}^{n+1}$$

and thereupon in view of (2.29) that

$$\nu'\big[\Gamma \cap C_{\hat{\varrho}}(\widehat{Z}, \hat{\tau}) \times (0, \hat{\varrho})\big] \leq c(M)(\delta_1^2 + \|\nu\|)\hat{\varrho}^{n+1} \tag{2.32}$$

which is the desired inequality when $C_{\hat{\varrho}}(\widehat{Z}, \hat{\tau}) \subset C_{20R}(X,t)$. To handle the other cases we make the following observations.

(2.33)(a) If $(\widehat{X}, \hat{t}) \in \Gamma$, and $0 < r < d\big(\{(\widehat{X}, \hat{t}\,)\}, C_{4R}(X,t)\big)$, then $\gamma'(\widehat{X}, \hat{t}, r) = 0$.

(2.33)(b) If $r \geq R, (\widehat{X}, \hat{t}\,) \in \Gamma$, then $\gamma'(\widehat{X}, \hat{t}, r) \leq c(M)\delta_1^2(R/r)^{n+3}$.

To prove (2.33)(a) recall that $\hat{\psi} \equiv 0$ on $\mathbf{R}^n \setminus C_{4R}(x,t)$. I.e., $d(\{(Y,s)\}, \{0\} \times \mathbf{R}^n) = 0$ for $(Y,s) \in C_r(\widehat{X}, \hat{t}\,)$ which clearly implies (a). (b) follows from the same recollection as in (a) and the fact that $h(\Gamma, \{0\} \times \mathbf{R}^n) \leq c\delta_1 R$. From (2.33) we see that if $\hat{\varrho} \geq R$, then

$$\begin{aligned}
\nu'\big(C_{\hat{\varrho}}(\widehat{Z}, \hat{\tau}) \times (0, \hat{\varrho})\big) &\leq \nu'\big(C_{\hat{\varrho}}(\hat{Z}, \hat{\tau}) \times (0, R)\big) + c(M)\delta_1^2\hat{\varrho}^{n+1} \\
&\leq \nu'\big(C_{10R}(X,t) \times (0,R)\big) + c(M)\delta_1^2\hat{\varrho}^{n+1} \\
&\leq c(M)(\delta_1^2 + \|\nu\|)\hat{\varrho}^{n+1}.
\end{aligned}$$

If $\hat{\varrho} \leq R$ and $C_{\hat{\varrho}}\big(\widehat{Z}, \hat{\tau})\big) \cap \big(\mathbf{R}^{n+1} \setminus \partial C_{20R}(X,t)\big) \neq \emptyset$, then $\nu\big(C_{\hat{\varrho}}(\widehat{Z}, \hat{\tau}) \times (0, \hat{\varrho})\big) = 0$ thanks to (2.33). We conclude that (2.32) holds whenever $(\widehat{Z}, \hat{\tau}) \in \Gamma$ and $\hat{\varrho} > 0$. Hence (2.27) is valid.

In order to study the implications of (2.27) we shall need some more notation. Let φ be an infinitely differentiable real-valued function on $\mathbf{R}^n$ with compact support in $C_1(0,0)$ (i.e, $\varphi \in C_0^\infty\big(C_1(0,0)\big)$) and set

$$\begin{aligned}
\varphi_\lambda(z,\tau) &\equiv \lambda^{-(n+1)}\varphi(z/\lambda, \tau/\lambda^2), \\
\|\varphi\|_\infty &= \max\big\{|\varphi(z,\tau)| : (z,\tau) \in \mathbf{R}^n\big\}.
\end{aligned}$$

Next for θ: $\mathbf{R}^n \to \mathbf{R}$, define the convolution of φ with θ by

$$\varphi * \theta(z,\tau) = \int_{\mathbf{R}^n} \varphi(z-y,\tau-s)\theta(y,s)\,dy\,ds$$

whenever this convolution makes sense. For short we write $\varphi_\lambda\theta$ for $\varphi_\lambda * \theta$. Suppose now that φ, as above, annihilates constants and the coordinate functions y_i, $0 \le i \le n-1$. That is

$$\int_{\mathbf{R}n} \varphi(y,s)\,dy\,ds = 0 \quad \text{and} \quad \int_{\mathbf{R}^n} y_i\varphi(y,s)\,dy\,ds = 0.$$

Observe for each linear function L of y only that

$$(\varphi_\lambda * \hat\psi)^2(\hat z,\hat\tau) = \big[\varphi_\lambda * (\hat\psi - L)\big]^2(\hat z,\hat\tau) \le c\|\varphi\|_\infty \lambda^{-(n+1)} \int_{C_\lambda(\hat z,\hat\tau)} |\hat\psi - L|^2\,dy\,ds.$$

From this observation, the definition of κ, (2.30), and (2.27) we see that

$$\begin{aligned}\int_0^\varrho \int_{C_\varrho(z,\tau)} \lambda^{-3}(\varphi_\lambda * \hat\psi)^2(y,s)\,dy\,ds\,d\lambda &\le c\|\varphi\|_\infty \int_0^\varrho \int_{C_\varrho(z,\tau)} \lambda^{-1}\kappa(y,s,\lambda)\,dy\,ds\,d\lambda \\ &\le c(M)\|\varphi\|_\infty(\delta_1^2 + \|\nu\|)\varrho^{n+1}.\end{aligned} \tag{2.34}$$

Let $P \in C_0^\infty\big(C_1(0,0)\big)$ be an even nonnegative function on $\mathbf{R}^n$ with $\int_{\mathbf{R}^n} P(w)\,dw \equiv 1$. We note that if $\varphi = \varphi(z,\tau)$ is any of the functions,

$$\lambda\frac{\partial P_\lambda}{\partial\lambda}, \quad \lambda^2\frac{\partial P_\lambda}{\partial\tau}, \quad \lambda^2\frac{\partial^2 P_\lambda}{\partial z_i^2}, \quad 1 \le i \le n-1, \tag{2.35}$$

then φ annihilates linear functions of y and $\|\varphi\|_\infty \le c$ as is easily seen. From (2.34), (2.35), we conclude

$$\begin{aligned}\int_0^\varrho \int_{C_\varrho(z,\tau)} \left(\lambda^{-1}\left[\frac{\partial(P_\lambda\hat\psi)}{\partial\lambda}\right]^2 + \lambda\left[\frac{\partial(P_\lambda\hat\psi)}{\partial\tau}\right]^2 + \lambda\left[\frac{\partial^2(P_\lambda\hat\psi)}{\partial z_i^2}\right]^2\right) dy\,ds\,d\lambda \\ \le c(\delta_1^2 + \|\nu\|)\varrho^{n+1} \quad \text{for } 1 \le i \le n-1.\end{aligned} \tag{2.36}$$

We now return to the proof of Theorem 1. In this case we claim that (2.36) holds with $\hat\psi$ replaced by ψ (see (2.17)) and δ_1 replaced by 1. Indeed, in view of (1.1) for ψ we can use the plan of the previous proof to conclude first that (2.27) holds with δ_1 replaced by 1 and second that (2.36) is valid. In fact the argument is much simpler as we do not have to make precise estimates on the constants.

Proof of Theorem 1. We use (2.36) with $\delta_1 = 1$ to show that

$$\|D^t_{1/2}\psi\|_* \le c(M)(1 + \|\nu\|)^{1/2} \tag{2.37}$$

thereby completing the proof of Theorem 1. Here $\|\cdot\|_*$ denotes the norm in the space $\mathrm{BMO}(\mathbf{R}^n)$ defined in the following way. Given $g\colon \mathbf{R}^n \to \mathbf{R}$, locally integrable with respect to Lebesgue n measure, let $C = C_r(z,\tau)$ and let $|C|$ be the Lebesgue n measure of C. Set

$$g_C = |C|^{-1} \int_C g\, dy\, ds.$$

Then $g \in \mathrm{BMO}(\mathbf{R}^n)$ with norm $\|g\|_*$ if and only if

$$\|g\|_* = \sup_C \Big\{ |C|^{-1} \int_C |g - g_C|\, dy\, ds \Big\} < \infty.$$

To prove (2.37), let $\beta \in C_0^\infty\big(C_{2\varrho}(z,\tau)\big)$, $0 \le \beta \le 1$, with $\beta \equiv 1$ on $C_{3\varrho/2}(z,\tau)$ and

$$\varrho^{-l}\left|\frac{\partial^l}{\partial y^l}\beta\right| + \varrho^{-2l}\left|\frac{\partial^l}{\partial s^l}\beta\right| \le c(l,n)$$

at each point of $\mathbf{R}^n$. If $C = C_\varrho(z,\tau)$, we put

$$\psi = \big[(\psi - \psi_C)\beta\big] + \big[\beta\psi_C + (1-\beta)\psi\big] = \psi_1 + \psi_2.$$

We first consider ψ_1. Given ε, $0 < \varepsilon < \varrho/100$, we note that

$$\begin{aligned}\int_{\mathbf{R}^n} (D^t_{1/2}P_\varepsilon\psi_1)^2\, dy\, ds &= \int_{\mathbf{R}^n} (D^t_{1/2}P_\varrho\psi_1)^2\, dy\, ds \\ &\quad - \int_\varepsilon^\varrho \int_{\mathbf{R}^n} 2D^t_{1/2}\Big(\frac{\partial}{\partial\lambda}P_\lambda\psi_1\Big) D^t_{1/2}(P_\lambda\psi_1)\, dy\, ds\, d\lambda \\ &= \int_{\mathbf{R}^n} (D^t_{1/2}P_\varrho\psi_1)^2\, dy\, ds \\ &\quad + a\int_\varepsilon^\varrho \int_{\mathbf{R}^n} \mathscr{H}\Big[\frac{\partial}{\partial\lambda}(P_\lambda\psi_1)\Big]\frac{\partial}{\partial t}(P_\lambda\psi_1)\, dy\, ds\, d\lambda \\ &= V_1 + V_2.\end{aligned} \tag{2.38}$$

In (2.38), a is a constant and $\mathscr{H}$ is the Hilbert transform on $\mathbf{R}$ defined by

$$\mathscr{H}k(s) = \mathrm{PV}\int_{\mathbf{R}} k(\hat\tau)(s-\hat\tau)^{-1}\, ds$$

for k a real valued function. In the above display we have $k = (\partial/\partial\lambda)(P_\lambda\psi_1)(y,\cdot\,)$. From (1.1) we get $|\partial(P_\varrho\psi_1)/\partial s| \le c(M)\varrho^{-1}$ at each point in $C_{8\varrho}(z,\tau)$ and this function vanishes elsewhere in $\mathbf{R}^n$. From this inequality we find for $|y-z| \le 4\varrho$ that

$$|D^t_{1/2}P_\varrho\psi_1|(y,s) = \left| c\int_{\mathbf{R}} \frac{P_\varrho\psi_1(y,s) - P_\varrho\psi_1(y,\hat{t}\,)}{|s-\hat{t}|^{3/2}}\, d\hat{t}\right| \le c(M)\min\left\{1, \frac{\varrho^3}{|s-\tau|^{3/2}}\right\}.$$

If $|y-z| > 4\varrho$, then $D^t_{1/2}P_\varrho\psi_1(y,s) \equiv 0$. Using these inequalities and integrating over $\mathbf{R}^n$, we conclude that

$$|V_1| \le c(M)\varrho^{n+1}. \tag{2.39}$$

To handle V_2 we note that $\mathscr{H}$ is a bounded operator with norm $\le c$ from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$—the usual space of Lebesgue square integrable functions on $\mathbf{R}$. Using this note and Hölder's inequality we get from (2.37),

$$\begin{aligned} |V_2|^2 &\le c\left(\int_0^\varrho\int_{\mathbf{R}^n} \lambda^{-1}\left[\frac{\partial}{\partial\lambda}(P_\lambda\psi_1)\right]^2 dy\,ds\right) \\ &\quad\cdot\left(\int_0^\varrho\int_{\mathbf{R}^n} \lambda\left[\frac{\partial}{\partial t}(P_\lambda\psi_1)\right)\right]^2 dy\,ds\right) \\ &= V_{21}\cdot V_{22}. \end{aligned} \tag{2.40}$$

To estimate V_{22} we observe for $(y,s)\in C_{4\varrho}(z,\tau)$, $0<\lambda\le\varrho$, and

$$L(\hat{x}) = \beta(y,s) + \sum_{i=1}^{n-1}\frac{\partial}{\partial y_i}\beta(y,s)(y_i - \hat{x}_i)$$

that

$$\begin{aligned} &\left|\frac{\partial}{\partial s}(P_\lambda\psi_1)\right|(y,s) \\ &\quad\le \left|\int_{\mathbf{R}^n}\frac{\partial}{\partial\hat{t}}(P_\lambda)(y-\hat{x},s-\hat{t}\,)\big(\psi(\hat{x},\hat{t}\,)-\psi_C)(\beta(\hat{x},\hat{t}\,)-L(\hat{x})\big)\,d\hat{x}\,d\hat{t}\right| \\ &\qquad+\sum_{i=1}^{n-1}\left|\frac{\partial}{\partial y_i}\beta(y,s)\int_{\mathbf{R}}(y_i-\hat{x}_i)\frac{\partial}{\partial\hat{t}}(P_\lambda)(y-\hat{x},s-\hat{t}\,)\big(\psi(\hat{x},\hat{t}\,)-\psi(y,s)\big)\,d\hat{x}\,d\hat{t}\right| \\ &\qquad+\left|\beta(y,s)\frac{\partial}{\partial s}P_\lambda\psi(y,s)\right| \\ &\quad\le c(M)/\varrho + c\left|\frac{\partial}{\partial s}P_\lambda\psi\right|(y,s), \end{aligned}$$

where we have used (1.1), (2.35), and Taylor's theorem to estimate the integrals. If $(y,s)\in \mathbf{R}^n\setminus C_{4\varrho}(z,\tau)$, then $(\partial/\partial s)(P_\lambda\psi_1)(y,s)=0$. Similarly, for $(y,s)\in C_{4\varrho}(z,\tau)$, $0<\lambda\leq\varrho$,

$$\left|\frac{\partial}{\partial\lambda}(P_\lambda\psi_1)\right|(y,s)\leq c(M)\lambda/\varrho+c\left|\frac{\partial}{\partial\lambda}P_\lambda\psi\right|(y,s)$$

and this function vanishes elsewhere in $\mathbf{R}^n$. Putting these inequalities into (2.40) and using (2.36) we obtain first that

$$|V_2|^2\leq c(M)(1+\|\nu\|)^2\varrho^{2n+2}$$

and second in view of (2.38), (2.39) that

$$\int_{\mathbf{R}^n}(D^t_{1/2}P_\varepsilon\psi_1)^2\,dy\,ds\leq c(M)(1+\|\nu\|)\varrho^{n+1}.$$

Using this inequality, the inequalities directly above (2.39), and letting $\varepsilon\to 0$, we deduce that $D^t_{1/2}P_\varepsilon\psi_1$ converges weakly in $L^2(\mathbf{R}^n)$ to a function ζ satisfying the same inequalities as $D^t_{1/2}P_\varepsilon\psi_1$. Using weak convergence, we see that $\psi_1(y,\cdot)=cI_{1/2}*\zeta(y,\cdot)$, where $I_{1/2}(s)=|s|^{-1/2}$. This equality implies that $\zeta=D^t_{1/2}\psi_1$ exists and satisfies

$$\int_{\mathbf{R}^n}(D^t_{1/2}\psi_1)^2\,dy\,ds\leq c(M)(1+\|\nu\|)\varrho^{n+1}. \tag{2.41}$$

Consider now integrals involving ψ_2. Observe that ψ_2 is constant on $C_{3\varrho/2}(z,\tau)$. Thus for $(y,s)\in C$,

$$D^t_{1/2}\psi_2(y,s)=c\int_{\{\hat t:|\hat t-\tau|\geq 9\varrho^2/4\}}\frac{\psi_2(y,\hat t\,)-\psi_2(y,s)}{|s-\hat t|^{3/2}}\,d\hat t.$$

Using this equality, $\psi_2=(\psi_C-\psi)\beta+\psi$, and (1.1) we deduce from estimates similar to the above that if

$$\alpha=c\int_{\{\hat t:|\hat t-\tau|\geq 9\varrho^2/4\}}\frac{\psi(z,\hat t\,)-\psi(z,\tau)}{|\tau-\hat t|^{3/2}}\,d\hat t,$$

then $|D^t_{1/2}\psi_2-\alpha|\leq c(M)$ on C. Thus,

$$\int_C|D^t_{1/2}\psi_2-\alpha|\,dy\,ds\leq c(M)\varrho^{n+1}. \tag{2.42}$$

From (2.42) and (2.41) we see that $D^t_{1/2}\psi$ exists and is integrable on C. Moreover, since

$$|\alpha - (D^t_{1/2}\psi)_C| \le |C|^{-1} \int |\alpha - D^t_{1/2}\psi(y,s)|\,dy\,ds$$

it follows from (2.41), (2.42) and Hölder's inequality that

$$\begin{aligned}\int_C |D^t_{1/2}\psi - (D^t_{1/2}\psi)_C|\,dy\,ds &\le 2\int_C |D^t_{1/2}\psi_2 - \alpha|\,dy\,ds + 2\int_C |D^t_{1/2}\psi_1|\,dy\,ds \\ &\le c(M)\varrho^{n+1} + c\varrho^{(n+1)/2}\Big(\int_{\mathbf{R}^n} |D^t_{1/2}\psi_1|^2\,dy\,ds\Big)^{1/2} \\ &\le c(M)(1+\|\nu\|)^{1/2}\varrho^{n+1}.\end{aligned} \tag{2.43}$$

Hence $D^t_{1/2}\psi \in \mathrm{BMO}\,(\mathbf{R}^n)$ with $\|D^t_{1/2}\psi\|_* \le c(M)(1+\|\nu\|)^{1/2}$. The proof of Theorem 1 is now complete. □

3. Proof of Theorem 2

We return to the proof of Theorem 2. We shall use (2.36) and the fact that $\|\nu\| \le \delta_1$ to show as in the proof of Theorem 1 that

$$\begin{aligned}&\text{(a)} \qquad \int_{\mathbf{R}^n} (D^t_{1/2}\hat\psi)^2\,dy\,ds \le c(M)\delta_1^2 R^{n+1}, \\ &\text{(b)} \qquad \sum_{i=1}^{n-1}\int_{\mathbf{R}^n}\Big(\frac{\partial\hat\psi}{\partial y_i}\Big)^2\,dy\,ds \le c(M)\delta_1^2R^{n+1}.\end{aligned} \tag{3.1}$$

The proof of (3.1)(a) is essentially the same as (2.41) with ψ_1, ϱ replaced by $\hat\psi$, R, only now we use the fact that $|\hat\psi| \le c(M)\delta_1 R$ to estimate V_1 while the estimate for V_2 follows directly from (2.36). We omit the details. As for (b) we have for $0 \le i \le n-1$,

$$\begin{aligned}\int_{\mathbf{R}^n}\Big(\frac{\partial}{\partial y_i}\hat\psi\Big)^2\,dy\,ds &= \int_{\mathbf{R}^n}\Big(\frac{\partial}{\partial y_i}P_R\hat\psi\Big)^2\,dy\,ds \\ &\quad - 2\int_0^R\int_{\mathbf{R}^n}\Big(\frac{\partial^2}{\partial y_i\partial\lambda}P_\lambda\hat\psi\Big)\Big(\frac{\partial}{\partial y_i}P_\lambda\hat\psi\Big)\,dy\,ds\,d\lambda \\ &\le c(M)\delta_1^2R^{n+1} + \Big|\int_0^R\int_{\mathbf{R}^n}\Big(\frac{\partial}{\partial\lambda}\widehat{P}_\lambda\psi\Big)\Big(\frac{\partial^2}{\partial y_i^2}\widehat{P}_\lambda\psi\Big)\,dy\,ds\,d\lambda\Big| \\ &\le c(M)\delta_1^2R^{n+1} + \Big(\int_0^R\int_{\mathbf{R}^n}\lambda^{-1}\Big(\frac{\partial}{\partial\lambda}P_\lambda\psi\Big)^2\,dy\,ds\,d\lambda\Big)^{1/2} \\ &\quad \cdot\Big(\int_0^R\int_{\mathbf{R}^n}\lambda\Big(\frac{\partial^2}{\partial y_i^2}P_\lambda\psi\Big)^2\,dy\,ds\,d\lambda\Big)^{1/2} \\ &\le c(M)\delta_1^2R^{n+1}.\end{aligned}$$

Thus (3.1) is true. From (3.1) we shall deduce the existence of a closed set $F_2 \subset F_1 \subset \Delta(X,t,R)$ with

$$(3.2) \qquad \begin{aligned} &(\alpha) \quad \sigma(F_2) \geq 2^{-(n+4)} R^{n+1}, \\ &(\beta) \quad |\hat{n}(Y,s,r) - e_0| \leq c(M)\delta_1^{1/[8(n-1)]} \text{ for all } (Y,s) \in F_2 \\ &\qquad\ \text{and } 0 < r \leq R. \end{aligned}$$

We remark that the introduction of $\hat{\psi}$ and most of our effort so far was made to get us in a position to prove (3.2). To this end if $g\colon \mathbf{R}^n \to \mathbf{R}$ and g is locally integrable on $\mathbf{R}^n$ we let $\mathscr{M} g$ be the n-dimensional Hardy–Littlewood maximal function defined by

$$\mathscr{M} g(z,\tau) = \sup_{r>0} |C_r(z,\tau)|^{-1} \int_{C_r(z,\tau)} |g|(y,s)\, dy\, ds.$$

Similarly let $\mathscr{M}^{(1)} g(\,\cdot\,,s)$ denote the $(n-1)$-dimensional Hardy–Littlewood maximal function of $g(\,\cdot\,,s)$ taken with respect to balls and let

$$\nabla\hat{\psi}(z,\tau) = \left(\frac{\partial}{\partial z_1}\hat{\psi}, \ldots, \frac{\partial}{\partial z_{n-1}}\hat{\psi}\right)(z,\tau), \quad (z,\tau) \in \mathbf{R}^n,$$

be the spatial gradient of $\hat{\psi}$ whenever these partial derivatives exist. Recall that $\hat{\psi}$ has compact support in $C_{4R}(X,t)$. From (3.1)(b), the Hardy–Littlewood maximal theorem, and weak type estimates we first deduce the existence of a closed set $J \subset (t - 16R^2, t + 16R^2)$ with

$$(3.3) \qquad \begin{aligned} &\text{(i)} \quad H^1\big[(t-16R^2, t+16R^2) \setminus J\big] \leq \delta_1 R^2, \\ &\text{(ii)} \quad \int_{C_{8R}(x,t)} \mathscr{M}^{(1)}(|\nabla\hat{\psi}|)(y,s)\, dy \leq c(M)\delta_1^{1/2} R^{n-1} \text{ for all } s \in J. \end{aligned}$$

If $s \in J$ we can again use weak type estimates and the Hardy–Littlewood maximal theorem to get a closed set $G_1(s) \subset C_{4R}(x,t) \cap (\mathbf{R}^{n-1} \times \{s\})$ with

$$(3.4) \qquad \begin{aligned} &(+) \quad H^{n-1}\big[C_{4R}(x,t) \cap (\mathbf{R}^{n-1} \times \{s\}) \setminus G_1(s)\big] \leq \delta_1^{1/4} R^{n-1}, \\ &(++) \quad \mathscr{M}^{(1)}(|\nabla\hat{\psi}|)(y,s) \leq c(M)\delta_1^{1/4} \text{ for all } (y,s) \in G_1(s). \end{aligned}$$

In (3.3), (3.4), H^1, H^{n-1} denote Hausdorff one and $(n-1)$-dimensional measure on $\mathbf{R}^n$. For fixed $s \in J$ let $\chi_1(\,\cdot\,,s)$ denote the characteristic function of $C_{4R}(x,t) \cap (\mathbf{R}^{n-1} \times \{s\}) \setminus G_1(s)$. Using (3.3), (3.4) and the above argument once again we get for each $s \in J$ a closed set $G_2(s) \subset G_1(s)$ with

$$(3.5) \qquad \begin{aligned} &(*) \quad H^{n-1}\big[C_{4R}(x,t) \cap (\mathbf{R}^{n-1} \times \{s\}) \setminus G_2(s)\big] \leq \delta_1^{1/8} R^{n-1}, \\ &(**) \quad \mathscr{M}^{(1)}(\chi_1)(y,s) \leq c(M)\delta_1^{1/8} \text{ for all } (y,s) \in G_2(s). \end{aligned}$$

We shall use (3.3)–(3.5) and (2.22) $(*)$ to show that if $s \in J$ and $(y,s) \in G_2(s) \cap C_{2R}(x,t)$, then

$$|\hat{\psi}(y,s) - \hat{\psi}(z,s)| \le c(M)\delta_1^{1/[8(n-1)]}|y-z|, \qquad \text{whenever } |y-z| \le R. \tag{3.6}$$

In fact if $(z,s) \in G_1(s)$ and $r = |y-z|$, then from basic Sobolev inequalities and (3.4) $(++)$ we find

$$|\hat{\psi}(y,s) - \hat{\psi}(z,s)| \le c|y-z|\big[\mathscr{M}^{(1)}(|\nabla\hat{\psi}|)(y,s) + \mathscr{M}^{(1)}(|\nabla\hat{\psi}|)(z,s)\big] \le c(M)\delta_1^{1/4} r.$$

Otherwise (i.e. $(z,s) \notin G_1(s)$) from (3.5) $(**)$ we see there exists $(\tilde{z},s) \in G_1(s)$ with $|z-\tilde{z}| < c(M)\delta_1^{1/[8(n-1)]} r$. Using (2.22) $(*)$ and the above inequalities we conclude that

$$\begin{aligned} |\hat{\psi}(y,s) - \hat{\psi}(z,s)| &\le |\hat{\psi}(y,s) - \hat{\psi}(\tilde{z},s)| + |\hat{\psi}(\tilde{z},s) - \hat{\psi}(z,s)| \\ &\le c(M)\delta_1^{1/4} r + c(M)\delta_1^{1/[8(n-1)]} r \le c(M)\delta_1^{1/[8(n-1)]} r. \end{aligned}$$

Thus (3.6) is true. Next we note from (3.1)(a) that

$$\int_{\mathbf{R}}\int_{\mathbf{R}^n} \frac{\big(\hat{\psi}(y,s) - \hat{\psi}(y,\tau)\big)^2}{(s-\tau)^2}\, dy\, ds\, d\tau = c\int_{\mathbf{R}^n} (D^t_{1/2}\hat{\psi})^2\, dy\, ds \le c(M)\delta_1^2 R^{n+1},$$

where the first equality can be proved by way of the Fourier transform. Using this inequality and weak type estimates we see that if

$$f_1(z,\tau) = \int_{-8R}^{8R} \frac{\big(\hat{\psi}(z,\tau+h) - \hat{\psi}(z,\tau)\big)^2}{h^2}\, dh,$$

then there exists G' closed with $G' \subset C_{4R}(x,t)$ and

$$\begin{aligned} &\text{(i)} \qquad |C_{4R}(x,t) \setminus G'| \le \delta_1 R^{n+1}, \\ &\text{(ii)} \qquad f_1(z,\tau) \le c(M)\delta_1 \quad \text{for all } (z,\tau) \in G'. \end{aligned} \tag{3.7}$$

We claim that if $(y,s) \in G' \cap C_{2R}(x,t)$, then

$$|\hat{\psi}(y,s) - \hat{\psi}(y,\tau)| \le c\delta_1^{1/4}|s-\tau|^{1/2} \qquad \text{whenever } |s-\tau| \le R^2. \tag{3.8}$$

In fact if $\tau - s = r^2 > 0$, then

$$r^{-4}\int_{r^2}^{2r^2} \big(\hat{\psi}(y,s+h) - \hat{\psi}(y,s)\big)^2\, dh \le cf_1(y,s) \le c(M)\delta_1,$$

thanks to (3.7)(ii). Using weak type estimates we see that there exists a closed set $J' \subset (r^2, 2r^2)$ with

$$\text{(3.9)} \qquad \begin{aligned} &\text{(A)} \qquad H^1\big[(r^2, 2r^2) \setminus J'\big] \le \delta_1^{1/2} r^2, \\ &\text{(B)} \qquad |\hat\psi(y,s) - \hat\psi(y,\hat s)| \le c(M)\delta_1^{1/4} r \quad \text{for all } \hat s \in J'. \end{aligned}$$

Using (3.9)(A) we conclude first the existence of $\hat s \in J$ with $0 \le \hat s - \tau \le 2\delta_1^{1/2} r^2$ and second from (2.22)$(*)$, (3.9)(B) that

$$\begin{aligned} |\hat\psi(y,s) - \hat\psi(y,\tau)| &\le |\hat\psi(y,s) - \hat\psi(y,\hat s)| + |\hat\psi(y,\hat s) - \hat\psi(y,\tau)| \\ &\le c(M)\delta_1^{1/4} r + c(M)\delta_1^{1/4} r \le c(M)\delta_1^{1/4} r. \end{aligned}$$

Thus (3.8) is valid. Let χ be the characteristic function of $C_{4R}(x,t) \setminus G'$ and note from (3.7) as well as our usual argument that there exists a closed set $G'' \subset G'$ with

$$\text{(3.10)} \qquad \begin{aligned} &\text{(a)} \qquad |C_{4R}(x,t) \setminus G''| \le \delta_1^{1/2} R^{n+1}, \\ &\text{(b)} \qquad \mathscr{M}(\chi)(z,\tau) \le c(M)\delta_1^{1/2} \quad \text{for all } (z,\tau) \in G''. \end{aligned}$$

We put $G_2 = \bigcup_{t \in J} G_2(t)$ and $G = G'' \cap G_2$. Observe from (3.3)(i), (3.5) that

$$|C_{4R}(x,t) \setminus G_2| \le c(M)(\delta_1 + \delta_1^{1/8}) R^{n+1}$$

which in view of (3.10) implies

$$\text{(3.11)} \qquad |C_{4R}(x,t) \setminus G| \le c(M)(\delta_1^{1/8} + \delta_1^{1/2}) R^{n+1} = c(M)\delta_1^{1/8} R^{n+1}.$$

Let $(y,s) \in G \cap C_{2R}(x,t)$ and $(z,\tau) \in C_R(y,s)$. If $r = |y - z| + |s - \tau|^{1/2}$, then from (3.10)(b) we see there exists $(\tilde z, \tilde \tau) \in G' \cap C_r(y,s)$ with $d\big(\{(z,\tau)\}, \{(\tilde z,\tilde\tau)\}\big) \le c(M)\delta_1^{1/[2(n+1)]} r$. From this inequality, (2.22)$(*)$, (3.6) and (3.8) we deduce

$$\text{(3.12)} \qquad \begin{aligned} |\hat\psi(y,s) - \hat\psi(z,\tau)| &\le |\hat\psi(y,s) - \hat\psi(\tilde z,\tilde\tau)| + |\hat\psi(\tilde z,\tilde\tau) - \hat\psi(z,\tau)| \\ &\le |\hat\psi(y,s) - \hat\psi(\tilde z,s)| + |\hat\psi(\tilde z,s) - \hat\psi(\tilde z,\tilde\tau)| + c(M)\delta_1^{1/[2(n+1)]} r \\ &\le c(M)\big(\delta_1^{1/[8(n-1)]} + \delta_1^{1/4} + \delta_1^{1/[2(n+1)]}\big) r \le c(M)\delta_1^{1/[8(n-1)]} r. \end{aligned}$$

We now can prove (3.2). We choose $F_2 \subset F_1$ closed with $p(F_2) \subset G$ and $|p(F_2)| \ge 2^{-(n+4)} R^{n+1}$. This choice is possible for $\delta_1 > 0$ sufficiently small, as we see from (2.14) and (3.11). From this choice, the fact that Hausdorff measure does not increase under a projection we get (3.2)(α). To prove (3.2)(β) let $(Y,s) \in F_2$ and $0 < r \le R$. Then $(y,s) \in G$ so from (3.12) we find that each point of $\Gamma \cap C_r(Y,s)$

lies within parabolic distance $c(M)\delta_1^{1/[8(n-1)]}r$ of a point of the plane through (Y,s) with normal e_0. Also from (2.23) every point in $\Gamma \cap C_r(Y,s)$ lies within parabolic distance $c(M)\delta_1 r$ of a point of $\partial\Omega$. Finally from (1.3), (2.18) we see that each point of $\Delta(Y,s,r)$ lies within parabolic distance $c\delta_1 r$ of the plane through (Y,s) with normal $\hat{n}(Y,s,r)$. Since $\partial\Omega$ separates $\mathbf{R}^{n+1}$ and $\hat{n}(X,t,R) = e_0$, we conclude from basic geometry that all of the above can only hold if (3.2)(β) is valid.

Next we use (3.2), a John–Nirenberg type argument and the fact that $(X,t) \in \partial\Omega$, $R > 0$ are arbitrary to prove

Lemma 3.13. *Let $\delta_2 = \delta_1^{1/[24(n-1)]}$ and for given $(X,t) \in \partial\Omega$, $R > 0$, let*

$$K = \{(Y,s) \in \Delta(X,t,R) : |\hat{n}(Y,s,r) - \hat{n}(X,t,R)| \le \delta_2 \text{ for } 0 < r \le R\}.$$

If $0 < \delta_2 \le \tilde{\delta}$ and $\tilde{\delta} = \tilde{\delta}(M)$ is small enough, then there exists $\bar{c} = \bar{c}(M) \ge 1$ with

$$\sigma\big(\Delta(X,t,R) \setminus K\big) \le e^{-1/(\bar{c}\delta_2)} R^{n+1}.$$

Proof. Again we assume that $e_0 = \hat{n}(X,t,R)$. Given $\widetilde{R}$, $R \le \widetilde{R} \le 2R$, set

$$\begin{aligned}\widehat{E}(\lambda) &= \widehat{E}(\lambda,\widetilde{R}) \\ &= \{(Y,s) \in \Delta(X,t,\widetilde{R}) : |\hat{n}(Y,s,r) - \hat{e}_0| > \lambda\delta_2^2 \text{ for some } r,\ 0 < r \le \widetilde{R}\}.\end{aligned}$$

We show there exists $\theta = \theta(M)$, $0 < \theta < 1$, and $c(M) \ge 1$ such that for some $\widetilde{R} \in [R, 2R]$,

$$\sigma\big(\widehat{E}(k+1)\big) \le \theta\sigma\big(\widehat{E}(k)\big) + c(M)e^{-1/[c(M)\delta_2]}R^{n+1} \tag{3.14}$$

for $k = 1, \dots, k_0 - 1$. Here k_0 is the largest positive integer for which $k_0 \le \delta_2^{-1}$. Once (3.14) is proved we can iterate this inequality $k_0 - 1$ times starting from $k = 1$ to get Lemma 3.13 by iteration. In fact if $(Y,s) \in \widehat{E}\big(k + \frac{1}{2}\big)$ we let

$$\varrho = \varrho(Y,s,k) = \inf\{r : 0 < r \le \widetilde{R} \text{ and } |\hat{n}(Y,s,r) - e_0| \le \big(k + \tfrac{1}{2}\big)\delta_2^2\}.$$

Then from (2.18) we see for $0 < \delta_2 \le \tilde{\delta}$, $(Y,s) \in \widehat{E}\big(k + \frac{1}{2}\big)$, and $\tilde{\delta} > 0$ sufficiently small that

$$k\delta_2^2 \le \big(k + \tfrac{1}{2}\big)\delta_2^2 - c(M)\delta_1 \le |\hat{n}(Z,\tau,5\varrho) - e_0| \le \big(k + \tfrac{1}{2}\big)\delta_2^2 + c(M)\delta_1 \tag{3.15}$$

for all $(Z,\tau) \in \Delta(Y,s,5\varrho)$. As above, using a well-known covering lemma, we find $\{\Delta(Y_i,s_i,\varrho_i)\}$ with

$$\begin{aligned}&\text{(a)} \qquad \widehat{E}\big(k + \tfrac{1}{2}\big) \subset \bigcup \Delta(Y_i,s_i,5\varrho_i), \\ &\text{(b)} \qquad \Delta(Y_j,s_j,\varrho_j) \cap \Delta(Y_i,s_i,\varrho_i) = \emptyset,\ i \ne j.\end{aligned} \tag{3.16}$$

Let ξ' be the set of positive integers i for which $\Delta(Y_i, s_i, \varrho_i) \cap \big(\mathbf{R}^{n+1} \setminus \Delta(X, t, \widetilde{R})\big) \neq \emptyset$ and let ξ be the rest of the positive integers that are indices in the above union. From (2.18) and the triangle inequality it is easily deduced that

$$|\hat{n}(Y_i, s_i, \varrho_i) - e_0| \leq c\delta_1 \log(R/\varrho_i).$$

Also from (3.15) we have $|\hat{n}(Y_i, s_i, \varrho_i) - e_0| \geq \delta_2^2$. Combining these inequalities we see that

$$\varrho_i \leq e^{-\delta_2^2/(c\delta_1)} R \leq e^{-1/(c\delta_2)} R.$$

Hence if $c_* = c_*(M)$ is large enough and $r' = e^{-1/(c_*\delta_2)} R$, then

$$\bigcup_{i \in \xi'} \Delta(Y_i, s_i, \varrho_i) \subset \Delta(X, t, \widetilde{R} + r') \setminus \Delta(X, t, \widetilde{R} - r').$$

We now choose $\widetilde{R} \in [R, 2R]$ so that

$$\sigma\big[\Delta(X, t, \widetilde{R} + r') \setminus \Delta(X, t, \widetilde{R} - r')\big] \leq M r' (1000R)^n.$$

The existence of $\widetilde{R}$ follows from Ahlfors regularity of $\partial\Omega$ (see (1.5)) and an easy counting argument using the fact that there are at least $1/(8r')$ disjoint 'rings' of the above type contained in $\Delta(X, t, 2R) \setminus \Delta(X, t, R)$. From the above inequalities it follows that

$$\sigma\Big(\bigcup_{i \in \xi'} \Delta(Y_i, s_i, \varrho_i)\Big) \leq M r' (1000R)^n. \tag{3.17}$$

If $i \in \xi$, then from (3.2) with R, X, t replaced by ϱ_i, Y_i, s_i, there exists $L_i \subset \Delta(Y_i, s_i, \varrho_i)$ such that for $0 < r \leq \varrho_i$ and $(Z, \tau) \in \Delta(Y_i, s_i, \varrho_i)$,

$$\begin{aligned} &\text{(i)} \quad |\hat{n}(Z, \tau, r) - \hat{n}(Y_i, s_i, \varrho_i)| \leq c(M)\delta_2^3, \\ &\text{(ii)} \quad \sigma(L_i) \geq 2^{-(n+4)} \varrho_i^{n+1}. \end{aligned} \tag{3.18}$$

Let $L = \bigcup_{i \in \xi} L_i$. Then from (3.18)(ii), (3.16)(a), and (1.5) we get

$$\begin{aligned} \sigma\big(\widehat{E}(k+1)\big) &\leq \sigma\Big(\bigcup \Delta(Y_i, s_i, 5\varrho_i)\Big) \leq c(M)\sigma\Big(\bigcup \Delta(Y_i, s_i, \varrho_i)\Big) \\ &\leq c(M)\sigma\Big(\bigcup_{i \in \xi} \Delta(Y_i, s_i, \varrho_i)\Big) + c(M) r' R^n \\ &\leq c(M)\sigma(L) + c(M) r' R^n. \end{aligned} \tag{3.19}$$

From the definition of ϱ_i and (3.18)(i) we deduce for $i \in \xi$ and $(Z, \tau) \in L_i$ that

$$|\hat{n}(Z, \tau, r) - e_0| < (k+1)\delta_2^2 \quad \text{for } 0 < r \leq \widetilde{R}$$

provided $0 < \delta_2 \le \tilde{\delta}$ and $\tilde{\delta} = \tilde{\delta}(M) > 0$ is small enough. Consequently,

$$L_i \cap \widehat{E}(k+1) = \emptyset$$

and from (3.15) we have $L_i \subset \widehat{E}(k)$. Thus $L \subset \widehat{E}(k) \setminus \widehat{E}(k+1)$ and so by (3.19)

$$\sigma\big(\widehat{E}(k)\big) \ge \sigma(L) + \sigma\big(\widehat{E}(k+1)\big) \ge \big(1 + c(M)^{-1}\big)\sigma\big(\widehat{E}(k+1)\big) - c(M) r' R^n.$$

Clearly this inequality is equivalent to (3.14). The proof of Lemma 3.13 is now complete. □

We shall also need

Lemma 3.20. *Let f be as defined below (2.12) and δ_2 as in Lemma 3.13. If $0 < \delta_2 \le \delta'$ and $\delta' = \delta'(M) > 0$ is sufficiently small, then there exists $c^* = c^*(M) \ge 1$ such that*

$$\sigma\big(\{(Z,\tau) \in \Delta(X,t,R) : f(Z,\tau) \ge \delta_2\}\big) \le e^{-1/(c^*\delta_2)} R^{n+1}.$$

Proof. Let

$$a(Y,s,\varrho) = \sigma\big(\Delta(Y,s,\varrho)\big)^{-1} \int_{\Delta(Y,s,\varrho)} f(Z,\tau)\, d\sigma(Z,\tau).$$

We shall show that $f \in \mathrm{BMO}\big[\Delta(X,t,R)\big]$ defined with respect to σ. In fact we prove

$$\sup_{\Delta(Y,s,r) \subset \Delta(X,t,R)} \left(\sigma\big(\Delta(Y,s,r)\big)^{-1} \int_{\Delta(Y,s,r)} |f - a(Y,s,r)|\, d\sigma \right) \le c(M)\delta_1^2 \tag{3.21}$$

It is well known that (3.21) implies the conclusion of Lemma 3.20 for Ahlfors regular sets and in fact one can prove this by an argument similar to the one used in proving Lemma 3.13. Thus we prove only (3.21). To do this we first claim that if $\varrho_1 = |Y - Z| + |s - \tau|^{1/2} \le \frac{1}{2}\varrho \le \frac{1}{2}R$, then

$$I = \left| \int_{\varrho}^{100R} \big[\gamma(Z,\tau,r) - \gamma(Y,s,r)\big] r^{-1}\, dr \right| \le c(M)\delta_1^2(\varrho_1/\varrho). \tag{3.22}$$

To prove (3.22) note for $r \ge \varrho$ that

$$\gamma(Y,s,r-\varrho_1) \le \gamma(Z,\tau,r) \le \gamma(Y,s,r+\varrho_1).$$

Thus,

$$
\begin{aligned}
I &\leq \left| \int_{\varrho}^{100R} \big[\gamma(Y,s,r-\varrho_1) - \gamma(Y,s,r)\big] r^{-1}\, dr \right| \\
&\quad + \left| \int_{\varrho}^{100R} \big[\gamma(Y,s,r+\varrho_1) - \gamma(Y,s,r)\big] r^{-1}\, dr \right| \qquad (3.23)\\
&= I_1 + I_2.
\end{aligned}
$$

From the definition of γ in (1.7), (2.18), and (1.3) we note that

$$
\gamma(Y,s,r') \leq (r')^{-n-3} \int_{\Delta(Y,s,r')} d\big(\{(\widehat{X},\hat{t}\,)\}, \widehat{P}(Y,s,r')\big)^2\, d\sigma(\widehat{X},\hat{t}\,) \leq c(M)\delta_1^2.
$$

Using this note and changing variables in the integral involving $r-\varrho_1$ of (3.23), we get

$$
\begin{aligned}
|I_1| &\leq \int_{\varrho-\varrho_1}^{\varrho} \gamma(Y,s,r')(r')^{-1}\, dr' + c(M)\delta_1^2 \varrho_1 \int_{\varrho}^{100R} (r')^{-2}\, dr' \\
&\quad + \int_{100R-\varrho_1}^{100R} \gamma(Y,s,r')(r')^{-1}\, dr' \\
&\leq c(M)\delta_1^2(\varrho_1/\varrho).
\end{aligned}
$$

A similar estimate holds for $|I_2|$. Putting these estimates in (3.23) we conclude (3.22). Next suppose $\Delta(Y,s,r) \subset \Delta(X,t,R)$ and put $f = h + k$ on $\Delta(Y,s,r)$ where

$$
h(Z,\tau) = \int_0^{100r} \gamma(Z,\tau,r')(r')^{-1}\, dr', \qquad (Z,\tau) \in \Delta(Y,s,r).
$$

From the hypotheses in Theorem 2

$$
\int_{\Delta(Y,s,r)} h\, d\sigma \leq c(M)\|\nu\| r^{n+1} \leq c(M)\delta_1^2 r^{n+1}.
$$

Also if $\hat{a} = k(Y,s)$, then from (3.22) with $\varrho = 100r$, we obtain

$$
\int_{\Delta(Y,s,r)} |k - \hat{a}|\, d\sigma \leq c(M)\delta_1^2 r^{n+1}.
$$

Putting these two estimates together and using

$$
|\hat{a} - a(Y,s,r)| \leq \sigma\big[\Delta(Y,s,r)\big]^{-1} \int_{\Delta(Y,s,r)} |f - a(Y,s,r)|\, d\sigma
$$

we get

$$
\begin{aligned}
\int_{\Delta(Y,s,r)} |f - a(Y,s,r)|\, d\sigma &\leq 2\int_{\Delta(Y,s,r)} h\, d\sigma + 2\int_{\Delta(Y,s,r)} |k - \hat{a}|\, d\sigma \\
&\leq c(M)\delta_1^2 r^{n+1}.
\end{aligned}
$$

Thus (3.21) is valid and the proof of Lemma 3.20 is complete. □

Armed with Lemmas 3.13 and 3.20 we can now retrace our earlier work and finally get Theorem 2. Indeed, from these lemmas we can choose F' closed,

$$F' \subset \{(Z,\tau) \in \Delta(X,t,2R) : f(Z,\tau) \le \delta_2\} \cap K,$$

where K is as in Lemma 3.13 with R replaced by $2R$, so that for some $c \ge 1$ and $0 \le \delta_2 \le \min\{\tilde{\delta}, \delta'\}$, we have

$$\sigma\big(\Delta(X,t,2R) \setminus F'\big) \le e^{-1/(c\delta_2)} R^{n+1}. \tag{3.24}$$

Again we assume that $e_0 = \hat{n}(X,t,R)$ and that $\widehat{P}(X,t,R) = \{(Y,s) \in \mathbf{R}^{n+1} : y_0 = -10\delta_1 R\}$. Arguing as in (2.9)–(2.11) we conclude the existence of $\tilde{\psi}$: $\mathbf{R}^n \to \mathbf{R}$ with

$$|\tilde{\psi}(y,s) - \tilde{\psi}(z,\tau)| \le c\delta_2\big[|y-z| + |s-\tau|^{1/2}\big] \tag{3.25}$$

whenever $(y,s),(z,\tau) \in p(F')$. Let $Q_i, \{v_i\}$ be as (2.15), (2.17) with F_1 replaced by F'. We can extend $\tilde{\psi}$ to $\mathbf{R}^n$ as in either (2.17) or (2.19). Since the extension in (2.17) is slightly simpler we use it and thus put

$$\psi(y,s) = \begin{cases} \tilde{\psi}(y,s), (y,s) \in p(F'), \\ \sum_{i\in\Lambda} (\tilde{\psi}(x_i',t_i') + c^+\delta_2\varrho_i) v_i(y,s) \text{ when } (y,s) \in \mathbf{R}^n \setminus p(F'). \end{cases} \tag{3.26}$$

Observe for each $\big(\psi(x_i',t_i'), x_i', t_i'\big) \in F'$ that there is a truncated parabolic cone with vertex at this point whose interior is $\subset \Omega$ and which has height $4R$, axis parallel to e_0, and angle opening $\frac{1}{2}\pi - c\delta_2$. Using this fact it is easily seen for c^+ large enough that if

$$\widetilde{\Omega} = \{(y_0,y,s) \in \mathbf{R}^{n+1} : y_0 > \psi(y,s)\} \text{ then } \widetilde{\Omega} \cap C_{100R}(X,t) \subset \Omega. \tag{3.27}$$

From the definition of F' above (3.24) it follows as earlier that

$$|\psi(y,s) - \psi(z,\tau)| \le c\delta_2\big[|z-y| + |\tau-s|^{1/2}\big]. \tag{3.28}$$

Also if $\Gamma = \{\big(\psi(z,\tau),z,\tau\big) : (z,\tau) \in \mathbf{R}^n\}, (y,s) \in Q_i,$ and

$$\begin{aligned}(Y,s) &= \big(\psi(y,s),y,s\big) \in \Gamma \cap C_{100R}(X,t),\\ (Y',s) &= (y_0,y,s) \in \Delta(X,t,100R)\end{aligned}$$

then from (1.3), (3.28), and the definition of F' we deduce

$$|y_0 - \psi(y,s)| \le |y_0 - \psi(x_i',t_i')| + |\psi(y,s) - \psi(x_i',t_i')| \le c\delta_2\varrho_i. \tag{3.29}$$

Clearly (3.29) yields

$$d\big(\{(Y,s)\},\partial\Omega\big)+d\big(\{(Y',s)\},\Gamma\big)\le c\delta_2\varrho_i. \tag{3.30}$$

Using (3.28), (3.30), we can now repeat the argument following (2.23) with $\|\nu\|$, δ_1 replaced by δ_2 and $\hat\psi$ by ψ to conclude first as in (2.27) that

$$\|\nu'\|\le c(M)\delta_2^2, \tag{3.31}$$

where ν' is defined relative to ψ and thereupon as in (2.36),

$$\int_0^{\varrho}\int_{C_\varrho(z,\tau)}\left(\lambda^{-1}\left[\frac{\partial(P_\lambda\psi)}{\partial\lambda}\right]^2+\lambda\left[\frac{\partial(P_\lambda\psi)}{\partial\tau}\right]^2+\lambda\left[\frac{\partial^2(P_\lambda\psi)}{\partial z_i^2}\right]^2\right)dy\,ds\,d\lambda \le c(M)\delta_2^2\varrho^{n+1}. \tag{3.32}$$

From (3.32) we obtain by the same argument as in (2.37),

$$\|D^t_{1/2}\psi\|_*\le c(M)\delta_2. \tag{3.33}$$

From (3.28) and (3.33) we see that (a) of Theorem 2 is valid with $\delta_2=\delta$. (c), (d) of Theorem 2 follow from (3.24), (3.30) with $G=F'$. (e) of this theorem is just (3.27). To prove (b) of Theorem 2, observe that

$$p\big(\partial\widetilde\Omega\cap C_R(X,t)\setminus\partial\Omega\big)\subset C_R(x,t)\setminus p(F')$$

and from (1.3), (3.24) that

$$|C_R(x,t)\setminus p(F')|\le\sigma\big(\Delta(X,t,2R)\setminus F'\big)\le e^{-1/(c\delta_2)}R^{n+1}. \tag{3.34}$$

To get (3.34) we have also used the fact (once again) that Hausdorff measure does not increase under a projection. From (2.24) we conclude that

$$\sigma\big(\partial\widetilde\Omega\cap C_R(X,t)\setminus\partial\Omega\big)\le\int_{C_R(x,t)\setminus p(F')}\sqrt{1+|\nabla\psi|^2}\,dy\,ds\le e^{-1/(c\delta_2)}R^{n+1}.$$

The above inequality and (3.24) give (b) of Theorem 2. Thus Theorem 2 is true when $n>1$.

If $n=1$ the only 'planes' allowed in (1.3) are lines parallel to the t axis. Using this fact it is easily shown for δ_0 small enough that (1.3) implies $\partial\Omega=\{(\psi(t),t):t\in\mathbf{R}\}$ for some ψ satisfying (1.1) with $(x,t),(y,s)$ replaced by s, t and $b_1\le c\delta_0$. From (2.30), (2.34) we see that (2.36) holds without the term in z_i. Using (2.36) and arguing as in the proof of (2.37) we get both Theorems 1 and 2 with $\widetilde\Omega=\Omega$. The proof of Theorems 1 and 2 are now complete. □

Remark. In [KT] the 'elliptic version' of Theorem 2 is proved for Ahlfors regular domains Ω which are Reifenberg flat and with locally small chord arc constant, where now chord arc has a different meaning. To make comparisions and simplify matters we give all definitions globally. More specifically suppose $\Omega \subset \mathbf{R}^n$. Then $\partial\Omega$ is said to be Ahlfors regular if (1.5) holds with σ = surface area or H^{n-1} measure on $\partial\Omega$. $\partial\Omega$ separates $\mathbf{R}^n$ and is (δ,∞) Reifenberg flat if (1.3) holds with $\delta=\delta_0$, only now all $(n-1)$-planes are allowed in the definition. Also if in addition to these assumptions,

$$\sigma\big[\Delta(X,r)\big]/(\omega_n r^{n-1}) \le 1+\delta \quad \text{for } 0<r<\infty,\ X\in\partial\Omega, \tag{3.35}$$

then Ω is said to be a (δ,∞) chord arc domain. Here ω_n is the volume of the unit ball in $\mathbf{R}^{n-1}$, $X\in\partial\Omega$, and $\Delta(X,r)=\{Y\in\partial\Omega : |Y-X|<r\}$. In [KT] it is shown that a theorem analogous to Theorem 2 which we call "Semmes theorem with small constants" holds for (δ,∞) chord arc domains provided $\delta>0$ is sufficiently small. One can also define as in (1.7), $\gamma(Z,r)$, only now the infimum is taken over all $(n-1)$-planes and integration is with respect to surface area on $\partial\Omega$. ν is defined relative to γ as in (1.8) and $\|\nu\|_+$ is as in (1.9) only with balls replacing rectangles. Using the same argument as in Theorem 2 it follows that this version of (1.9) with small constant implies "Semmes theorem with small constants". On the other hand it is obvious that "Semmes theorem with small constants" (on all scales) implies that Ω is a (δ,∞) chord arc domain for some small δ. Moreover it can be shown that "Semmes theorem with small constants" implies (1.9) with $\|\nu\|_+$ small. For a proof of this, see the argument at the end of Section 7 in [HLN]. A proof in a more general situation is given in [DS1, Part IV, Theorem 1.3]. Thus the 'elliptic version' of (1.9) with small constant, the chord arc conditions in [KT], and "Semmes theorem with small constants" are all equivalent in the sense that small constants in one implies small constants in the other. As a consequence of this equivalence it is easily seen that the elliptic analogue of a chord arc domain with vanishing constant as defined in Section 1 is equivalent to the definition in [KT].

On the other hand the parabolic analogue of the global chord arc conditions in [KT] is weaker than (1.9). In fact in [LS] it was shown for $n=1$ that if ω is any non increasing function on $(0,\infty)$ with $\omega(0)=0$, $\omega(2\tau)\le 2\omega(\tau)$, $\tau\ge 0$, and $\omega\equiv 1$ for $\tau\ge 1$, then there exists ψ: $\mathbf{R}\to\mathbf{R}$, satisfying

$$|\psi(t)-\psi(s)|\le \omega(|t-s|), \quad s,t\in\mathbf{R}. \tag{3.36}$$

Moreover, if $\int_0^1 \tau^{-2}\omega^2(\tau)\,d\tau=\infty$ and $D=\{(x_0,t) : x_0>\psi(t)\}$, then it follows first that $\|D^t_{1/2}\psi\|_*=\infty$ and second that the Carleson norm in (1.9) calculated relative to ∂D is infinite. Let $\Omega=\{(x_0,t) : x_0>\delta\psi(t)\}$ and suppose also that $\omega(\tau)\le c\tau^{1/2}$, for $\tau>0$. Then (1.3) holds with δ_0 replaced by $c\delta$ as follows from (3.36) and

$$\sigma\big[\Delta(x_0,t,r)\big]/(2r^2)=1$$

which is (3.35) with $\delta = 0$ for our σ so that Ω is a parabolic analogue of a (δ, ∞) chord arc domain in the sense of [KT]. Furthermore (1.9) is false for $\partial\Omega$ since it is false for ∂D. Finally from the above equality and (2.4) it follows for $n = 1$ that (1.9) with small constant implies the conditions in [KT] for a chord arc domain. Thus (1.9) is stronger for $n = 1$ than the parabolic analogue of the chord arc conditions in [KT]. To get $n > 1$ examples simply put $\Omega = \{(X,t) \in \mathbf{R}^{n+1} : x_0 > \psi(t)\theta(x)\}$ where $\theta \in C_0^\infty(\mathbf{R}^{n-1})$ with $\theta \equiv 1$ on $\{x : |x| \le N\}$, N large and ψ is as in any of the above examples. Finally

$$\omega(\tau) = \begin{cases} \tau^{1/2}\big[\log(1/\tau)\big]^{-1/2} \text{ for } 0 < \tau \le e^{-4}, \\ \min\{(e^2/2)\tau, 1\} \text{ for } \tau > e^{-4} \end{cases}$$

is an example of an ω which satisfies all of the above conditions.

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