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# A GENERALISED CONICAL DENSITY THEOREM FOR UNRECTIFIABLE SETS

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**Abstract.** We provide a strengthening of an elementary technique in geometric measure theory. Given an $s$-set $S \subset \mathbf{R}^n$, in the language of tangent measures, this technique establishes the existence of tangent measures to the measure $H^s_{\lfloor S}$ on one side of an $(n-1)$-plane.

If $S$ is purely unrectifiable or of dimension less than $n-1$, our strengthening consists of being able to find tangent measures on one side of an $(n-1)$-plane containing a vector of our choice. We give an application of this result to the problem of characterising removable sets for harmonic functions.

## 1. Introduction

In this note we give a strengthening of a well-known technique in the subfield of geometric measure theory known as "Rectifiability and densities". Arguments using this technique are usually called "touching point arguments", they are used to establish conical density theorems of the following type: Given set $S$, we can (roughly speaking) show that for almost all points $x \in S$, there exists a sequence of radii $r_n \to 0$ such that $B_{r_n}(x) \cap S$ lies mostly on one side of an $(n-1)$-plane intersecting $x$. This essentially follows from the existence of large empty sub-balls in any ball centered on any point of $S$, which in turn follows from basic density estimates. The technique originates from work of Besicovitch [1], and has been used in an essential way in [6], [8] and [12].

For those familiar with the language of tangent measures, these arguments establish the existence of a tangent measure to the measure $H^s_{\lfloor S}$ that exists on one side of an $(n-1)$-plane. For sets $S$ that are unrectifiable or of dimension less than $n-1$, our strengthening consists of being able to show that $B_{r_n}(x) \cap S$ lies increasingly on one side of an $(n-1)$-plane containing a vector in $S^{n-1}$ of our choice.

This result comes from a reworking of the basic idea using Besicovitch–Federer projection theorem instead of density estimates. The proof is intricate, but in our opinion the method is classical. This work arose from the author's attempts to generalise rectifiability and density theorems outside Euclidean space, [5]. Our

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main lemma is an $n$-dimensional version of Lemma 14 from [5]. Touching point arguments have application in a variety of geometrical problems. We hope the results of this note might also be of some utility; by way of example, we apply the theorem to the results of Mattila and Paramonov, to give a strengthening (under more restrictive hypotheses) of one of their theorems related to the problem of characterising removable sets for harmonic functions.

First some notation. Let $B_r(z) := \{y \in \mathbf{R}^n : |y-z| < r\}$. Let $H^s$ denote Hausdorff $s$-measure. Let $P_V$ be the orthogonal projection onto subspace $V$.

We define the cone $X(x,v,\alpha) = \{y \in \mathbf{R}^n : \alpha |P_{v^\perp}(y-x)| \le |P_{\langle v\rangle}(y-x)|\}$; see Figure 1. Let $X^+(x,v,\alpha) = \{y \in \mathbf{R}^n : y \in X(x,v,\alpha) \text{ and } (y-x)\cdot v > 0\}$.

**Theorem 1.** *Given an integer $n \ge 3$ and real numbers $s \in (0, n-1]$ and $\varrho > 0$, let $S \subset \mathbf{R}^n$ be a set (purely unrectifiable in the case $s = n-1$) of positive finite $H^s$ measure.*

*Then for any $\psi \in S^{n-1}$ we have that for $H^s$ a.e. $x \in S$*

$$\liminf_{r\to 0} \frac{H^s\big(B_r(x) \cap S \cap X^+(x,\phi,\varrho)\big)}{r^s} = 0 \tag{1}$$

*for some $\phi \in \psi^\perp \cap S^{n-1}$.*

Note that for a set $S \subset \mathbf{R}^2$ that is purely unrectifiable or of dimension $s < 1$ a much stronger result holds: Given $\phi \in S^1$ and $\varrho > 0$, for $H^s$ a.e. $x \in S$ equation (1) holds true. This can be seen by simplified versions of the arguments of this paper. However in higher dimensions this stronger result is not true, a counter example is given by taking the cartesian product of a purely unrectifiable set with a interval.

**1.1. Application: Removable sets for harmonic functions.** A compact set $E \subset \mathbf{C}$ is said to be *removable* for bounded analytic functions if and only if the following holds true:

If $U \subset \mathbf{C}$ is open, $E \subset U$ and $f\colon U\backslash E \to \mathbf{C}$ is a bounded analytic function, then $f$ has an analytic extension to $U$.

The long standing problem of geometrically characterising removable sets was resolved for sets of finite $H^1$ measure in [2], building on work of [3] and [10]. It is known and relatively easy to show that (see [9]) sets of zero $H^1$ measure are removable and sets of Hausdorff dimension greater than 1 are not removable. So the characterisation is needed for sets of dimension 1. The complete characterisation for sets of dimension 1 (i.e. including sets of infinite $H^1$ measure) was achieved in [13].

First some notation: An *$m$-rectifiable* set $S$ in $\mathbf{R}^n$ is a set that can be covered by countably many $C^1$ submanifolds of dimension $m$. A *purely $m$-unrectifiable* set $T$ is a set with the property; $H^m(T \cap M) = 0$ for all $C^1$ $m$-dimensional submanifolds $M$.

It was proved in [2] that if $S$ has finite $H^1$ measure and is not removable then $S$ must have a 1-rectifiable subset with positive $H^1$ measure. The converse is well known, so sets of finite $H^1$ measure are removable if and only if they are purely 1-unrectifiable. Analogous questions can be asked about harmonic functions.

Following [11] a compact subset $E$ of $\mathbf{R}^n$ is called $\mathrm{Lip}_1$-removable for harmonic functions, (abbreviated $L_1RH$) if for each domain $D \subset \mathbf{R}^n$ every locally Lipschitz function $f\colon D \to \mathbf{R}$ which is harmonic in $D \backslash E$ is harmonic in $D$.

It would be reasonable to conjecture that $E$ is $\mathrm{Lip}_1$-removable if and only if it is purely $(n-1)$ rectifiable.

The only results on this conjecture are from [11] (Theorem 5.5) where it is proved:

**Theorem 2** (Mattila–Paramonov). *Let $X \subset \mathbf{R}^n$ be a compact set of finite $H^{n-1}$ measure such that for some constant $A > 0$,*

$$H^{n-1}\big(X \cap B_r(a)\big) \le A r^{n-1} \tag{2}$$

*for $a \in \mathbf{R}^n$ and $r > 0$. Suppose that the following holds at $H^{n-1}$ almost all points $a \in X$: For every $v \in S^{n-1}$ there is a $\delta > 0$ such that*

$$\liminf_{r\to 0} r^{1-n} H^{n-1}\big(y \in X \cap B_r(a) : |(y-a)\cdot v| > \delta |y-a|\big) > 0. \tag{3}$$

*Then $X$ is* $\mathrm{Lip}_1$*-removable.*

Given a set $S$ of positive finite $H^{n-1}$ measure, if $a \in S$ has the property that

$$\liminf_{r\to 0} r^{1-n} H^{n-1}\big(y \in S \cap B_r(a) : |(y-a)\cdot v| > \delta |y-a|\big) = 0 \tag{4}$$

for any $\delta > 0$ then we say $S$ has a *weak tangent* $V := v^{\perp}$ at $a$. Existence of weak tangents is a well-known partial result to rectifiability (see [7] where the term was coined); if the $\liminf$ of (4) is replaced by a $\limsup$, then this condition holding for almost every point $a \in S$ and for any $\delta > 0$ is equivalent to $(n-1)$-rectifiability, see [9].

So Theorem 2 says that any set (with density bound (2)) which is not $\mathrm{Lip}_1$-removable must have a subset $B$ of positive $H^{n-1}$ measure such that for each $a \in B$ there exists $v \in S^1$ with condition (4) holding for every $\delta > 0$. Informally: a non $\mathrm{Lip}_1$ removable set must have weak tangents on some subset of positive measure. This result suggests that these sets should be rectifiable.

Suppose the conjecture for harmonic functions is not true for sets with density bound (2) and so we have a purely $(n-1)$ unrectifiable set $X$ which is not $\mathrm{Lip}_1$-removable satisfying (2). Then by directly inserting the statement of Theorem 1 into the proof of Theorem 5.5 of Mattila–Paramonov [11] (instead of their Lemma 5.2) we have the following result:

**Theorem 3.** *Suppose* $S \subset \mathbf{R}^n$ *is a purely unrectifiable* $(n-1)$*-set which is not* $\mathrm{Lip}_1$*-removable and has the property*: *For some constant* $A > 0$

$$H^{n-1}\big(S \cap B_r(a)\big) \le A r^{n-1} \tag{5}$$

*for all* $a \in \mathbf{R}^n$, $r > 0$.

*Then there must exist a subset* $B \subset S$ *of positive* $H^{n-1}$ *measure such that for any* $\psi \in S^{n-1}$ *we have that for almost all* $x \in B$ *the set* $S$ *must have a weak tangent* $V$ *at* $x$ *with the property that* $\psi \in V$.

So if there exists a purely unrectifiable $(n-1)$-set $S$ with density bound (5) which is not $\mathrm{Lip}_1$ removable then it must have a subset $B$ of positive measure such that for almost every point $z \in B$ there exists an infinite collection of $(n-1)$ subspaces $\big\{V_1^z, V_2^z, \dots\big\}$ each of which is a weak tangent to $S$ at $z$.

*Acknowledgement.* I would like to thank the referee for reading the paper with great care and in particular for pointing out a hypothesis in Theorem 1 was not necessary.

## 2. Background and notation

First we will need to introduce some more notation. If $x \in v^{\perp}$, $v \in S^{n-1}$, $s > 0$, let $K(x,v,s) := P_{v^{\perp}}^{-1}\big(B_s(x) \cap v^{\perp}\big)$ be the standard definition of a cylinder.

We will also need to define a kind of double cone object, see Figure 1. Given $x \in \mathbf{R}^n$, $v \in S^{n-1}$, $s \in (0,1)$ and $r > 0$ define:

$$\Psi^u(x,v,s,r) := \big\{z \in B_r(x) : |P_{v^{\perp}}(z-x)| \le s|P_v(z-(x+rv))|\big\},$$

$$\Psi^d(x,v,s,r) := \big\{z \in B_r(x) : |P_{v^{\perp}}(z-x)| \le s|P_v(z-(x-rv))|\big\},$$

and $\Psi(x,v,s,r) = \Psi^u(x,v,s,r) \cap \Psi^d(x,v,s,r)$. Let $A(x,\alpha,\beta) := B_\beta(x) \backslash \overline{B_\alpha(x)}$.

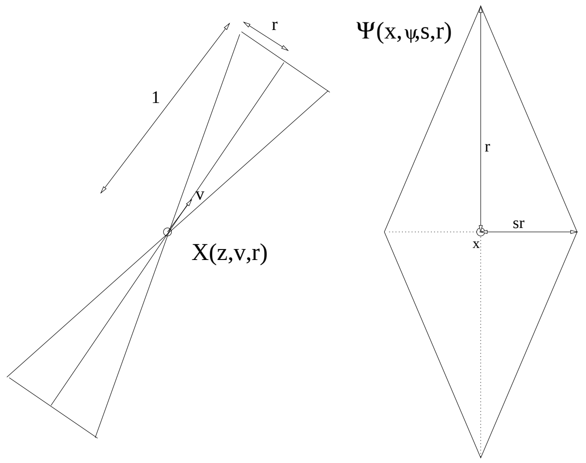


Figure 1.

## 3. Proof

We will first prove that Theorem 1 holds true for a.e. $\psi \in S^{n-1}$ (in the sense of $H^{n-1}$ measure on $S^{n-1}$), and then show how this implies the result for *every* $\psi \in S^{n-1}$.

Suppose the statement of Theorem 1 was false for some set $Y \subset S^{n-1}$ of positive measure. Then there must exist $\psi \in Y$ for which the following two statements hold true:

– We can take some subset $S_0 \subset S$ such that for some $\varrho > 0$ and some $\lambda_1 > 0$ we have for all $\phi \in \psi^\perp \cap S^{n-1}$

$$\liminf_{r\to 0} \frac{H^s\big(B_r(x) \cap X^+(x,\phi,\varrho) \cap S\big)}{r^s} > 2\lambda_1,$$

for all $x \in S_0$.

– By the fact that $S$ is unrectifiable or of dimension less than $n-1$, from the Besicovitch–Federer projection theorem ([9, Theorem 18.1]) (or in the latter case by an elementary result, ([9, Theorem 7.5])) we have

$$L^2\big(P_{\psi^\perp}(S)\big) = 0. \tag{6}$$

Now we can take a closed subset $S_1 \subset S_0$ and some small $r_0 > 0$ such that

$$\frac{H^s\big(B_r(x) \cap X^+(x,\phi,\varrho) \cap S\big)}{r^s} > \lambda_1 \tag{7}$$

for all $\phi \in \psi^\perp \cap S^{n-1}$, $x \in S_1$, $r \in (0, r_0)$. And (using [9, Theorem 6.2] for the upper bound on the density)

$$\lambda_1 < \frac{H^s\big(B_r(x) \cap S\big)}{r^s} < 2 \tag{8}$$

for all $x \in S_1$, $r \in (0, r_0)$.

First we prove the following.

**Lemma 1.** *Given an integer $n \geq 3$ and a real number $s \in (0,n)$ suppose we have a set $S \subset \mathbf{R}^n$ of positive finite $H^s$ measure and a closed subset $S_1 \subset S$ with the following two properties:*

*Firstly, there exists a vector $\psi \in S^{n-1}$ and a number $r_0 > 0$ such that for some small $\lambda_1 > 0$ and $\varrho > 0$ we have*

$$\frac{H^s\big(B_r(x) \cap X^+(x,\phi,\varrho) \cap S\big)}{r^s} > \lambda_1$$

*for all* $\phi \in \psi^{\perp} \cap S^{n-1}$ *and all* $x \in S_1$, $r \in (0, r_0)$.

*Secondly, we have*

$$\lambda_1 < \frac{H^s\big(B_r(x) \cap S\big)}{r^s} < 2 \tag{9}$$

*for all* $x \in S_1$, $r \in (0, r_0)$.

*Then we can find constants* $\kappa_{\varrho}^{\lambda_1} > 0$ *and* $\vartheta_{\varrho}^{\lambda_1} > 0$ *such that the following statement holds true*: *Suppose* $x \in S_1$ *and* $d \in (0, r_0)$ *is such that*

$$\frac{H^s\big(B_{4d}(x) \backslash S_1\big)}{d^s} \le \varepsilon \tag{10}$$

*then for all* $z \in (\psi^{\perp} + x) \cap K(x, \psi, \kappa_{\varrho}^{\lambda_1} d)$ *we have that*

$$K\big(z, \psi, \varepsilon^{1/s} \vartheta_{\varrho}^{\lambda_1} d\big) \cap S_1 \cap B_{2d}(x) \neq \emptyset.$$

*Proof.* Firstly to simplify the expressions, we let $\mu := H^s_{\lfloor S}$. Let $r \in (0, r_0)$. We start by showing the following:

If $a \in \partial K(x, \psi, r)$ let $W_a$ denote the $(n-1)$-dimensional tangent plane of the boundary of the cylinder $K(x, \psi, r)$ at point $a$. Let $n_a \in W_a^{\perp} \cap S^{n-1}$ be the unit normal pointing "inwards" towards the center of cylinder $K(x, \psi, r)$.

*Step* 1: We will show that we have

$$X^{+}(a, n_a, \varrho) \cap B_{2r\varrho/(1+\varrho)}(a) \subset K(x, \psi, r). \tag{11}$$

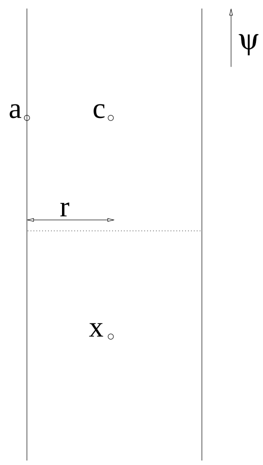


Figure 2.

To see this, it makes the calculations a lot easier if we change our orthonormal basis. Let $\varepsilon_1 = n_a$ and $\varepsilon_2 = \psi$ and complete this to get an orthonormal set of vectors $\{\varepsilon_j\}$ where $\{\varepsilon_2, \dots, \varepsilon_n\}$ span $W_a$. Let $c = a + r\varepsilon_1$. See Figure 2. Now in our new basis (using point $c$) the definition of the cylinder is as follows

$$K(x, \varepsilon_2, r) = \left\{ z \in \mathbf{R}^n : \big((z-c)\cdot\varepsilon_1\big)^2 + \sum_{j=3}^{n} \big((z-c)\cdot\varepsilon_j\big)^2 < r^2 \right\}.$$

And in our basis we have that

$$X^+(a, \varepsilon_1, \varrho) = \left\{ z \in \mathbf{R}^n : (z-a)\cdot\varepsilon_1 \ge \varrho \sqrt{\left( \sum_{k=2}^{n} \big((z-a)\cdot\varepsilon_k\big)^2 \right)} \right\}.$$

So if $z \in X^+(a, \varepsilon_1, \varrho) \cap B_{2r\varrho/(1+\varrho)}(a)$ we have that $(z-a)\cdot\varepsilon_1 \in (0, 2r\varrho/(1+\varrho))$ and so as $\varrho$ is small $(z-a)\cdot\varepsilon_1 \in (0, r)$ and we have

$$\begin{aligned}
\big((z-c)\cdot\varepsilon_1\big)^2 + \sum_{k=3}^{n}\big((z-c)\cdot\varepsilon_k\big)^2 &= \big((c-a)\cdot\varepsilon_1 - (z-a)\cdot\varepsilon_1\big)^2 \\
&\quad + \sum_{k=3}^{n}\big((z-c)\cdot\varepsilon_k\big)^2 \\
&= \big(r - (z-a)\cdot\varepsilon_1\big)^2 + \sum_{k=3}^{n}\big((z-c)\cdot\varepsilon_k\big)^2 \\
&\le \left( r - \varrho\sqrt{\left(\sum_{k=2}^{n}\big((z-a)\cdot\varepsilon_k\big)^2\right)} \right)^2 \\
&\quad + \sum_{k=3}^{n}\big((z-c)\cdot\varepsilon_k\big)^2 \qquad (12) \\
&= r^2 - 2r\varrho\sqrt{\left(\sum_{k=2}^{n}\big((z-a)\cdot\varepsilon_k\big)^2\right)} \\
&\quad + \varrho^2\left(\sum_{k=2}^{n}\big((z-a)\cdot\varepsilon_k\big)^2\right) \\
&\quad + \sum_{k=3}^{n}\big((z-c)\cdot\varepsilon_k\big)^2.
\end{aligned}$$

Now as $a = c - r\varepsilon_1$ we know that $(c-a)\cdot\varepsilon_k = 0$ for $k \in \{2,3,\dots,n\}$ so

$$\sum_{k=3}^{n}\big((z-a)\cdot\varepsilon_k\big)^2 = \sum_{k=3}^{n}\big((z-c)\cdot\varepsilon_k + (c-a)\cdot\varepsilon_k\big)^2 = \sum_{k=3}^{n}\big((z-c)\cdot\varepsilon_k\big)^2.$$

So putting this into (12) we get that

$$\big((z-c)\cdot\varepsilon_1\big)^2 + \sum_{k=3}^{n}\big((z-c)\cdot\varepsilon_k\big)^2 \le r^2 - 2r\varrho\sqrt{\bigg(\sum_{k=2}^{n}\big((z-a)\cdot\varepsilon_k\big)^2\bigg)} + (1+\varrho^2)\bigg(\sum_{k=2}^{n}\big((z-a)\cdot\varepsilon_k\big)^2\bigg). \tag{13}$$

Thus if

$$2r\varrho\sqrt{\bigg(\sum_{k=2}^{n}\big((z-a)\cdot\varepsilon_k\big)^2\bigg)} \ge (1+\varrho^2)\bigg(\sum_{k=2}^{n}\big((z-a)\cdot\varepsilon_k\big)^2\bigg) \tag{14}$$

then $z \in K(x,\varepsilon_2,r)$. Now (14) is equivalent to

$$2r\varrho \ge (1+\varrho^2)\sqrt{\bigg(\sum_{k=2}^{n}\big((z-a)\cdot\varepsilon_k\big)^2\bigg)} \tag{15}$$

and as $z \in B_{2r\varrho/(1+\varrho^2)}(a)$, (15) obviously holds and so (14) also holds and $z \in K(x,\psi,r)$; we have shown Step 1.

*Step 2*: Now suppose $a \in \partial K(x,\psi,r)$ and $n_a \in S^{n-1}\cap\psi^{\perp}$ such that

$$X^+(a,n_a,\varrho)\cap B_{r\varrho/(1+\varrho^2)}(a) \subset K(x,\psi,r)$$

and we have that

$$\mu\big(X^+(a,n_a,\varrho)\cap B_{r\varrho/(1+\varrho^2)}(a)\cap S_1\big) \ge \frac{\lambda_1}{2}\bigg(\frac{r\varrho}{(1+\varrho^2)}\bigg)^s. \tag{16}$$

We will show there exists some number $\zeta_\varrho^{\lambda_1} > 0$ with the property that

$$B_{r\varrho/(1+\varrho^2)}(a)\cap X^+(a,n_a,\varrho)\cap K\big(x,\psi,(1-\zeta_\varrho^{\lambda_1})r\big)\cap S_1 \ne \emptyset. \tag{17}$$

Now note that for any $\beta > 0$, if $z \in X^+(a, n_a, \varrho) \cap B_\beta(a)$ then $X^+(a, n_a, \varrho) \cap B_\beta(a) \subset B_{2\beta}(z)$. So if $X^+(a, n_a, \varrho) \cap B_\beta(a) \cap S_1 \neq \emptyset$ then by (9)

$$\mu\big(X^+(a, n_a, \varrho) \cap B_\beta(a)\big) \leq 2(2\beta)^s$$

hence there exists some small number $\sigma_{\lambda_1} \in \big(0, \frac{1}{8}\varrho\big)$ such that

$$\mu\bigg(A\bigg(a, \sigma_{\lambda_1} r, \frac{r\varrho}{(1+\varrho^2)}\bigg) \cap X^+(a, n_a, \varrho) \cap S_1\bigg) \geq \frac{\lambda_1}{4}\bigg(\frac{r\varrho}{(1+\varrho^2)}\bigg)^s.$$

We are going to use this and (13) to show Step 2. Firstly let

$$g(p) = -2r\varrho p + (1+\varrho^2)p^2,$$

the two zeros of this quadratic are at 0 and $2r\varrho/(1+\varrho^2)$. Since $\sigma_{\lambda_1}$ is smaller than $\frac{1}{8}\varrho$ its easy to see that $g(\sigma_{\lambda_1} r) = \big((1+\varrho^2)\sigma_{\lambda_1}^2 - 2\varrho\sigma_{\lambda_1}\big)r^2$ and $g\big(r\varrho/(1+\varrho^2)\big) = -(r\varrho)^2/(1+\varrho^2)$ are both less than zero.

If we let

$$\zeta_\varrho^{\lambda_1} = \frac{1}{2}\min\bigg\{-\frac{g(\sigma_{\lambda_1} r)}{r^2}, -\frac{g\big(r\varrho/(1+\varrho^2)\big)}{r^2}\bigg\}$$

then we have that for $q \in \big(\sigma_{\lambda_1} r, r\varrho/(1+\varrho^2)\big)$

$$g(q) \leq -2\zeta_\varrho^{\lambda_1} r^2.$$

Let $z \in A\big(a, \sigma_{\lambda_1} r, r\varrho/(1+\varrho^2)\big) \cap X^+(a, n_a, \varrho) \cap S_1$. Now in our notation inequality (13) becomes

$$\begin{aligned}\big((z-c)\cdot\varepsilon_1\big)^2 + \sum_{j=3}^n \big((z-c)\cdot\varepsilon_j\big)^2 &\leq r^2 + g\Bigg(\sqrt{\sum_{k=2}^n \big((z-a)\cdot\varepsilon_k\big)^2}\Bigg)\\ &\leq (1-2\zeta_\varrho^{\lambda_1})r^2\\ &\leq (1-\zeta_\varrho^{\lambda_1})^2 r^2\end{aligned}$$

so $z \in K\big(x, \psi, (1-\zeta_\varrho^{\lambda_1})r\big)$ and this establishes Step 2.

Let $\xi_\varrho = \varrho/(1+\varrho^2)$ and let $\kappa_\varrho^{\lambda_1} = \zeta_\varrho^{\lambda_1}/\xi_\varrho$. Given $x \in S_1$ let $d > 0$ such that for some small $\varepsilon > 0$

$$\frac{\mu\big(B_{4d}(x)\backslash S_1\big)}{d^s} \leq \varepsilon. \tag{18}$$

Let $z \in \psi^\perp + x$ be such that $\Psi(z, \psi, \kappa_\varrho^{\lambda_1}, d) \subset B_{2d}(x)$.

Define

$$H(z, \psi^\perp, \alpha) := \bigcup_{h\in(-\alpha,\alpha)} z + h\psi + \psi^\perp.$$

Let $\phi_k = \sum_{j=0}^k \xi_\varrho \kappa_\varrho^{\lambda_1} d(1-\zeta_\varrho^{\lambda_1})^j$ where we let $\phi_{-1} = 0$. Define

$$\Gamma_k := H(z, \psi^\perp, \phi_k) \cap K\big(z, \psi, (1-\zeta_\varrho^{\lambda_1})^k \kappa_\varrho^{\lambda_1} d\big). \tag{19}$$

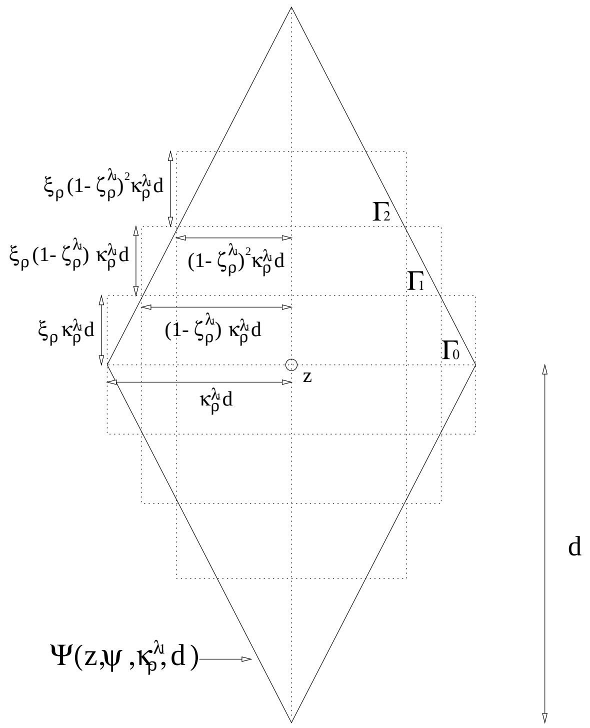


Figure 3.

*Step* 3: We will show

$$\Psi(z,\psi,\kappa_\varrho^{\lambda_1},d)\subset\bigcup_{j=0}^{\infty}\Gamma_j. \tag{20}$$

See Figure 3.

We argue inductively. Firstly recall that $\phi_0=\xi_\varrho\kappa_\varrho^{\lambda_1}d$ and note that it is obvious that

$$\Psi(z,\psi,\kappa_\varrho^{\lambda_1},d)\cap H(z,\psi^{\perp},\xi_\varrho\kappa_\varrho^{\lambda_1}d)\subset\Gamma_0.$$

Suppose we have that

$$\Psi(z,\psi,\kappa_\varrho^{\lambda_1},d)\cap H(z,\psi^{\perp},\phi_k)\subset\bigcup_{j=0}^{k}\Gamma_j. \tag{21}$$

Let $\vartheta_{k+1}$ be the radius of the two congruent spheres given by $\partial\Psi(z,\psi,\kappa_\varrho^{\lambda_1},d)\cap$

$\partial H(z,\psi^\perp,\phi_k)$. So by definition of $\Psi(z,\psi,\kappa_\varrho^{\lambda_1},d)$ we know

$$\begin{aligned}\vartheta_{k+1} &= \kappa_\varrho^{\lambda_1} d - \kappa_\varrho^{\lambda_1}\phi_k \\ &= \kappa_\varrho^{\lambda_1} d - \kappa_\varrho^{\lambda_1}\left(\sum_{j=0}^{k}\xi_\varrho\kappa_\varrho^{\lambda_1} d(1-\zeta_\varrho^{\lambda_1})^j\right) \\ &= \kappa_\varrho^{\lambda_1} d - \xi_\varrho(\kappa_\varrho^{\lambda_1})^2 d\left(\sum_{j=0}^{k}(1-\zeta_\varrho^{\lambda_1})^j\right).\end{aligned} \tag{22}$$

And note

$$\sum_{j=0}^{k}(1-\zeta_\varrho^{\lambda_1})^j = \frac{1}{\zeta_\varrho^{\lambda_1}}\big(1-(1-\zeta_\varrho^{\lambda_1})^{k+1}\big).$$

Recall $\kappa_\varrho^{\lambda_1} = \zeta_\varrho^{\lambda_1}/\xi_\varrho$ so putting the above expression into (22) we get

$$\vartheta_{k+1} = \kappa_\varrho^{\lambda_1} d(1-\zeta_\varrho^{\lambda_1})^{k+1}$$

which is exactly the width of the cylinder $\Gamma_{k+1}$.

Now $\vartheta_{k+1}$ is the biggest radius of the spheres given by $\psi^\perp$ slices of

$$\partial\Psi(z,\psi,\kappa_\varrho^{\lambda_1},d)\cap\big(H(z,\psi^\perp,\phi_{k+1})\backslash H(z,\psi^\perp,\phi_k)\big).$$

So we know that $\Psi(z,\psi,\kappa_\varrho^{\lambda_1},d)\cap\big(H(z,\psi^\perp,\phi_{k+1})\backslash H(z,\psi^\perp,\phi_k)\big)$ is "thin" enough to fit into $\Gamma_{k+1}$. It is also, by definition, "short" enough to fit into $\Gamma_{k+1}$. So by inductive assumption (21) we have

$$\Psi(z,\psi,\kappa_\varrho^{\lambda_1},d)\cap H(z,\psi^\perp,\phi_{k+1})\subset\bigcup_{j=0}^{k+1}\Gamma_j.$$

This establishes Step 3.

*Step* 4: Recall $z\in\psi^\perp+x$ such that $\Psi(z,\psi,\kappa_\varrho^{\lambda_1},d)\subset B_{2d}(x)$.

Suppose $\Psi(z,\psi,\kappa_\varrho^{\lambda_1},d)\cap S_1\neq\emptyset$. Let $W:=\big\{k:\Gamma_k\cap\Psi(z,\psi,\kappa_\varrho^{\lambda_1},d)\cap S_1\neq\emptyset\big\}$. If $W$ is finite we define $k_1:=\max\big\{k:k\in W\big\}$, otherwise we define $k_1$ to be any positive integer such that $\lambda_1(1-\zeta_\varrho^{\lambda_1})^{s(k_1+1)}(\xi_\varrho\kappa_\varrho^{\lambda_1}d)^s<2\varepsilon d^s$.

In the case where $W$ is finite we will show $k_1$ is sufficiently big so that

$$\lambda_1(1-\zeta_\varrho^{\lambda_1})^{s(k_1+1)}(\xi_\varrho\kappa_\varrho^{\lambda_1}d)^s<2\varepsilon d^s.$$

Suppose not and

$$\lambda_1(1-\zeta_\varrho^{\lambda_1})^{s(k_1+1)}(\xi_\varrho\kappa_\varrho^{\lambda_1}d)^s\geq 2\varepsilon d^s. \tag{23}$$

We know that $\Gamma_{k_1+1}\cap\Psi(z,\psi,\kappa_\varrho^{\lambda_1},d)\cap S_1=\emptyset$. Let $y_1\in\Gamma_{k_1}\cap\Psi(z,\psi,\kappa_\varrho^{\lambda_1},d)\cap S_1$, let $h=|P_{\psi^\perp}(y_1-z)|$ so we know that

$$h\in\big((1-\zeta_\varrho^{\lambda_1})^{k_1+1}\kappa_\varrho^{\lambda_1}d,(1-\zeta_\varrho^{\lambda_1})^{k_1}\kappa_\varrho^{\lambda_1}d\big). \tag{24}$$

Now $y_1\in\Psi(z,\psi,\kappa_\varrho^{\lambda_1},d)\subset B_{2d}(x)$ and as $h\xi_\varrho\le\xi_\varrho(1-\zeta_\varrho^{\lambda_1})^{k_1}\kappa_\varrho^{\lambda_1}d=\zeta_\varrho^{\lambda_1}(1-\zeta_\varrho^{\lambda_1})d$ so as in Step 1, letting $n_{y_1}$ be the “inwards” pointing unit normal of $\partial K(z,\psi,h)$ at point $y_1$, we have

$$X^+(y_1,n_{y_1},\varrho)\cap B_{h\xi_\varrho}(y_1)\subset B_d(y_1)\subset B_{4d}(x). \tag{25}$$

Now by the fact that $y_1\in S_1$ (recall (7)) we know that

$$\mu\big(B_{h\xi_\varrho}(y_1)\cap X^+(y_1,n_{y_1},\varrho)\big)\ge\lambda_1(\xi_\varrho h)^s.$$

By using (23), (24) we know $\lambda_1(\xi_\varrho h)^s\ge 2\varepsilon d^s$ and so from (25) and the density estimate (10) we know that we have

$$\mu\big(B_{h\xi_\varrho}(y_1)\cap X^+(y_1,n_{y_1},\varrho)\cap S_1\big)\ge\lambda_1(\xi_\varrho h)^s-\varepsilon d^s\ge\tfrac12\lambda_1(\xi_\varrho h)^s$$

and so by equations (16) and (17) (recall $\xi_\varrho=\varrho/(1+\varrho^2)$) we can pick $y_2\in X^+(y_1,n_{y_1},\varrho)\cap B_{h\xi_\varrho}(y_1)\cap S_1$ such that

$$y_2\in K\big(z,\psi,(1-\zeta_\varrho^{\lambda_1})h\big)\subset K\big(z,\psi,(1-\zeta_\varrho^{\lambda_1})^{k_1+1}\kappa_\varrho^{\lambda_1}d\big). \tag{26}$$

Suppose $y_1\in\Psi^u(z,\psi,\kappa_\varrho^{\lambda_1},d)$. This means

$$h=|P_{\psi^\perp}(y_1-z)|\le\kappa_\varrho^{\lambda_1}|P_{\langle\psi\rangle}(y_1-(z+d\psi))|.$$

So from (26)

$$|P_{\psi^\perp}(y_2-z)|\le(1-\zeta_\varrho^{\lambda_1})h\le(1-\zeta_\varrho^{\lambda_1})\kappa_\varrho^{\lambda_1}\big|P_{\langle\psi\rangle}\big(y_1-(z+d\psi)\big)\big|.$$

And since $y_2\in B_{h\xi_\varrho}(y_1)$

$$\begin{aligned}\big|P_{\langle\psi\rangle}\big(y_2-(z+d\psi)\big)\big|&\ge\big|P_{\langle\psi\rangle}\big(y_1-(z+d\psi)\big)\big|-h\xi_\varrho\\&\ge\big|P_{\langle\psi\rangle}\big(y_1-(z+d\psi)\big)\big|-\xi_\varrho\kappa_\varrho^{\lambda_1}\big|P_{\langle\psi\rangle}\big(y_1-(z+d\psi)\big)\big|\\&=\big|P_{\langle\psi\rangle}\big(y_1-(z+d\psi)\big)\big|(1-\zeta_\varrho^{\lambda_1}).\end{aligned}$$

Putting these two together we have

$$|P_{\psi^\perp}(y_2-z)|\le\kappa_\varrho^{\lambda_1}\big|P_{\langle\psi\rangle}\big(y_2-(z+d\psi)\big)\big|.$$

So $y_2 \in \Psi^u(z, \psi, \kappa_\varrho^{\lambda_1}, d)$ and similarly

$$y_1 \in \Psi^d(z, \psi, \kappa_\varrho^{\lambda_1}, d) \quad \text{implies} \quad y_2 \in \Psi^d(z, \psi, \kappa_\varrho^{\lambda_1}, d)$$

so we have $y_2 \in \Psi(z, \psi, \kappa_\varrho^{\lambda_1}, d)$.

Now we want to show that $y_2 \in \left(\bigcup_{j=k_1+1}^{\infty} \Gamma_j\right)$. We know from (20) that $y_2 \in \bigcup_{j=1}^{\infty} \Gamma_j$. However we also know from (26) that $|P_{\psi_\perp}(y_2 - z)|$ is sufficiently small (recall the definition of $\Gamma_{k_1+1}$, see (19)) so that we must have $y_2 \in \left(\bigcup_{j=k_1+1}^{\infty} \Gamma_j\right)$ and hence

$$y_2 \in \Psi(z, \psi, \kappa_\varrho^{\lambda_1}, d) \cap \left(\bigcup_{j=k_1+1}^{\infty} \Gamma_j\right) \cap S_1,$$

contradicting the maximality of $k_1$. So we have established Step 4, and so we know (recall (23)) that

$$\lambda_1 (1 - \zeta_\varrho^{\lambda_1})^{s(k_1+1)} (\xi_\varrho \kappa_\varrho^{\lambda_1} d)^s < 2\varepsilon d^s. \tag{27}$$

Let $\theta = (1 - \zeta_\varrho^{\lambda_1})^{k_1} \kappa_\varrho^{\lambda_1} d$ and recall that this is the width of cylinder $\Gamma_{k_1}$. So

$$2\varepsilon d^s > \lambda_1 (1 - \zeta_\varrho^{\lambda_1})^{s(k_1+1)} (\xi_\varrho \kappa_\varrho^{\lambda_1} d)^s = \lambda_1 (1 - \zeta_\varrho^{\lambda_1})^s \xi_\varrho^s \theta^s.$$

And so

$$\theta < \frac{(2\varepsilon)^{1/s} d}{\lambda_1^{1/s} (1 - \zeta_\varrho^{\lambda_1}) \xi_\varrho}.$$

To summarise what we have proved: we have shown that for any $z \in \psi^\perp + x$ such that $\Psi(z, \psi, \kappa_\varrho^{\lambda_1}, d) \subset B_{2d}(x)$, if we know that

$$\Psi(z, \psi, \kappa_\varrho^{\lambda_1}, d) \cap S_1 \neq \emptyset,$$

then

$$K\left(z, \psi, \frac{(2\varepsilon)^{1/s} d}{\lambda_1^{1/s} (1 - \zeta_\varrho^{\lambda_1}) \xi_\varrho}\right) \cap B_{2d}(x) \cap S_1 \neq \emptyset. \tag{28}$$

As we know that for any $z \in (\psi^\perp + x) \cap K(x, \psi, \kappa_\varrho^{\lambda_1} d)$, (recall this is the hypothesis on point $z$ in the statement of Lemma 1), we have

$$x \in \Psi(z, \psi, \kappa_\varrho^{\lambda_1}, d)$$

and thus $\Psi(z, \psi, \kappa_\varrho^{\lambda_1}, d) \cap S_1 \neq \emptyset$ and so by (28) for

$$\vartheta_\varrho^{\lambda_1} = \frac{2^{1/s}}{\lambda_1^{1/s} (1 - \zeta_\varrho^{\lambda_1}) \xi_\varrho}$$

we have $K(z, \psi, \varepsilon^{1/s} \vartheta_\varrho^{\lambda_1} d) \cap S_1 \cap B_{2d}(x) \neq \emptyset$ which gives us the conclusion of the lemma. Thus we have completed the proof. □

**Proof of Theorem 1 continued**

Let $\varepsilon > 0$ be some very small number. Again we simplify expressions by letting $\mu := H^s_{\lfloor S}$. Let $x \in S_1$ be a density point of $S_1$, let $d \in (0, r_0)$ such that

$$\frac{\mu\big(B_{16d}(x)\backslash S_1\big)}{d^s} \leq \varepsilon. \tag{29}$$

We can partition $[2d, 4d]$ into $M := \big[2/4\varepsilon^{1/s}\vartheta_\varrho^{\lambda_1}\big]$ subintervals. Call them $I_1$, $I_2, \dots, I_M$ and let $c_k$ be the center of interval $I_k$. Now for $k \in \{1, 2, \dots, M\}$ we know the following is true: For all $z \in (\psi^\perp + x) \cap K\big(x, \psi, \kappa_\varrho^{\lambda_1}\frac{1}{2}c_k\big)$ by Lemma 1 we have

$$K\big(z, \psi, \varepsilon^{1/s}\vartheta_\varrho^{\lambda_1}\tfrac{1}{2}c_k\big) \cap S_1 \cap B_{c_k}(x) \neq \emptyset. \tag{30}$$

Now fix $k \in \{1, 2, \dots, M\}$. From (30) we have that for every $z \in (\psi^\perp + x) \cap K(x, \psi, \kappa_\varrho^{\lambda_1} d)$ we have $K(z, \psi, 2\varepsilon^{1/s}\vartheta_\varrho^{\lambda_1} d) \cap S_1 \cap B_{c_k}(x) \neq \emptyset$.

Let $E_k = (\psi^\perp + x)\backslash P_{\psi^\perp}^{-1}\big(P_{\psi^\perp}(S_1 \cap \overline{B_{c_k}(x)})\big)$.

Now recall since $S_1$ is purely unrectifiable or of Hausdorff dimension $< n-1$ by our choice of $\psi$ (recall we are first proving the theorem for a.e. $\psi \in S^{n-1}$, see (5)) we have

$$L^{n-1}(E_k) \geq L^{n-1}\big(B_1(0)\big)(\kappa_\varrho^{\lambda_1} d)^{n-1}.$$

Since $S_1$ is closed, $E_k$ is open. Thus for each $z \in E_k$ we can find

$$r_z \in (0, 2\varepsilon^{1/s}\vartheta_\varrho^{\lambda_1} d) \tag{31}$$

such that

- $K(z, \psi, r_z) \cap B_{c_k}(x) \cap (\psi^\perp + x) \cap S_1 = \emptyset$,
- $\partial K(z, \psi, r_z) \cap B_{c_k}(x) \cap (\psi^\perp + x) \cap S_1 \neq \emptyset$.

Now by the $5r$ covering theorem ([9, Theorem 2.1]) from the covering of $E_k$ given by $\{B_{r_z}(z) : z \in E_k\}$ we can find a disjoint collection of balls

$$\big\{B_{r^k_m}(z^k_m) : z^k_m \in E_k,\ m = 1, 2, \dots, N_k\big\}$$

which are centered on points of $E_k$ and are such that

$$\sum_{m=1}^{N_k} (r^k_m)^{n-1} > \frac{(\kappa_\varrho^{\lambda_1} d)^{n-1}}{5^{n-1}}. \tag{32}$$

Now each $m \in \{1, 2, \dots, N_k\}$ we can pick a point

$$x^k_m \in \partial K(z^k_m, \psi, r^k_m) \cap \overline{B_{c_k}(x)} \cap S_1.$$

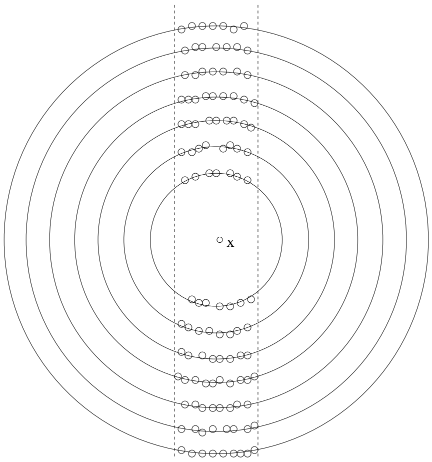


Figure 4.

The essential point is that by the fact that the points $\{x_m^k : m = 1,2,\dots,N_k\}$ satisfy inequality (7) in every direction perpendicular to $\psi$, unless the points $\{x_m^k : m = 1,\dots,N_k\}$ are very near the surface of the ball $B_{c_k}(x)$, there will be some fraction of the set $S_1$ in the interior of the cylinders $\{K(z_m^k,\psi,r_m^k),\ m = 1,\dots,N_k\}$ of Hausdorff measure approximately $\sum_{k=1}^{N_k}(r_m^k)^s$.

Now since $s \in (0, n-1]$, by the well-known inequality ([4, Theorem 19])

$$\left(\sum_{m=1}^{N_k}(r_m^k)^s\right)^{1/s} \geq \left(\sum_{m=1}^{N_k}(r_m^k)^{n-1}\right)^{1/(n-1)} \geq \frac{\kappa_\varrho^{\lambda_1} d}{5} \tag{33}$$

so we see this can not happen too often, see Figure 4. Proceeding formally, for each $k \in \{1,2,\dots,M\}$, $m \in \{1,2,\dots,N_k\}$ let $n_{x_m^k}$ denote the inner normal to the boundary of the cylinder $K(z_m^k,\psi,r_m^k)$ at point $x_m^k$. Again let us fix $k \in \{1,2,\dots,M\}$. We know that for each $m \in \{1,\dots,N_k\}$

$$\mu\big(X^+(x_m^k, n_{x_m^k}, \varrho) \cap B_{\xi_\varrho r_m^k}(x_m^k)\big) \geq \lambda_1(\xi_\varrho r_m^k)^s. \tag{34}$$

Let $I_1^k = \{m \in \{1,2,\dots,N_k\} : X^+(x_m^k, n_{x_m^k}, \varrho) \cap B_{\xi_\varrho r_m^k}(x_m^k) \subset B_{c_k}(x)\}$. Let $I_2^k = \{1,2,\dots,N_k\}\backslash I_1^k$. Now firstly for any $m \in I_1^k$ as

$$X^+(x_m^k, n_{x_m^k}, \varrho) \cap B_{\xi_\varrho r_m^k}(x_m^k) \subset K(z_m^k,\psi,r_m^k) \cap B_{c_k}(x)$$

and so $X^+(x_m^k, n_{x_m^k}, \varrho) \cap B_{\xi_\varrho r_m^k}(x_m^k) \subset B_{c_k}(x)\backslash S_1$, thus we have

$$\mu\left(\bigcup_{m\in I_1^k} X^+(x_m^k, n_{x_m^k}, \varrho) \cap B_{\xi_\varrho r_m^k}(x_m^k)\right) \leq \varepsilon d^s.$$

So

$$\sum_{m\in I_1^k}(r_m^k)^s \le \frac{\varepsilon d^s}{\lambda_1\xi_\varrho^s}.$$

Thus given sufficient smallness of $\varepsilon$, from (33) we have

$$\sum_{m\in I_2^k}(r_m^k)^s \ge \frac{1}{5^s}(\kappa_\varrho^{\lambda_1}d)^s - \frac{\varepsilon d^s}{\lambda_1\xi_\varrho^s} \ge \frac{1}{2\times 5^s}(\kappa_\varrho^{\lambda_1}d)^s. \tag{35}$$

However by definition of $I_2^k$ we know that for $k=1,2,\ldots,M$ (recall (31))

$$\bigl\{B_{\xi_\varrho r_m^k}(x_m^k), m\in I_2^k\bigr\} \subset N_{2\varepsilon^{1/s}\vartheta_\varrho^{\lambda_1}d}\bigl(\partial B_{c_k}(x)\bigr).$$

So using the density estimate of subset $S_1$ (i.e. (8)) and (35)

$$\begin{aligned}\mu\bigl(B_{5d}(x)\bigr) &\ge \mu\biggl(\bigcup_{k=1}^M N_{2\varepsilon^{1/s}\vartheta_\varrho^{\lambda_1}d}\bigl(\partial B_{c_k}(x)\bigr)\biggr)\\ &\ge \sum_{k=1}^M\sum_{m\in I_2^k}\mu\bigl(B_{\xi_\varrho r_m^k}(x_m^k)\bigr)\\ &\ge \frac{\lambda_1\xi_\varrho^s}{2}\biggl(\frac{M}{5^s}\biggr)(\kappa_\varrho^{\lambda_1}d)^s\end{aligned}$$

and since $M$ can be made as large as we like by reducing $\varepsilon$ (recall that $M=\bigl[2/4\varepsilon^{1/s}\vartheta_\varrho^{\lambda_1}\bigr]$) we end up contradicting the density bound (8) and thus Theorem 1 is proved for almost every $\psi\in S^{n-1}$.

Now for arbitrary $\psi\in S^{n-1}$ we can argue in the following way: Suppose the conclusion of Theorem 1 is false for $\psi$. Then as before we must have some subset $S_1\subset S$ and some $\varrho>0$, $\lambda_1>0$ such that for all $\phi\in\psi^\perp\cap S^{n-1}$

$$\liminf_{r\to 0}\frac{H^s\bigl(B_r(x)\cap X^+(x,\phi,\varrho)\cap S\bigr)}{r^s} > \lambda_1,$$

for all $x\in S_1$.

Now by elementary geometry we can see that there exists a small number $a(\varrho)>0$ such that any $\tilde\psi\in S^{n-1}$ for which $|\tilde\psi-\psi|<a(\varrho)$ has the following property:

For any $\tilde\phi\in S^{n-1}\cap\tilde\psi^\perp$ there exists $\phi\in S^{n-1}\cap\psi^\perp$ such that

$$X^+(z,\phi,\varrho)\subset X^+\bigl(z,\tilde\phi,\tfrac12\varrho\bigr) \tag{36}$$

for any $z\in\mathbf{R}^n$.

Since we have proved Theorem 1 for almost every $\tilde{\psi} \in S^{n-1}$ we can certainly find a $\tilde{\psi} \in S^{n-1}$ for which it is true and $|\tilde{\psi} - \psi| < a(\varrho)$. Then for $H^n$ a.e. $x_0 \in S_1$ we have that for some $\tilde{\phi} \in S^{n-1} \cap \tilde{\psi}^{\perp}$:

$$\liminf_{r \to 0} \frac{H^s\bigl(B_r(x_0) \cap X^+\bigl(x_0, \tilde{\phi}, \tfrac{1}{2}\varrho\bigr) \cap S\bigr)}{r^s} = 0.$$

However by (36) this implies that

$$\liminf_{r \to 0} \frac{H^s\bigl(B_r(x_0) \cap X^+(x_0, \phi, \varrho) \cap S\bigr)}{r^s} = 0,$$

which contradicts the definition of $S_1$ so we are done. □

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