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POTENTIAL THEORY OF QUASIMINIMIZERS

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Abstract. We study nonlinear potential theory related to quasiminimizers on a metric measure space equipped with a doubling measure and supporting a Poincaré inequality. Our objective is to show that quasiminimizers create an interesting potential theory with new features although from the potential theoretic point of view they have several drawbacks: They do not provide a unique solution to the Dirichlet problem, they do not obey the comparison principle and they do not form a sheaf. However, many potential theoretic concepts such as harmonic functions, superharmonic functions and the Poisson modification have their counterparts in the theory of quasiminimizers and, in particular, we are interested in questions related to regularity, convergence and polar sets.

1. Introduction

Quasiminimizers minimize a variational integral only up to a multiplicative constant. More precisely, let $\Omega \subset \mathbf{R}^n$ be an open set, $K \geq 1$ and $1 < p < \infty$. In the case of the p-Dirichlet integral, a function u belonging to the Sobolev space $W^{1,p}_{\text{loc}}(\Omega)$ is a K-quasiminimizer, if

$$(1.1)\qquad \int_{\Omega'} |\nabla u|^p \, dx \leq K \int_{\Omega'} |\nabla v|^p \, dx$$

for all functions $v \in W^{1,p}(\Omega')$ with $v - u \in W^{1,p}_0(\Omega')$ and for all open sets Ω' with compact closure in Ω. A 1-quasiminimizer, called a minimizer, is a weak solution of the corresponding Euler equation

$$(1.2)\qquad \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0.$$

Clearly being a weak solution of (1.2) is a local property. However, being a K-quasiminimizer is not a local property as one-dimensional examples easily show. This indicates that the theory for quasiminimizers usually differs from the theory for minimizers and that there are some unexpected difficulties.

Quasiminimizers have been previously used as tools in studying regularity of minimizers of variational integrals, see [GG1–2]. The advantage of this approach is that it covers a wide range of applications and that it is based only

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on the minimization of the variational integral instead of the corresponding Euler equation. Hence regularity properties as Hölder continuity and L^p-estimates are consequences of the quasiminimizing property. For us an important fact is that nonnegative quasiminimizers satisfy the Harnack inequality, see [DT].

Instead of using quasiminimizers as tools, our objective is to show that quasiminimizers have a fascinating theory themselves. In particular, they form a basis for nonlinear potential theoretic model with interesting features. From the potential theoretic point of view quasiminimizers have several drawbacks: They do not provide unique solutions to the Dirichlet problem, they do not obey the comparison principle, they do not form a sheaf and they do not have a linear structure even when the corresponding Euler equation is linear. However, quasiminimizers form a wide and flexible class of functions in calculus of variations under very general circumstances. To emphasize this we study potential theory of quasiminimizers in metric measure spaces although most of the results are new even in the Euclidean setup. Observe that the quasiminimizing condition (1.1) applies not only to one particular variational integral but the whole class of variational integrals at the same time. For example, if a variational kernel $F(x,\nabla u)$ satisfies

$$\alpha|h|^p \le F(x,h) \le \beta|h|^p \tag{1.3}$$

for some $0<\alpha\le\beta<\infty$, then the minimizers of

$$\int F(x,\nabla u)\,dx$$

are quasiminimizers of the p-Dirichlet integral

$$\int |\nabla u|^p\,dx. \tag{1.4}$$

Hence the potential theory for quasiminimizers includes all minimizers of all variational integrals similar to (1.4). The essential feature of the theory is the control provided by the bounds in (1.3). We also mention that quasiminimizers are preserved under a bilipschitz change of coordinates, but the constant K may change.

This work is organized as follows. In Section 2 we recall the basic properties of the Sobolev spaces on metric measure spaces. Quasiminimizers and quasisuperminimizers are studied in Section 3. Quasisuperminimizers replace the class of superminimizers in the classical setup.

In Section 4 we introduce the Poisson modification of a quasisuperminimizer. The Poisson modification lies below the original quasisuperminimizer, but it need not be a minimizer inside the set of modification.

A minimizer satisfies a Harnack inequality and is locally Hölder continuous after a redefinition on a set of measure zero. In Section 5 we show, using the celebrated De Giorgi method adapted to metric spaces, that a quasisuperminimizer satisfies the weak Harnack inequality. From this it follows that quasisuperminimizers are lower semicontinuous.

Section 6 is devoted to convergence properties of quasisuperminimizers. We show that the class of quasisuperminimizers is closed under monotone convergence, provided the limit function is locally bounded. This will be a crucial fact for us in Section 7 where we define quasisuperharmonic functions. Quasisuperharmonic functions play a central role in the potential theory of quasiminimizers. However, it is not obvious how quasisuperharmonic functions should be defined. The standard definition based on the comparison principle is useless because the Dirichlet problem does not have a unique solution. Our definition has a global nature because of the lack of the sheaf property. In Section 8 we study equivalent definitions of superharmonicity. The Poisson modification of a quasisuperharmonic function is considered in Section 9. Finally, Section 10 deals with polar sets.

2. Newtonian spaces

Let X be a metric space and let μ be a Borel measure on X. Throughout the paper we assume that the measure of every nonempty open set is positive and that the measure of every bounded set is finite. Later we impose further requirements on the space and on the measure, see 2.9.

2.1. Upper gradients. Let u be a real-valued function on X. A nonnegative Borel measurable function g on X is said to be an *upper gradient* of u if for all rectifiable paths γ joining points x and y in X we have

$$|u(x) - u(y)| \le \int_\gamma g\, ds. \tag{2.2}$$

See [C], [He] and [Sh1] for a discussion on upper gradients.

A property is said to hold for *p-almost all* paths, if the set of paths for which the property fails is of zero p-modulus. If (2.2) holds for p-almost all paths γ, then g is said to be a *p-weak upper gradient* of u.

2.3. Newtonian spaces. Let $1 \le p < \infty$. We define the space $\widetilde{N}^{1,p}(X)$ to be the collection of all p-integrable functions u on X that have a p-integrable p-weak upper gradient g on X. This space is equipped with the seminorm

$$\|u\|_{\widetilde{N}^{1,p}(X)} = \|u\|_{L^p(X)} + \inf \|g\|_{L^p(X)},$$

where the infimum is taken over all p-weak upper gradients of u.

We define an equivalence relation in $\widetilde{N}^{1,p}(X)$ by saying that $u \sim v$ if

$$\|u - v\|_{\widetilde{N}^{1,p}(X)} = 0.$$

The *Newtonian space* $N^{1,p}(X)$ is defined to be the space $\widetilde{N}^{1,p}(X)/\sim$ with the norm

$$\|u\|_{N^{1,p}(X)} = \|u\|_{\widetilde{N}^{1,p}(X)}.$$

For basic properties of the Newtonian spaces we refer to [Sh1]. Cheeger [C] gives an alternative definition which leads to the same space when $1 < p < \infty$, see [Sh1].

We recall that if $1 < p < \infty$, every function u that has a p-integrable p-weak upper gradient has a *minimal p-integrable p-weak upper gradient* in X, denoted g_u, in the sense that if g is another p-weak upper gradient of u, then $g_u \le g$ μ-almost everywhere in X.

2.4. Capacity. The *p-capacity* of a set $E \subset X$ is defined by

$$\mathrm{C}_p(E) = \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$, with $u = 1$ on E. The discussion in [KM2] can easily be adapted to show that the capacity is an outer measure, see also [Sh1]. The p-capacity is the natural measure for exceptional sets of Sobolev functions.

Let Ω be an open subset of X. We say that a subset E of Ω is *compactly contained* in Ω, abbreviated $E \Subset \Omega$, if the closure of E is a compact subset of Ω. Let $\Omega \subset X$ be bounded and $E \Subset \Omega$. The *relative p-capacity* of E with respect to Ω is the number

$$\mathrm{C}_p(E, \Omega) = \inf \int_X g_u^p \, d\mu,$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$ such that $u = 1$ on E and $u = 0$ on $X \setminus \Omega$.

2.5. Newtonian spaces with zero boundary values. Let E be an arbitrary subset of X. We define $N_0^{1,p}(E)$ to be the set of functions $u \in N^{1,p}(X)$ for which

$$\mathrm{C}_p\big(\{x \in X \setminus E : u(x) \ne 0\}\big) = 0.$$

The space $N_0^{1,p}(E)$ equipped with the norm

$$\|u\|_{N_0^{1,p}(E)} = \|u\|_{N^{1,p}(X)},$$

is the *Newtonian space with zero boundary values.* The norm is unambiguously defined by [Sh1] and the space $N_0^{1,p}(E)$ with this norm is a Banach space.

2.6. Local Newtonian spaces. Let Ω be an open subset of X. We say that u belongs to the *local Newtonian space* $N^{1,p}_{\mathrm{loc}}(\Omega)$ if $u \in N^{1,p}(E)$ for every measurable set $E \Subset \Omega$. If $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ with $1 < p < \infty$, then u has a minimal p-weak upper gradient g_u in Ω in the following sense: If $\Omega' \Subset \Omega$ is an open set and g is the minimal upper gradient of u in Ω', then $g_u = g$ μ-almost everywhere in Ω'.

2.7. Doubling property. The measure μ is *doubling*, if there is a constant $c_d \geq 1$ so that

$$\mu\big(B(z,2r)\big) \leq c_d \mu\big(B(z,r)\big)$$

for every open ball $B(z,r)$ in X. The constant c_d is called the *doubling constant* of μ.

A metric space X is said to be *doubling* if there is a constant $c < \infty$ such that every ball $B(x,r)$, $x \in X$, $r > 0$, can be covered by at most c balls with the radii $r/2$. If X is equipped with a doubling measure, then X is doubling.

2.8. Poincaré inequalities. Let $1 \leq q < \infty$. The space X is said to *support a weak* $(1,q)$*-Poincaré inequality* if there are constants $c > 0$ and $\tau \geq 1$ such that

$$\fint_{B(z,r)} |u - u_{B(z,r)}| \, d\mu \leq cr \biggl(\fint_{B(z,\tau r)} g^q \, d\mu \biggr)^{1/q}$$

for all balls $B(z,r)$ in X, for all integrable functions u in $B(z,\tau r)$ and for all q-weak upper gradients g of u. In a doubling measure space a weak $(1,q)$-Poincaré inequality implies a weak (t,q)-Poincaré inequality for some $t > q$ possibly with a different τ, see [BCLS] and [HaK].

2.9. Assumptions. Throughout the work we make the following rather standard assumptions:

From now on we assume, without further notice, that the complete metric measure space X is equipped with a doubling Borel measure for which the measure of every nonempty open set is positive and the measure of every bounded set is finite. Furthermore we assume that the space supports a weak $(1,q)$-Poincaré inequality for some q with $1 < q < p$.

The assumption on the Poincaré inequality is needed in the regularity theory for quasiminimizers of variational integrals on metric spaces, see [KS1].

3. Quasiminimizers and quasisuperminimizers

Suppose that $\Omega \subset X$ is open and $1 < p < \infty$. Let $K \geq 1$. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is called a K*-quasiminimizer* if for all open $\Omega' \Subset \Omega$ and all functions $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ with $v - u \in N^{1,p}_0(\Omega')$ we have

$$\int_{\Omega'} g_u^p \, d\mu \leq K \int_{\Omega'} g_v^p \, d\mu. \tag{3.1}$$

Here g_u and g_v are the minimal p-weak upper gradients of u and v in Ω, respectively.

A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is called a K*-quasisuperminimizer* if (3.1) holds for all open $\Omega' \Subset \Omega$ and all functions $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ such that $v \geq u$ μ-almost everywhere in Ω' and $v - u \in N^{1,p}_0(\Omega')$. A function u is called a K*-quasisubminimizer* if $-u$ is a K-quasisuperminimizer. It is easy to see that a function is a K-quasiminimizer if and only if it is both K-quasisubminimizer and K-quasisuperminimizer. In this work we concentrate on properties of K-quasisuperminimizers.

If $K = 1$, then 1-quasiminimizers and 1-quasisuperminimizers are called *minimizers* and *superminimizers*, respectively. If $C_p(X \setminus \Omega) > 0$ and $f \in N^{1,p}(\Omega)$, then there is a unique minimizer $u \in N^{1,p}(\Omega)$ such that $u - f \in N_0^{1,p}(\Omega)$. In other words, the Dirichlet problem has a unique solution with the given boundary values, see [C], [KM2] and [Sh2]. Observe, that for minimizers and superminimizers it is enough to test (3.1) with $\Omega' = \Omega$.

The next lemma shows that open sets in (3.1) can be replaced by μ-measurable sets.

3.2. Lemma. *A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a K-quasisuperminimizer if and only if for all μ-measurable sets $E \Subset \Omega$ and all functions $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ such that $v \ge u$ μ-almost everywhere in E and $v - u \in N_0^{1,p}(E)$ we have*

$$\int_E g_u^p \, d\mu \le K \int_E g_v^p \, d\mu. \tag{3.3}$$

Proof. Only the converse needs a proof. For this let $E \Subset \Omega$ be μ-measurable, $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ such that $v \ge u$ μ-almost everywhere in E and $v - u \in N_0^{1,p}(E)$. Define $\tilde v \colon \Omega \to [-\infty, \infty]$,

$$\tilde v(x) = \begin{cases} v(x), & x \in E, \\ u(x), & x \in \Omega \setminus E. \end{cases}$$

Let $\varepsilon > 0$ and choose an open set Ω' so that $E \subset \Omega' \Subset \Omega$ and

$$\int_{\Omega' \setminus E} g_{\tilde v}^p \, d\mu < \frac{\varepsilon}{K}.$$

Then $\tilde v - u \in N_0^{1,p}(\Omega')$ and $\tilde v \ge u$ μ-almost everywhere in Ω'. Therefore we can apply (3.1) and we obtain

$$\begin{aligned} \int_E g_u^p \, d\mu &\le \int_{\Omega'} g_u^p \, d\mu \le K \int_{\Omega'} g_{\tilde v}^p \, d\mu \\ &\le K \int_E g_{\tilde v}^p \, d\mu + K \int_{\Omega' \setminus E} g_{\tilde v}^p \, d\mu \le K \int_E g_v^p \, d\mu + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary we obtain (3.3). □

The condition (3.3) can be stated in a slightly different form. Observe that this is immediate for $K = 1$.

3.4. Lemma. *A function* $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ *is a* K*-quasisuperminimizer if and only if for all open* $\Omega' \Subset \Omega$ *and all functions* $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ *such that* $v \geq u$ μ*-almost everywhere in* Ω' *and* $v - u \in N^{1,p}_0(\Omega')$ *we have*

$$\int_{\Omega' \cap \{v>u\}} g_u^p \, d\mu \leq K \int_{\Omega' \cap \{v>u\}} g_v^p \, d\mu. \tag{3.5}$$

Proof. First suppose that (3.5) holds. Since $g_u = g_v$ μ-almost everywhere on the set $\{x \in \Omega' : u(x) = v(x)\}$ and $v \geq u$ μ-almost everywhere on Ω', we have

$$\begin{aligned}\int_{\Omega'} g_u^p \, d\mu &= \int_{\Omega' \cap \{v>u\}} g_u^p \, d\mu + \int_{\Omega' \cap \{v=u\}} g_u^p \, d\mu \\ &\leq K \int_{\Omega' \cap \{v>u\}} g_v^p \, d\mu + \int_{\Omega' \cap \{v=u\}} g_v^p \, d\mu \leq K \int_{\Omega'} g_v^p \, d\mu.\end{aligned}$$

Then suppose that u is a K-quasisuperminimizer. Let $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ be such that $v \geq u$ μ-almost everywhere in Ω' and $v - u \in N^{1,p}_0(\Omega')$. Then the set

$$E = \{x \in \Omega' : v(x) > u(x)\}$$

is μ-measurable, $v \geq u$ μ-almost everywhere in E and $v - u \in N^{1,p}_0(E)$. From Lemma 3.2 we conclude that

$$\int_E g_u^p \, d\mu \leq K \int_E g_v^p \, d\mu$$

and this is what we wanted to prove. □

We observe that if u is a K-quasisuperminimizer, then αu and $u + \beta$ are K-quasisuperminimizers when $\alpha \geq 0$ and $\beta \in \mathbf{R}$. However, the sum of two K-quasisuperminimizers is not a K-quasisuperminimizer in general.

3.6. Lemma. *Suppose that* u_i *is a* K_i*-quasisuperminimizer in* Ω, $i = 1, 2$. *Then* $\min(u_1, u_2)$ *is a* $(K_1 + K_2)$*-quasisuperminimizer in* Ω.

Proof. Let $u = \min(u_1, u_2)$. Since $N^{1,p}_{\mathrm{loc}}(\Omega)$ is a lattice we have $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$. Let $\Omega' \Subset \Omega$ be an open set, and let $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ be such that $v - u \in N^{1,p}_0(\Omega')$ and $v \geq u$ μ-almost everywhere in Ω'. Set

$$E_1 = \{x \in \Omega' : v(x) \geq u_1(x)\} \quad \text{and} \quad E_2 = \{x \in \Omega' : v(x) > u_2(x)\}.$$

Then

$$\begin{aligned}\int_{\Omega'} g_u^p \, d\mu &\leq \int_{\Omega' \cap \{u_1 \leq u_2\}} g_{u_1}^p \, d\mu + \int_{\Omega' \cap \{u_1 > u_2\}} g_{u_2}^p \, d\mu \\ &\leq \int_{E_1} g_{u_1}^p \, d\mu + \int_{E_2} g_{u_2}^p \, d\mu.\end{aligned}$$

We observe that $v-u_1 \in N_0^{1,p}(E_1)$, $v \ge u_1$ μ-almost everywhere on E_1, $v-u_2 \in N_0^{1,p}(E_2)$, $v \ge u_2$ μ-almost everywhere on E_2. Then Lemma 3.2 implies

$$\int_{\Omega'} g_u^p\,d\mu \le K_1 \int_{E_1} g_v^p\,d\mu + K_2 \int_{E_2} g_v^p\,d\mu \le (K_1+K_2)\int_{\Omega'} g_v^p\,d\mu.$$

This completes the proof. □

3.7. Lemma. *Suppose that u_i is a K_i-quasisuperminimizer in Ω, $i=1,2$. Then $\min(u_1,u_2)$ is a K_1K_2-quasisuperminimizer in Ω.*

Proof. Let u and v be as in the proof of Lemma 3.6. Then

$$\int_{\Omega'} g_u^p\,d\mu \le \int_{\Omega'\cap\{u_1\le u_2\}} g_{u_1}^p\,d\mu + \int_{\Omega'\cap\{u_1>u_2\}} g_{u_2}^p\,d\mu.$$

Let $w = \max\bigl(\min(u_2,v),u_1\bigr)$. Then $w-u_1 \in N_0^{1,p}(\Omega'\cap\{u_1\le u_2\})$, $w\ge u_1$ μ-almost everywhere in Ω' and since u_1 is a K_1-quasisuperminimizer we have

$$\begin{aligned}\int_{\Omega'\cap\{u_1\le u_2\}} g_{u_1}^p\,d\mu &\le K_1\int_{\Omega'\cap\{u_1\le u_2\}} g_w^p\,d\mu\\ &\le K_1\int_{\Omega'\cap\{u_1\le u_2\}\cap\{v<u_2\}} g_v^p\,d\mu + K_1\int_{\Omega'\cap\{u_1\le u_2\}\cap\{v\ge u_2\}} g_{u_2}^p\,d\mu.\end{aligned}$$

This implies that

$$\begin{aligned}\int_{\Omega'} g_u^p\,d\mu \le& K_1\int_{\Omega'\cap\{u_1\le u_2\}\cap\{v<u_2\}} g_v^p\,d\mu + K_1\int_{\Omega'\cap\{u_1\le u_2\}\cap\{v\ge u_2\}} g_{u_2}^p\,d\mu\\ &+\int_{\Omega'\cap\{u_1>u_2\}} g_{u_2}^p\,d\mu\\ \le& K_1\int_{\Omega'\cap\{u_1\le u_2\}} g_v^p\,d\mu + K_1\int_{\Omega'\cap\{v\ge u_2\}} g_{u_2}^p\,d\mu.\end{aligned}$$

Since $\max(u_2,v)-u_2 \in N_0^{1,p}(\Omega'\cap\{v\ge u_2\})$ and $\max(u_2,v)\ge u_2$ μ-almost everywhere in Ω', we obtain

$$\int_{\Omega'\cap\{v\ge u_2\}} g_{u_2}^p\,d\mu \le K_2\int_{\Omega'\cap\{v\ge u_2\}} g_v^p\,d\mu$$

and we conclude that

$$\begin{aligned}\int_{\Omega'} g_u^p\,d\mu &\le K_1\int_{\Omega'\cap\{u_1\le u_2\}\cap\{v<u_2\}} g_v^p\,d\mu + K_1K_2\int_{\Omega'\cap\{v\ge u_2\}} g_v^p\,d\mu\\ &\le K_1K_2\int_{\Omega'} g_v^p\,d\mu. \ \square\end{aligned}$$

From Lemmas 3.6 and 3.7 we obtain:

3.8. Corollary. *Suppose that u_i is a K_i-quasisuperminimizer in Ω, $i=1,2$. Then $\min(u_1,u_2)$ is a $\min(K_1+K_2,K_1K_2)$-quasisuperminimizer in Ω.*

The next corollary is important in our constructions. It provides a construction method which does not increase the constant K. Recall that 1-quasisuperminimizers are called superminimizers.

3.9. Corollary. *Suppose that u is a K-quasisuperminimizer and that h is a superminimizer in Ω. Then $\min(u,h)$ is a K-quasisuperminimizer in Ω.*

Corollary 3.9 applied to constant functions h gives the following result.

3.10. Lemma. *Suppose that $u\in N^{1,p}_{\mathrm{loc}}(\Omega)$. Then u is a K-quasisuperminimizer if and only if $\min(u,c)$ is a K-quasisuperminimizer for every $c\in\mathbf{R}$.*

Proof. If u is a K-quasisuperminimizer, then it follows from Corollary 3.9 that $\min(u,c)$ is a K-quasisuperminimizer for each $c\in\mathbf{R}$, since constants are minimizers.

For the converse let $\Omega'\Subset\Omega$, $v-u\in N^{1,p}_0(\Omega')$ and $v\ge u$ μ-almost everywhere in Ω'. Write $v_i=\min(v,i)$ and $u_i=\min(u,i)$, $i=1,2,\dots$. Then $v_i-u_i\in N^{1,p}_0(\Omega')$ and $v_i\ge u_i$ μ-almost everywhere in Ω'. By assumption u_i is a K-quasisuperminimizer and hence

$$\int_{\Omega'} g_{u_i}^p\,d\mu\le K\int_{\Omega'} g_{v_i}^p\,d\mu$$

for $i=1,2,\dots$. Since $u,v\in N^{1,p}(\Omega')$, we obtain

$$\int_{\Omega'} g_u^p\,d\mu\le K\int_{\Omega'} g_v^p\,d\mu$$

as $i\to\infty$. This shows that u is a K-quasisuperminimizer in Ω. □

4. Poisson modification of a quasisuperminimizer

Suppose that u is a superminimizer, i.e. 1-quasisuperminimizer, in Ω and that $\Omega'\Subset\Omega$ is open. Define

$$P(u,\Omega')(x)=\begin{cases} h(x), & x\in\Omega',\\ u(x), & x\in\Omega\setminus\Omega',\end{cases}$$

where h is a minimizer in Ω' with $h-u\in N^{1,p}_0(\Omega')$. Then $P(u,\Omega')\in N^{1,p}_{\mathrm{loc}}(\Omega)$ and the next lemma is well known in classical potential theory.

4.1. Lemma. *Suppose that u is a superminimizer in Ω and that $\Omega'\Subset\Omega$ is open. Then the function $P(u,\Omega')$ is a superminimizer in Ω and $u\ge h$ μ-almost everywhere in Ω'.*

Proof. We first show that $u \ge h$ μ-almost everywhere in Ω'. Let

$$A = \{x \in \Omega' : u(x) < h(x)\}.$$

Now $u - h \in N_0^{1,p}(A)$ and by the superminimizing property of u

$$\int_A g_u^p \, d\mu \le \int_A g_h^p \, d\mu. \tag{4.2}$$

Now (4.2) holds as an equality because h is a minimizer in A with boundary values u and this means that u and h are both minimizers in A with the same boundary values. Since the minimizer with given boundary values is unique, see Theorem 7.14 in [C], this means that $\mu(A) = 0$ as required.

To prove the superminimizing property of $P(u, \Omega')$ write $w = P(u, \Omega')$ and let $\Omega'' \Subset \Omega$ be open and let $v \in N_{\mathrm{loc}}^{1,p}(\Omega)$ such that $v \ge w$ μ-almost everywhere with $w - v \in N_0^{1,p}(\Omega'')$. We show that

$$\int_{\Omega''} g_w^p \, d\mu \le \int_{\Omega''} g_v^p \, d\mu. \tag{4.3}$$

To this end let $A_1 = \{x \in \Omega'' : v(x) \ge u(x)\}$ and $A_2 = \Omega'' \setminus A_1$. Then $A_2 \subset \Omega'$ and define

$$v_1(x) = \begin{cases} v(x), & x \in A_2, \\ u(x), & x \in (\Omega' \cap \Omega'') \setminus A_2. \end{cases} \tag{4.4}$$

Since

$$v_1 = \min(v, u) = \min\bigl(\max(h, v), u\bigr)$$

in $\Omega' \cap \Omega''$, it follows that $v_1 - h \in N_0^{1,p}(\Omega' \cap \Omega'')$. The minimizing property of h in $\Omega' \cap \Omega''$ implies

$$\int_{\Omega' \cap \Omega''} g_h^p \, d\mu \le \int_{\Omega' \cap \Omega''} g_{v_1}^p \, d\mu = \int_{A_2} g_v^p \, d\mu + \int_{(\Omega' \cap \Omega'') \setminus A_2} g_u^p \, d\mu$$

and the superminimizing property of u in A_1 gives

$$\int_{\Omega'' \setminus \Omega'} g_u^p \, d\mu + \int_{\Omega' \cap \Omega'' \cap A_1} g_u^p \, d\mu = \int_{A_1} g_u^p \, d\mu \le \int_{A_1} g_v^p \, d\mu$$

because $v \ge u$ in A_1 and $v - u \in N_0^{1,p}(A_1)$. Combining these inequalities we obtain

$$\begin{aligned} \int_{\Omega''} g_w^p \, d\mu &= \int_{\Omega'' \setminus \Omega'} g_u^p \, d\mu + \int_{\Omega'' \cap \Omega'} g_h^p \, d\mu \\ &\le \int_{A_1} g_v^p \, d\mu - \int_{\Omega' \cap \Omega'' \cap A_1} g_u^p \, d\mu + \int_{A_2} g_v^p \, d\mu + \int_{(\Omega' \cap \Omega'') \setminus A_2} g_u^p \, d\mu \\ &= \int_{\Omega''} g_v^p \, d\mu, \end{aligned}$$

since $\Omega' \cap \Omega'' \cap A_1 = (\Omega' \cap \Omega'') \setminus A_2$. This is (4.3) and the lemma follows. □

The next theorem shows how the counterpart of the Poisson modification is constructed for K-quasisuperminimizers.

4.5. Theorem. *Suppose that u is a K-quasisuperminimizer in Ω and that $\Omega' \Subset \Omega$ is open. If h is a minimizer in Ω' with $h-u \in N_0^{1,p}(\Omega')$, then the function $w\colon \Omega \to [-\infty,\infty]$,*

$$w(x) = \begin{cases} \min\bigl(u(x), h(x)\bigr), & x \in \Omega', \\ u(x), & x \in \Omega \setminus \Omega', \end{cases} \tag{4.6}$$

is a K-quasisuperminimizer in Ω. Moreover, $w \le u$ in Ω.

Proof. Clearly $w \le u$ in Ω and it remains to show the K-quasisuperminimizing property of w. To this end, let $\Omega'' \Subset \Omega$ be an open set and $v \in N_{\mathrm{loc}}^{1,p}(\Omega)$ such that $v \ge w$ μ-almost everywhere in Ω'' and $v - w \in N_0^{1,p}(\Omega'')$. We show that

$$\int_{\Omega''} g_w^p \, d\mu \le K \int_{\Omega''} g_v^p \, d\mu.$$

Let

$$U_1 = \{x \in \Omega'' : v(x) < u(x)\} \quad \text{and} \quad U_2 = \{x \in \Omega'' : v(x) \ge u(x)\}.$$

Then $v - u \in N_0^{1,p}(U_2)$. Since $v \ge u$ in U_2, the quasisuperminimizing property of u and Lemma 3.2 give

$$\int_{U_2} g_u^p \, d\mu \le K \int_{U_2} g_v^p \, d\mu. \tag{4.7}$$

We define $h = u$ in $\Omega \setminus \Omega'$. Let $D = \{x \in \Omega'' : u(x) > h(x)\}$. Then $U_1 \subset D$ and we define

$$\psi(x) = \begin{cases} v(x), & x \in U_1, \\ u(x), & x \in D \setminus U_1. \end{cases}$$

Then $\psi - h \in N_0^{1,p}(D)$. Since $D \subset \Omega'$ and h is a minimizer in Ω', h is a minimizer in D as well and we obtain

$$\int_D g_h^p \, d\mu \le \int_D g_\psi^p \, d\mu. \tag{4.8}$$

The inequality (4.7) implies

$$\begin{aligned} K \int_{\Omega''} g_v^p \, d\mu &= K \int_{U_2} g_v^p \, d\mu + K \int_{U_1} g_v^p \, d\mu \\ &\ge K \int_{U_2} g_v^p \, d\mu + \int_{U_1} g_v^p \, d\mu \\ &\ge \int_{U_2} g_u^p \, d\mu + \int_{U_1} g_v^p \, d\mu + \int_{D\setminus U_1} g_u^p \, d\mu - \int_{D\setminus U_1} g_u^p \, d\mu \\ &= \int_{U_2} g_u^p \, d\mu + \int_D g_\psi^p \, d\mu - \int_{D\setminus U_1} g_u^p \, d\mu. \end{aligned}$$

Since $U_1 \subset D$, the above inequality together with (4.8) imply

$$
\begin{aligned}
K\int_{\Omega''} g_v^p\,d\mu &\ge \int_{U_2} g_u^p\,d\mu + \int_D g_h^p\,d\mu - \int_{D\setminus U_1} g_u^p\,d\mu \\
&= \int_{U_2\setminus(D\setminus U_1)} g_u^p\,d\mu + \int_D g_h^p\,d\mu.
\end{aligned}
\tag{4.9}
$$

Finally we observe that $\Omega'' \setminus D \subset U_2 \setminus (D\setminus U_1)$, because if $x \in \Omega''\setminus D$, then x belongs either to $U_2\setminus D$ or to $U_1\setminus D$. However, we have $U_1 \subset D$ and hence $x\in U_2\setminus D$ and $x \in U_2\setminus(D\setminus U_1)$. This shows that

$$
\int_{U_2\setminus(D\setminus U_1)} g_u^p\,d\mu \ge \int_{\Omega''\setminus D} g_u^p\,d\mu
$$

and together with (4.9) this implies

$$
K\int_{\Omega''} g_v^p\,d\mu \ge \int_{\Omega''\setminus D} g_u^p\,d\mu + \int_D g_h^p\,d\mu = \int_{\Omega''} g_w^p\,d\mu.
$$

This is the required inequality. □

As for the minimizers we write $P(u,\Omega') = w$. This is the *Poisson modification* of u in Ω'. Note that there is no ambiquity with the previous definition of $P(u,\Omega')$ for a minimizer u since the definitions coincide for a minimizer u by Lemma 4.1.

The Poisson modification $P(u,\Omega')$ of a K-quasisuperminimizer u in $\Omega' \Subset \Omega$ has the following properties:

(1) $P(u,\Omega')$ is a K-quasisuperminimizer in Ω,
(2) $P(u,\Omega') = u$ in $\Omega\setminus\Omega'$,
(3) $P(u,\Omega') \le u$ in Ω,
(4) $P(u,\Omega') = h$, if $P(u,\Omega') < u$, and where h is the minimizer with boundary values u in Ω'.

Note that in contrast to the classical potential theory there is no semicontinuity requirements for $P(u,\Omega')$. Also Ω' can be any open set such that $\Omega' \Subset \Omega$ and it need not be regular for the Dirichlet problem.

5. Reqularity of quasisuperminimizers

Quasiminimizers satisfy the Harnack inequality and are locally Hölder continuous after redefinition on a set of measure zero, see [KS1]. Here we show that quasisuperminimizers satisfy the weak Harnack inequality and are lower semicontinuous as in the case of supersolutions and superharmonic functions. Our proof is based on the De Giorgi method. The basic work has been done in [KS1] and [KM2] since for this regularity result the cases $K > 1$ and $K = 1$ are similar.

We begin by recalling the definition of the De Giorgi class. Let $k_0 \in \mathbf{R}$ and $0 < \varrho < R$. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ belongs to the *De Giorgi class* $\mathrm{DG}_p(\Omega, k_0)$, if there is a constant $c < \infty$ such that for all $k \ge k_0$, $z \in \Omega$ such that $B(z,R) \Subset \Omega$ the function u satisfies

$$\int_{A_z(k,\varrho)} g_u^p \, d\mu \le c(R-\varrho)^{-p} \int_{A_z(k,R)} (u-k)^p \, d\mu$$

where

$$A_z(k,r) = \{x \in B(z,r) : u(x) > k\}$$

and g_u is the minimal p-weak upper gradient of u in Ω. If the inequality above holds for all $k \in \mathbf{R}$, then we simply write $u \in \mathrm{DG}_p(\Omega)$.

5.1. Lemma. *Suppose that u is a K-quasisuperminimizer in Ω. Then $-u$ belongs to $\mathrm{DG}_p(\Omega)$.*

The proof of Lemma 5.1 is similar to the proof of Lemma 4.1 in [KM2].

The next result shows that a nonnegative quasisuperminimizer satisfies a weak Harnack inequality.

5.2. Lemma. *Suppose that $u \ge 0$ is a K-quasisuperminimizer in an open set $\Omega \subset X$. Then for every ball $B(z,R)$ with $B(z,5R) \subset \Omega$ we have*

$$\left(\fint_{B(z,R)} u^\sigma \, d\mu \right)^{1/\sigma} \le c \operatorname*{ess\,inf}_{B(z,3R)} u,$$

where $c < \infty$ and $\sigma > 0$ depend only on K and on the constants in the doubling condition and Poincaré inequality.

For the proof we refer to the proof of Lemma 4.7 in [KM2].

Next we observe that a quasisuperminimizer has a lower semicontinuous representative. We denote

$$\operatorname*{ess\,lim\,inf}_{y \to x} u(y) = \lim_{r \to 0} \operatorname*{ess\,inf}_{B(x,r)} u.$$

5.3. Lemma. *Suppose that u is a K-quasisuperminimizer in Ω. Then the function $u^*\colon \Omega \to [-\infty, \infty]$ defined by*

$$u^*(x) = \operatorname*{ess\,lim\,inf}_{y \to x} u(y)$$

is a lower semicontinuous function in Ω and it belongs to the same equivalence class as u in $N^{1,p}_{\mathrm{loc}}(\Omega)$.

The proof is similar to the proof of Theorem 5.1 in [KM2].

6. Convergence results for quasisuperminimizers

We show that the quasisuperminimizing property is preserved under increasing convergence if the limit function is locally bounded above or belongs to $N^{1,p}_{\mathrm{loc}}(\Omega)$. The corresponding result for superminimizers has been studied in [KM2]. There are some unexpected difficulties for quasiminimizers and the argument is more involved in this case.

6.1. Theorem. *Suppose that (u_i) is an increasing sequence of K-quasisuperminimizers in Ω and $u=\lim_{i\to\infty} u_i$ such that either*
(i) *u is locally bounded above or*
(ii) *$u\in N^{1,p}_{\mathrm{loc}}(\Omega)$.*
Then u is a K-quasisuperminimizer in Ω.

We first consider the case (i). In this case it follows from the De Giorgi type upper bound

$$\int_{B(x,\varrho)} g_{u_i}^p\,d\mu \le c(R-\varrho)^{-p}\int_{B(x,R)}(u_i-k)^p\,d\mu,$$

where

$$k< -\sup\Bigl\{\operatorname*{ess\,sup}_{B(x,R)} u_i : i=1,2,\dots\Bigr\},$$

$0<\varrho<R$ and $B(x,R)\Subset\Omega$, that the sequence (g_{u_i}) is uniformly bounded in $L^p(\Omega')$ for every $\Omega'\Subset\Omega$. This implies that $u\in N^{1,p}_{\mathrm{loc}}(\Omega)$ and we may assume that (g_{u_i}) converges weakly to g_u in $L^p(\Omega')$, where g_u is an upper gradient of u.

We need a couple of lemmas. The first one is technical.

6.2. Lemma. *Let $u\in N^{1,p}_{\mathrm{loc}}(\Omega)$ and $K\ge 1$. Suppose that for every open set $\Omega'\Subset\Omega$ and for every $v\in N^{1,p}_{\mathrm{loc}}(\Omega)$ such that $v-u\in N^{1,p}_0(\Omega')$ and $v\ge u$ μ-almost everywhere in Ω' we have*

$$\int_{\Omega'} g_u^p\,d\mu \le K\int_{\Omega'} g_v^p\,d\mu.$$

Then for every open set $\Omega''\Subset\Omega$ and for every $v\in N^{1,p}_{\mathrm{loc}}(\Omega)$ such that $v-u\in N^{1,p}_0(\Omega'')$ and $v\ge u$ μ-almost everywhere in Ω'' we have

$$\int_{\Omega''} g_u^p\,d\mu \le K\int_{\Omega''} g_v^p\,d\mu.$$

Proof. Let $\Omega''\Subset\Omega$ be open and $v\in N^{1,p}_{\mathrm{loc}}(\Omega)$ such that $v-u\in N^{1,p}_0(\Omega'')$ and $v\ge u$ μ-almost everywhere in Ω''. Let $\varepsilon>0$. By Theorem 5.2.5 of [Sh1] there is a Lipschitz function $\varphi\ge 0$ such that $\operatorname{spt}\varphi\Subset\Omega''$ and

$$\|\varphi-(v-u)\|_{N^{1,p}(\Omega'')}<\varepsilon.$$

Choose an open set Ω' such that $\operatorname{spt}\varphi \Subset \Omega' \Subset \Omega''$.

By the assumption

$$\begin{aligned}\left(\int_{\overline{\Omega'}} g_u^p\,d\mu\right)^{1/p} &\le K^{1/p}\left(\int_{\overline{\Omega'}} g_{u+\varphi}^p\,d\mu\right)^{1/p} \\ &\le K^{1/p}\left[\left(\int_{\Omega''} g_v^p\,d\mu\right)^{1/p} + \left(\int_{\Omega'} g_{\varphi-(v-u)}^p\,d\mu\right)^{1/p}\right] \\ &\le K^{1/p}\left(\int_{\Omega''} g_v^p\,d\mu\right)^{1/p} + K^{1/p}\varepsilon.\end{aligned}$$

Since $\mu(\Omega''\setminus\overline{\Omega'})$ can be made arbitrarily small, we see from the absolute continuity of the integral that

$$\left(\int_{\Omega''} g_u^p\,d\mu\right)^{1/p} \le K^{1/p}\left(\int_{\Omega''} g_v^p\,d\mu\right)^{1/p} + K^{1/p}\varepsilon.$$

We obtain the claim by letting $\varepsilon \to 0$. □

6.3. Lemma. *Let u and u_i be as in Theorem* 6.1. *Then for every ball $B(x,r) \Subset \Omega$ we have*

$$\limsup_{i\to\infty} \int_{B(x,r)} g_{u_i}^p\,d\mu \le c\int_{B(x,r)} g_u^p\,d\mu,$$

where the constant c depends only on K and p.

Proof. Let $B(x,\varrho) \Subset B(x,r) \Subset \Omega$ and choose a Lipschitz cut-off function η such that $0 \le \eta \le 1$, $\eta = 0$ in $\Omega \setminus B(x,r)$ and $\eta = 1$ in $B(x,\varrho)$. Let

$$w_i = u_i + \eta(u-u_i), \quad i = 1,2,\dots.$$

Then $w_i - u_i \in N_0^{1,p}\big(B(x,r)\big)$ and $w_i \ge u_i$ μ-almost everywhere in $B(x,r)$. Hence the quasisuperminimizing property of u_i gives

$$\begin{aligned}\int_{B(x,\varrho)} g_{u_i}^p\,d\mu &\le \int_{B(x,r)} g_{u_i}^p\,d\mu \le K\int_{B(x,r)} g_{w_i}^p\,d\mu \\ &\le \alpha K\left(\int_{B(x,r)} (1-\eta)^p g_{u_i}^p\,d\mu + \int_{B(x,r)} g_\eta^p (u-u_i)^p\,d\mu + \int_{B(x,r)} \eta^p g_u^p\,d\mu\right),\end{aligned}$$

where $\alpha = 2^{p-1}$ and we used the fact that

$$g_{w_i} \le (1-\eta)g_{u_i} + g_\eta(u-u_i) + \eta g_u,$$

see Lemma 2.4 in [KM2]. Adding the term

$$\alpha K \int_{B(x,\varrho)} g_{u_i}^p \, d\mu$$

to both sides and taking into account that $\eta = 1$ in $B(x,\varrho)$ we obtain

$$\begin{aligned}(1+\alpha K)\int_{B(x,\varrho)} g_{u_i}^p \, d\mu \le \alpha K \int_{B(x,r)} g_{u_i}^p \, d\mu + \alpha K \int_{B(x,r)} g_\eta^p (u-u_i)^p \, d\mu \\ + \alpha K \int_{B(x,r)} g_u^p \, d\mu.\end{aligned} \tag{6.4}$$

Set $\Psi\colon \big(0, \operatorname{dist}(x,\partial\Omega)\big) \to \mathbf{R}$,

$$\Psi(r) = \limsup_{i\to\infty} \int_{B(x,r)} g_{u_i}^p \, d\mu.$$

Since $-u_i$ belongs to the De Giorgi class (see Lemma 5.1), we observe that Ψ is a finite-valued and increasing function of r. Hence the points of discontinuities form a countable set. Let r, $0 < r < \operatorname{dist}(x,\partial\Omega)$, be a point of continuity of Ψ. Letting $i \to \infty$, we obtain from (6.4) the estimate

$$(1+\alpha K)\Psi(\varrho) \le \alpha K \Psi(r) + \alpha K \int_{B(x,r)} g_u^p \, d\mu, \tag{6.5}$$

because

$$\int_{B(x,r)} g_\eta^p (u-u_i)^p \, d\mu \to 0$$

as $i \to \infty$. Since r is a point of continuity of Ψ, we conclude from (6.5) that

$$(1+\alpha K)\Psi(r) \le \alpha K \Psi(r) + \alpha K \int_{B(x,r)} g_u^p \, d\mu,$$

or in other words

$$\Psi(r) \le \alpha K \int_{B(x,r)} g_u^p \, d\mu. \tag{6.6}$$

This holds at each point of continuity of Ψ. Since Ψ is increasing and

$$r \mapsto \int_{B(x,r)} g_u^p \, d\mu$$

is a continuous function of r, it is easy to see that (6.6) holds for every r with $0 < r < \operatorname{dist}(x,\partial\Omega)$. This is the required estimate. □

Proof of Theorem 6.1. Case (i). As noted before $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and hence it suffices to prove inequality (3.1). To this end let $\Omega' \Subset \Omega$ be open and $v \in N^{1,p}_{\mathrm{loc}}(\Omega')$, $v \geq u$ μ-almost everywhere and $v-u \in N^{1,p}_0(\Omega')$. By Lemma 6.2 it suffices to show that

$$\int_{\overline{\Omega'}} g_u^p \, d\mu \leq K \int_{\overline{\Omega'}} g_v^p \, d\mu.$$

Choose an open set Ω'' such that $\Omega' \Subset \Omega'' \Subset \Omega$ and a Lipschitz cut-off function η with the properties $\eta = 1$ on Ω', $0 \leq \eta \leq 1$ and $\eta = 0$ on $\Omega \setminus \Omega''$. Set

$$w_i = u_i + \eta(v - u_i), \quad i = 1, 2, \dots .$$

Then $w_i - u_i \in N^{1,p}_0(\Omega'')$ and $w_i \geq u_i$. By Lemma 2.4 in [KM2] we have

$$g_{w_i} \leq (1-\eta) g_{u_i} + \eta g_v + g_\eta (v - u_i),$$

and we obtain

$$\begin{aligned}\left(\int_{\Omega''} g_{w_i}^p \, d\mu\right)^{1/p} &\leq \left(\int_{\Omega''} \big((1-\eta) g_{u_i} + \eta g_v\big)^p \, d\mu\right)^{1/p} + \left(\int_{\Omega''} g_\eta^p (v-u_i)^p \, d\mu\right)^{1/p} \\ &= \alpha_i + \beta_i.\end{aligned}$$

Now $(\alpha_i + \beta_i)^p \leq \alpha_i^p + p\beta_i(\alpha_i+\beta_i)^{p-1}$ and hence

$$\int_{\Omega''} g_{w_i}^p \, d\mu \leq \int_{\Omega''} (1-\eta) g_{u_i}^p \, d\mu + \int_{\Omega''} \eta g_v^p \, d\mu + p\beta_i(\alpha_i+\beta_i)^{p-1}, \tag{6.7}$$

where we also used the convexity of the function $t \mapsto t^p$. We estimate the terms on the right-hand side separately.

First, since $g_\eta = 0$ μ-almost everywhere in Ω', by the Lebesgue monotone convergence theorem

$$\beta_i^p = \int_{\Omega'' \setminus \Omega'} g_\eta^p (v-u_i)^p \, d\mu \to 0 \tag{6.8}$$

as $i \to \infty$, because $v = u$ in $\Omega'' \setminus \Omega'$ and $u_i \to u$ in Ω.

Next we consider

$$\int_{\Omega''} (1-\eta) g_{u_i}^p \, d\mu.$$

Since $\eta = 1$ on $\overline{\Omega'}$, we obtain

$$\int_{\Omega''} (1-\eta) g_{u_i}^p \, d\mu \leq \int_{\Omega'' \setminus \overline{\Omega'}} g_{u_i}^p \, d\mu. \tag{6.9}$$

Choose balls $B(x_j,r_j)$, $j=1,2,\dots$, such that $B(x_j,r_j)\subset\Omega''\setminus\overline{\Omega'}$,

$$\Omega''\setminus\overline{\Omega'}\subset\bigcup_{j=1}^{\infty}B(x_j,r_j)\qquad\text{and}\qquad\sum_{j=1}^{\infty}\chi_{B(x_j,r_j)}\le N<\infty,$$

where N depends only on the doubling constant, see [He].

Define

$$\Psi_{x_j}(r)=\limsup_{i\to\infty}\int_{B(x_j,r)}g_{u_i}^p\,d\mu$$

and let $\varepsilon>0$. By Lemma 6.3

$$\Psi_{x_j}(r)\le c\int_{B(x_j,r_j)}g_u^p\,d\mu$$

and letting $i\to\infty$ we obtain

$$\begin{aligned}\limsup_{i\to\infty}\int_{\Omega''\setminus\overline{\Omega'}}g_{u_i}^p\,d\mu&\le\limsup_{i\to\infty}\sum_{j=1}^{\infty}\int_{B(x_j,r_j)}g_{u_i}^p\,d\mu\\&\le\sum_{j=1}^{\infty}\Psi_{x_j}(r_j)\le c\sum_{j=1}^{\infty}\int_{B(x_j,r_j)}g_u^p\,d\mu\\&\le c\sum_{j=1}^{\infty}\int_{\Omega''\setminus\overline{\Omega'}}\chi_{B(x_j,r_j)}g_u^p\,d\mu\le cN\int_{\Omega''\setminus\overline{\Omega'}}g_u^p\,d\mu.\end{aligned}\tag{6.10}$$

Next we choose $\mu(\Omega''\setminus\overline{\Omega'})$ so small that

$$\int_{\Omega''\setminus\overline{\Omega'}}g_u^p\,d\mu<\varepsilon$$

and from (6.9) and (6.10) we obtain

$$\limsup_{i\to\infty}\int_{\Omega''}(1-\eta)g_{u_i}^p\,d\mu\le cN\varepsilon.\tag{6.11}$$

Note that c and N are independent of ε. Since α_i remains bounded as $i\to\infty$, we obtain from (6.7), (6.8) and (6.11) that

$$\limsup_{i\to\infty}\int_{\Omega''}g_{w_i}^p\,d\mu\le cN\varepsilon+\int_{\Omega''}\eta g_v^p\,d\mu.\tag{6.12}$$

Now u_i is a K-quasisuperminimizer and hence

$$\int_{\overline{\Omega'}}g_{u_i}^p\,d\mu\le\int_{\Omega''}g_{u_i}^p\,d\mu\le K\int_{\Omega''}g_{w_i}^p\,d\mu.$$

Together with the lower semicontinuity of the L^p-norm, $p>1$, and (6.12) we have

$$\begin{aligned}\int_{\overline{\Omega'}} g_u^p\,d\mu &\le \limsup_{i\to\infty}\int_{\overline{\Omega'}} g_{u_i}^p\,d\mu \le KcN\varepsilon + K\int_{\Omega''}\eta g_v^p\,d\mu \\ &\le KcN\varepsilon + K\int_{\overline{\Omega'}} g_v^p\,d\mu + K\int_{\Omega''\setminus\overline{\Omega'}} g_v^p\,d\mu.\end{aligned}$$

We can still choose $\mu(\Omega''\setminus\overline{\Omega'})$ so small that

$$\int_{\overline{\Omega'}\setminus\Omega''} g_v^p\,d\mu < \varepsilon$$

and hence

$$\int_{\overline{\Omega'}} g_u^p\,d\mu \le KcN\varepsilon + K\varepsilon + K\int_{\overline{\Omega'}} g_v^p\,d\mu.$$

Finally letting $\varepsilon\to 0$ we obtain the claim. This completes the proof of the case (i).

Proof of Theorem 6.1. Case (ii). In Case (ii) we consider functions

$$u_{i,c}=\min(u_i,c),\quad i=1,2,\dots,\ c\in\mathbf{R}.$$

Since $u_{i,c}$ is a K-quasisuperminimizer, it follows from Case (i) that $u_c=\min(u,c)$ is a K-quasisuperminimizer. Finally, since $u\in N^{1,p}_{\mathrm{loc}}(\Omega)$, Lemma 3.10 implies that u is a K-quasisuperminimizer. □

7. Quasisuperharmonic functions

In the classical potential theory a superharmonic function can be defined as a limit of an increasing sequence of supersolutions of the Laplace equation, provided the limit is not identically ∞. If u is a superharmonic function, then one of the aforementioned sequences is $\min(u,i)$, $i=1,2,\dots$. Our defininition for a quasisuperharmonic function has a global nature. We show later that the cutoff method leads to an equivalent definition for quasisuperharmonic functions.

7.1. Definition. Let $\Omega\subset X$ be open and $K\ge 1$. We say that a function $u\colon\Omega\to(-\infty,\infty]$ is K-*quasisuperharmonic* in Ω, if u is not identically ∞ in any component of Ω and there is a sequence of open sets Ω_i and K-quasiminimizers $v_i\colon\Omega_i\to(-\infty,\infty]$ such that

(i) $\Omega_i\Subset\Omega_{i+1}$,
(ii) $\bigcup_{i=1}^\infty\Omega_i=\Omega$,
(iii) $v_i\le v_{i+1}$ in Ω_i,
(iv) $\lim_{i\to\infty}v_i^*=u$ in Ω.

Here v_i^* is the lower semicontinuous representative of v_i given by Lemma 5.3.

Since v_i^* is lower semicontinuous, a K-quasisuperharmonic function is lower semicontinuous.

7.2. Proposition. *If u is a K-quasisuperminimizer in Ω such that*

$$u(x) = \operatorname{ess} \liminf_{y \to x} u(y)$$

for every $x \in \Omega$, then u is K-quasisuperharmonic in Ω.

Proof. Let Ω_i be any sequence of open sets satisfying (i) and (ii) in Definition 7.1. Choose $v_i = u^*$ for $i = 1, 2, \ldots$. Then v_i satisfy (iii) and (iv) in Definition 7.1. □

7.3. Theorem. *Suppose that u is a K-quasisuperharmonic function in Ω and u is locally bounded above. Then $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and u is a K-quasisuperminimizer in Ω.*

Proof. It suffices to show that u is a K-quasisuperminimizer in each open $\Omega' \Subset \Omega$. Let $\Omega' \Subset \Omega$. Then there is an increasing sequence of K-quasisuperminimizers v_i in Ω' such that $u = \lim_{i\to\infty} v_i^*$ in Ω'. Since u is locally bounded above, Theorem 6.1 implies that u is a K-quasisuperminimizer in Ω'. □

7.4. Theorem. *Suppose that u is a K-quasisuperharmonic function in Ω and h is a continuous minimizer in Ω. Then $\min(u,h)$ is a K-quasisuperminimizer and K-quasisuperharmonic in Ω.*

Proof. First we show that $\min(u,h)$ is K-quasisuperharmonic. Since $\min(u,h)$ is lower semicontinuous and locally bounded, it suffices to check the conditions (i)–(iv) in Definition 7.1. Let Ω_i and v_i be as in Definition 7.1. Then each $\min(v_i^*, h)$ is a K-quasisuperminimizer in Ω_i by Corollary 3.9 and

$$\min(v_i, h) = \min(v_i^*, h) \to \min(u, h)$$

in Ω as $i \to \infty$. Hence $\min(u,h)$ is K-quasisuperharmonic and by Theorem 6.1 it is also a K-quasisuperminimizer in Ω. □

7.5. Corollary. *Suppose that u is a K-quasisuperharmonic function in Ω and $c \in \mathbf{R}$. Then $\min(u,c)$ is a K-quasisuperminimizer and K-quasisuperharmonic in Ω.*

If u is K-quasisuperharmonic, then αu and $u+\beta$ are K-quasisuperharmonic when $\alpha \geq 0$ and $\beta \in \mathbf{R}$. However, the sum of two quasisuperharmonic functions is not quasisuperharmonic in general.

7.6. Theorem. *Suppose that u_j is K_j-quasisuperharmonic in Ω, $j = 1, 2$. Then $\min(u_1, u_2)$ is $\min(K_1 + K_2, K_1K_2)$-quasisuperharmonic.*

Proof. Clearly $\min(u_1, u_2)$ is lower semicontinuous. Let $\Omega_{j,i}$ and $v_{j,i}$ be as in Definition 7.1 for u_j, $j = 1, 2$. Then $\Omega_i = \Omega_{1,i} \cap \Omega_{2,i}$, $i = 1, 2, \ldots$, is an increasing sequence of open sets and $\min(v_{1,i}, v_{2,i})$ is a $\min(K_1 + K_2, K_1K_2)$-quasisuperminimizer in Ω_i by Lemmas 3.6 and 3.7. Moreover

$$\min(v_{1,i}, v_{2,i})^* = \min(v_{1,i}^*, v_{2,i}^*) \to \min(v, v)$$

in Ω as $i \to \infty$. This proves the claim. □

7.7. Theorem. *Suppose that u is K-quasisuperharmonic in Ω. Then u is locally integrable to a power $\sigma > 0$ and, in particular, $|u| < \infty$ μ-almost everywhere in Ω.*

Proof. Let $z \in \Omega$ be such that $u(z) < \infty$. Let $\Omega' \Subset \Omega$ such that $B(z, 5R) \Subset \Omega'$. Let Ω_i and v_i be as in Definition 7.1. Then $\inf_{B(z,R)} u < \infty$ whenever $R > 0$ is such that $B(z, R) \subset \Omega'$. By Lemma 5.2 for every ball $B(z, R)$ with $B(z, 5R) \subset \Omega'$ we have

$$\left(\fint_{B(z,R)} v_i^\sigma \, d\mu\right)^{1/\sigma} \le c \inf_{B(z,3R)} v_i \le c \inf_{B(z,3R)} v,$$

where $c < \infty$ and $\sigma > 0$ are as in Lemma 5.2. In particular, they are independent of i. Letting $i \to \infty$ we conclude that

$$\left(\fint_{B(z,R)} u^\sigma \, d\mu\right)^{1/\sigma} \le c \inf_{B(z,3R)} u < \infty. \qquad \square$$

We can use the same reasoning as in [KM2] and obtain the following two results.

7.8. Theorem. *If u is K-quasisuperharmonic in Ω, then*

$$u(x) = \operatorname{ess\,lim\,inf}_{y \to x} u(y)$$

for every $x \in \Omega$.

7.9. Theorem. *If u and v are K-quasisuperharmonic in Ω and $u = v$ μ-almost everywhere in Ω, then $u = v$ in Ω.*

We close this section by a characterization of K-quasisuperharmonic functions and by a description of K-quasisuperminimizers among K-quasisuperharmonic functions.

7.10. Theorem. *Suppose that $u\colon \Omega \to (-\infty, \infty]$ is not identically ∞ in any component of Ω. Then u is K-quasisuperharmonic if and only if $\min(u^*, k)$ is a K-quasisuperminimizer for every $k \in \mathbf{R}$.*

Proof. First suppose that u is K-quasisuperharmonic in Ω. It follows from Theorem 7.8 and Corollary 7.5 that $\min(u^*, k)$ is a K-quasisuperminimizer in Ω for every $k \in \mathbf{R}$.

Then suppose that $\min(u^*, k)$ is a K-quasisuperminimizer for every $k \in \mathbf{R}$. If Ω_i is a sequence of open sets satisfying (i) and (ii) in Definition 7.1, then the sequence $v_i = \min(u^*, i)$, $i = 1, 2, \ldots$, satisfies (iii) and (iv) in Definition 7.1. $\square$

7.11. Theorem. *Suppose that u is K-quasisuperharmonic in Ω. Then u is a K-quasisuperminimizer in Ω if and only if there is $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ such that $u \le v$ μ-almost everywhere.*

Proof. If u is a K-quasisuperminimizer, then we can choose $v = u$. For the converse let $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ with $u \le v$ μ-almost everywhere. Fix $\Omega' \Subset \Omega$ and choose a Lipschitz cutoff function φ such that $0 \le \varphi \le 1$, $\varphi = 1$ in Ω' and $\operatorname{spt}\varphi$ is compact in Ω. By the lower semicontinuity of u we may assume that $u > 0$ on $\operatorname{spt}\varphi$. Fix $c > 0$ and write

$$A = \{x \in \operatorname{spt}\varphi : u(x) \le \varphi(x)v(x)\}.$$

Now A is μ-measurable and $A \Subset \Omega$. The function $u_c = \min(u, c)$ is a K-quasisuperminimizer and $u_c - (\varphi v)_c \in N^{1,p}_0(A)$. The quasiminimizing property of u_c yields

$$\int_A g_{u_c}^p \, d\mu \le K \int_A g_{(\varphi v)_c}^p \, d\mu \le K \int_A g_{\varphi v}^p \, d\mu \le K \int_{\operatorname{spt}\varphi} g_{\varphi v}^p \, d\mu = M < \infty.$$

Since $A \cap \{u < c\} \supset \Omega' \cap \{u < c\}$, we obtain

$$\int_{\Omega' \cap \{u<c\}} g_{u_c}^p \, d\mu \le M.$$

Letting $c \to \infty$ and noting that $\mu(\{u = \infty\}) = 0$ by Theorem 7.7 we see that $g_u = \lim_{c\to\infty} g_{u_c}$ defines an $L^p(\Omega')$-function. Since $u \le v$ μ-almost everywhere, $v \in L^p(\Omega')$ and u is locally bounded below, u belongs to $L^p(\Omega')$ and since it is easy to see that g_u is a weak upper gradient of u in Ω', it follows that $u \in N^{1,p}(\Omega')$. The definition of quasisuperharmonicity together with Theorem 6.1 implies that u is a K-quasisuperminimizer. □

8. Definitions of superharmonicity

There are alternative definitions for 1-quasisuperminimizers. One possible choice has been studied in [KM2]. In this section we study how these definitions are related to Definition 7.1. We begin with a definition, which is slightly different from that of [KM2].

8.1. Definition. A function $u\colon \Omega \to (-\infty, \infty]$ is called *superharmonic* in Ω, if

(i) u is lower semicontinuous in Ω,
(ii) u is not identically ∞ in any component of Ω,
(iii) for every open $\Omega' \Subset \Omega$ the comparison principle holds: if $v \in C(\overline{\Omega'}) \cap N^{1,p}(\Omega')$ and $v \le u$ on $\overline{\Omega'}$, then $\mathscr{O}(v, \Omega') \le u$ in Ω'.

Here $\mathscr{O}(v, \Omega')$ denotes the unique solution to the $\mathscr{K}_{v,v}(\Omega')$-obstacle problem in Ω'. This means that the function $\mathscr{O}(v, \Omega')$ minimizes the p-Dirichlet integral among all functions w such that $w - v \in N^{1,p}_0(\Omega')$ and $w \ge v$ μ-almost everywhere in Ω', see [KM2].

Observe that $\mathscr{O}(v,\Omega')$ is a superminimizer and continuous in Ω' by [KM2, Theorem 5.5].

8.2. Lemma. *Suppose that* u *is a superharmonic function in* Ω *and let* Ω' *be an open set such that* $\Omega' \Subset \Omega$. *Then there is an increasing sequence of continuous superminimizers* u_i, $i = 1, 2, \dots$, *in* Ω' *such that* $u = \lim_{i\to\infty} u_i$ *everywhere in* Ω'.

The proof of this lemma is similar to the proof of [KM2, Theorem 7.7].

8.3. Lemma. *If* u *is a superminimizer in* Ω *such that*

$$u(x) = \operatorname*{ess\,lim\,inf}_{y\to x} u(y)$$

for every $x \in \Omega$, *then* u *is superharmonic.*

Proof. It suffices to prove the comparison principle (iii) in Definition 8.1. Let Ω' be an open set such that $\Omega' \Subset \Omega$ and $v \in C(\overline{\Omega'}) \cap N^{1,p}(\Omega')$ and $v \le u$ on $\overline{\Omega'}$. Let

$$A = \{x \in \Omega' : \mathscr{O}(v,\Omega')(x) > u(x)\}.$$

Since $v - \mathscr{O}(v,\Omega') \in N_0^{1,p}(\Omega')$ and $u \ge v$, we have $\mathscr{O}(v,\Omega') - u \in N_0^{1,p}(A)$. Now $\mathscr{O}(v,\Omega') \ge u$ in A and the minimizing property of u yields

$$\int_A g^p_{\mathscr{O}(v,\Omega')}\,d\mu \ge \int_A g_u^p\,d\mu. \tag{8.4}$$

On the other hand $\mathscr{O}(v,\Omega')$ is the unique solution to the obstacle problem $\mathscr{K}_{v,v}(\Omega')$ and hence (8.4) holds as an equality. This implies that $\mathscr{O}(v,\Omega') = u$ μ-almost everywhere in A and thus $\mu(A) = 0$. This means that $\mathscr{O}(v,\Omega') \le u$ μ-almost everywhere in Ω'. Since u satisfies

$$u(x) = \operatorname*{ess\,lim\,inf}_{y\to x} u(y)$$

for every $x \in \Omega$ and since $\mathscr{O}(v,\Omega')$ is continuous in Ω', the condition $\mathscr{O}(v,\Omega') \le u$ μ-almost everywhere in Ω' implies $\mathscr{O}(v,\Omega') \le u$ everywhere in Ω'. □

Next we show that the condition (iii) is stronger than the comparison condition:

(iii′) for every open $\Omega' \Subset \Omega$ the comparison principle holds: if $v \in C(\overline{\Omega'}) \cap N^{1,p}(\Omega')$ and $v \le u$ on $\partial\Omega'$, then $\mathscr{H}(v,\Omega') \le u$ in Ω'.

Here $\mathscr{H}(v,\Omega')$ denotes the unique harmonic function in Ω' with $v - \mathscr{H}(v,\Omega') \in N_0^{1,p}(\Omega')$.

8.5. Lemma. *Suppose that* u *is superharmonic in the sense of Definition* 8.1. *Then* u *satisfies* (iii′).

Proof. Let $v \in C(\overline{\Omega'}) \cap N^{1,p}(\Omega')$ with $v \le u$ on $\partial\Omega'$. For every $\varepsilon > 0$ we construct a function v_1 such that $v - \varepsilon - v_1 \in N_0^{1,p}(\Omega')$, $v_1 \in C(\overline{\Omega'})$ and $v_1 \le u$ in $\overline{\Omega'}$.

Since the function u is lower semicontinuous, there is $m > -\infty$ such that $u(x) \ge m$ in $\overline{\Omega'}$. Let $\varepsilon > 0$. The set

$$U_1 = \{x \in \overline{\Omega'} : v(x) - \varepsilon < u(x)\}$$

is open in $\overline{\Omega'}$ and contains $\partial\Omega'$. Choose another open set U_2 in $\overline{\Omega'}$ such that $\partial\Omega' \Subset U_2 \Subset U_1$. Let

$$\lambda(x) = \min\bigl(1, \operatorname{dist}\bigl(x, (X \setminus U_1) \cap \Omega'\bigr)/\operatorname{dist}\bigl(U_2, (X \setminus U_1) \cap \Omega'\bigr)\bigr).$$

Then λ is a Lipschitz continuous function in Ω', $\lambda = 1$ in U_2 and $\lambda = 0$ in $(X \setminus U_1) \cap \Omega'$. Let

$$v_1(x) = \bigl(1 - \lambda(x)\bigr)m + \lambda(x)\bigl(v(x) - \varepsilon\bigr).$$

Then $v_1 \in C(\overline{\Omega'})$ and $v - \varepsilon - v_1 \in N_0^{1,p}(\Omega')$ because $v - \varepsilon = v_1$ in a neighbourhood of $\partial\Omega'$ in Ω'. Moreover, it easily follows that $v_1 \le u$ in $\overline{\Omega'}$.

Now (iii) in Definition 8.1 implies that $\mathscr{O}(v_1, \Omega') \le u$ in Ω'. On the other hand $\mathscr{O}(v_1, \Omega') \ge \mathscr{H}(v_1, \Omega')$ in Ω'. Since $v - \varepsilon - v_1 \in N_0^{1,p}(\Omega')$ and the minimizer is unique, we obtain

$$\mathscr{H}(v_1, \Omega') = \mathscr{H}(v - \varepsilon, \Omega') = \mathscr{H}(v, \Omega') - \varepsilon.$$

Hence

$$\mathscr{H}(v, \Omega') - \varepsilon \le \mathscr{O}(v_1, \Omega') \le u$$

in Ω'. Since $\varepsilon > 0$ was arbitrary, we conclude that $\mathscr{H}(v, \Omega') \le u$ in Ω' as required. □

8.6. Remark. Lemma 8.5 shows that all results in [KM2] which deal with superharmonic functions apply to superharmonic functions in the sense of Definition 8.1.

8.7. Lemma *Suppose that* $u\colon \Omega \to (-\infty, \infty]$ *is superharmonic in* Ω *and* Ω_i *is a sequence of open sets such that* $\Omega_1 \Subset \Omega_2 \Subset \cdots$ *and* $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$. *Then there is a sequence* (v_i) *of functions such that*

(i) $v_i \in C(\overline{\Omega_i}) \cap N^{1,p}(\Omega_i)$,
(ii) v_i *is a superminimizer in* Ω_i,
(iii) $v_i \le v_j$ *in* $\overline{\Omega_i}$ *if* $j \ge i$,
(iv) $\lim_{i\to\infty} v_i = u$ *in* Ω.

Proof. Let Ω_i be a sequence of open sets satisfying the assumptions of the lemma. By the lower semicontinuity of u for every i there is a sequence of functions $\varphi_{i,j}\colon \overline{\Omega_i} \to \mathbf{R}$ such that $\varphi_{i,j} < \varphi_{i,j+1}$ in $\overline{\Omega_i}$, $\varphi_{i,j}$ is Lipschitz and $\lim_{j\to\infty} \varphi_{i,j} = u$ in $\overline{\Omega_i}$. Set $v_1 = \mathscr{O}(\varphi_{1,1}, \Omega_1)$ and if $v_1, \dots, v_i$ have been chosen, then choose j such that $\varphi_{i+1,j} \geq v_i$ in $\overline{\Omega_i}$ and take $v_{i+1} = \mathscr{O}(\varphi_{i+1,j}, \Omega_{i+1})$. The sequence (v_i) has the required properties. □

In particular, Lemma 8.7 implies that if u is superharmonic in the sense of Definition 8.1, then it is 1-quasisuperharmonic in the sense of Definition 7.1. The converse is also true:

8.8. Theorem. *A function is superharmonic in the sense of Definition* 8.1 *if and only if it is* 1*-quasisuperharmonic in the sense of Definition* 7.1.

Proof. Let u be 1-quasisuperharmonic. It suffices to prove (iii) in Definition 8.1. To this end, fix an open set $\Omega' \Subset \Omega$ and let $v \in C(\overline{\Omega'}) \cap N^{1,p}(\Omega')$ with $v \leq u$ on $\overline{\Omega'}$. By Definition 7.1 there is Ω'' with $\Omega' \Subset \Omega'' \Subset \Omega$ and an increasing sequence of 1-quasisuperminimizers (that is, superminimizers) v_i^* in Ω'' such that $v_i^* \to u$ in Ω''.

Now for each $\varepsilon > 0$ there is i such that $v_i^* \geq v - \varepsilon$ in $\overline{\Omega'}$. To prove this, let $\varepsilon > 0$. If there are points x_i, $i = 1, 2, \dots$, in $\overline{\Omega'}$ such that $v_i^*(x_i) < v(x_i) - \varepsilon$, then possibly passing to a subsequence we may assume that $x_i \to x_0 \in \overline{\Omega'}$. For $j \leq i$ we have

$$v_j^*(x_i) \leq v_i^*(x_i) < v(x_i) - \varepsilon$$

and since v is continuous in $\overline{\Omega'}$, this yields

$$\liminf_{i\to\infty} v_j^*(x_i) \leq v(x_0) - \varepsilon \leq u(x_0) - \varepsilon$$

for each $j = 1, 2, \dots$. Now each v_j^* is lower semicontinuous and thus

$$v_j^*(x_0) \leq v(x_0) - \varepsilon \leq u(x_0) - \varepsilon$$

for each j. If $u(x_0) < \infty$, then this is impossible because

$$\lim_{j\to\infty} v_j^*(x_0) = u(x_0) > u(x_0) - \varepsilon,$$

and if $u(x_0) = \infty$, then this is also impossible because $v(x_0) < \infty$. Hence there is i such that $v_i^* \geq v - \varepsilon$ in $\overline{\Omega'}$.

Consider the function $\mathscr{O}(v, \Omega')$. Now $\mathscr{O}(v - \varepsilon, \Omega') = \mathscr{O}(v, \Omega') - \varepsilon$ and the function $\min(\mathscr{O}(v, \Omega') - \varepsilon, v_i^*)$ is a superminimizer (the minimum of two superminimizers is a superminimizer by Lemma 3.7). If $\mathscr{O}(v, \Omega') - \varepsilon > v_i^*$ in some open subset of Ω', then this leads to a contradiction with the uniqueness of the solution to an obstacle problem. Consequently $\mathscr{O}(v, \Omega') - \varepsilon \leq v_i^*$ in Ω'. We thus have

$$\mathscr{O}(v, \Omega') - \varepsilon \leq v_i^* \leq u$$

in Ω'. Since this holds for all $\varepsilon > 0$, we obtain $\mathscr{O}(v, \Omega') \leq u$ in Ω' as required for (iii) in Definition 8.1. This completes the proof. □

9. Poisson modification of a quasisuperharmonic function

Suppose that $u\colon \Omega \to (-\infty, \infty]$ is a K-quasisuperharmonic function and that $\Omega' \Subset \Omega$ is open. Then there are sequences Ω_i and v_i as in Definition 7.1. Fix i_0 such that $\Omega' \Subset \Omega_i$ for $i \geq i_0$. Let $P(v_i, \Omega')$ denote the Poisson modification of v_i in Ω', see Section 4. Then $P(v_i, \Omega')$ is a K-quasisuperminimizer in Ω_i.

Let $P(v_i, \Omega')^*$ denote the lower semicontinuous representative of $P(v_i, \Omega')$ in Ω_i as in Lemma 5.3. Then

$$P(v_{i+1}, \Omega')^* \geq P(v_i, \Omega')^*$$

in Ω_i and

$$P(u, \Omega') = \lim_{i\to\infty} P(v_i, \Omega')^*$$

defines a K-quasisuperharmonic function in Ω. Note that $P(u, \Omega') \leq u$ in Ω and hence $P(u, \Omega')$ cannot be ∞ in any component of Ω. The function $P(u, \Omega')$ is called the *Poisson modification* of u in Ω'. The following theorem summarizes the properties of $P(u, \Omega')$.

9.1. Theorem. *Suppose that $u\colon \Omega \to (-\infty, \infty]$ is a K-quasisuperharmonic function and that $\Omega' \Subset \Omega$ is open. The function $P(u, \Omega')\colon \Omega \to (-\infty, \infty]$ has the properties:*

(1) *$P(u, \Omega')$ is K-quasisuperharmonic in Ω,*
(2) *$P(u, \Omega') = u$ in $\Omega \setminus \Omega'$,*
(3) *$P(u, \Omega') \leq u$ in Ω, and*
(4) *$P(u, \Omega')$ is a minimizer in the open set $A = \{x \in \Omega' : P(u, \Omega')(x) < u(x)\}$.*

Proof. Only the property (4) needs a proof. Let v_i and Ω_i, $i = 1, 2, \ldots$, be as above. We may assume that $\Omega' \Subset \Omega_i$ for all i and that Ω' is connected, otherwise we consider a component of Ω'. Let h_i be the minimizer with boundary values v_i in Ω'.

Suppose that $x_0 \in A$. Then $P(u, \Omega')(x_0) < u(x_0)$ and, in particular, we have $P(u, \Omega')(x_0) < \infty$. The sequence (h_i) is an increasing sequence of minimizers in Ω' and let $h = \lim_{i\to\infty} h_i$. Now

$$h(x_0) = P(u, \Omega')(x_0) < \infty \tag{9.2}$$

since

$$P(u, \Omega')(x_0) = \lim_{i\to\infty} P(v_i, \Omega')^*(x_0) = \lim_{i\to\infty} \min(v_i, h_i)(x_0) = \lim_{i\to\infty} h_i(x_0) = h(x_0),$$

because $x_0 \in \Omega'$ and

$$\lim_{i\to\infty} v_i(x_0) = u(x_0) > P(u, \Omega')(x_0).$$

The Harnack convergence theorem, see [KM2], and (9.2) imply that h is a minimizer in Ω'.

Suppose that $u(x_0) < \infty$ and write

$$\varepsilon = u(x_0) - P(u, \Omega')(x_0) > 0.$$

Since u is lower semicontinuous and h is continuous, there is a neighbourhood U of x_0 such that

$$u(x) > u(x_0) - \tfrac{1}{2}\varepsilon \qquad \text{and} \qquad h(x) < h(x_0) + \tfrac{1}{2}\varepsilon$$

for $x \in U$. Because $P(v_i, \Omega')^* \le h_i$ in Ω', we have $P(u, \Omega') \le h$ in Ω' and thus for all $x \in U$ we have

$$u(x) > u(x_0) - \tfrac{1}{2}\varepsilon = P(u, \Omega')(x_0) + \tfrac{1}{2}\varepsilon = h(x_0) + \tfrac{1}{2}\varepsilon > h(x) \ge P(u, \Omega')(x).$$

Hence $U \subset A$ and thus A is open. An obvious modification takes care of the case $u(x_0) = \infty$. Now (9.2) holds for each $x \in A$ and (4) follows. □

10. Polar sets

Here we show that the set where a quasisuperharmonic function is equal to ∞ is of zero p-capacity. For superharmonic functions on metric spaces this question has been studied in [KS2]. We start with a lemma which gives a characterization of compact sets of zero p-capacity. See 2.4 for the definition of the relative p-capacity.

10.1. Lemma. *Suppose that $C \subset X$ is a compact set. Then $\mathrm{C}_p(C) > 0$ if and only if*

$$\lim_{t \to 0} \mathrm{C}_p(C, C_t) = \infty. \tag{10.2}$$

Here $C_t = \{x \in X : \operatorname{dist}(x, C) < t\}$.

Proof. If (10.2) holds, then clearly $\mathrm{C}_p(C) > 0$. Suppose then that $\mathrm{C}_p(C) > 0$. For $t > 0$, let $E(t)$ denote the condenser (C, C_t) and for $0 < t < r$, let $E(t, r)$ be the the condenser $(\overline{C_t}, C_r)$. We claim that

$$\lim_{t \to 0} \mathrm{C}_p\big(E(t, r)\big) = \mathrm{C}_p\big(E(r)\big). \tag{10.3}$$

Indeed, let u be an admissible Lipschitz function for the condenser $E(r)$. For $\varepsilon > 0$ the function $(1+\varepsilon)u$ is admissible for $E(t, r)$ provided t is small and hence

$$\mathrm{C}_p\big(E(t, r)\big) \le (1+\varepsilon)^p \int_X g_u^p \, d\mu.$$

Now X is proper and thus admissible Lipschitz functions are dense in the class of admissible functions for $E(r)$, see [KaS]. Hence we obtain

$$\mathrm{C}_p\big(E(t,r)\big) \le (1+\varepsilon)^p\,\mathrm{C}_p\big(E(r)\big)$$

for small t. This implies

$$\lim_{t\to 0}\mathrm{C}_p\big(E(t,r)\big) \le \mathrm{C}_p\big(E(r)\big)$$

and since $\mathrm{C}_p\big(E(t,r)\big) \ge \mathrm{C}_p\big(E(r)\big)$ for every t, $0<t<r$, we obtain (10.3).

Then we claim that

$$\mathrm{C}_p\big(E(r)\big)^{-1/(p-1)} \ge \mathrm{C}_p\big(E(t)\big)^{-1/(p-1)} + \mathrm{C}_p\big(E(t,r)\big)^{-1/(p-1)}. \tag{10.4}$$

Indeed, let $u_{E(t)}$ and $u_{E(t,r)}$ be the capacitary potentials of $E(t)$ and $E(t,r)$, respectively. This means that $u_{E(t)}$ is the unique minimizer in $C_t\setminus C$ with boundary values 0 in $X\setminus C_t$ and 1 in C. Let

$$u = \alpha u_{E(t)} + \beta u_{E(t,r)},$$

where $\alpha,\beta\ge 0$, $\alpha+\beta=1$. Then u is admissible for $E(r)$ and hence

$$\mathrm{C}_p\big(E(r)\big) \le \alpha^p\,\mathrm{C}_p\big(E(t)\big) + \beta^p\,\mathrm{C}_p\big(E(t,r)\big).$$

Note that $g_{u_{E(t)}}=0$ μ-almost everywhere in $X\setminus C(t)$ and $g_{u_{E(t,r)}}=0$ μ-almost everywhere in $C(t)$. If $\mathrm{C}_p\big(E(t)\big)>0$ and $\mathrm{C}_p\big(E(t,r)\big)>0$, we set

$$\alpha = \mathrm{C}_p\big(E(t)\big)^{-1/(p-1)}\Big(\mathrm{C}_p\big(E(t)\big)^{-1/(p-1)} + \mathrm{C}_p\big(E(t,r)\big)^{-1/(p-1)}\Big)^{-1}$$

and

$$\beta = \mathrm{C}_p\big(E(t,r)\big)^{-1/(p-1)}\Big(\mathrm{C}_p\big(E(t)\big)^{-1/(p-1)} + \mathrm{C}_p\big(E(t,r)\big)^{-1/(p-1)}\Big)^{-1}$$

and the claim follows. If either $\mathrm{C}_p\big(E(t)\big)=0$ or $\mathrm{C}_p\big(E(t,r)\big)=0$, then (10.4) is obvious.

Letting $t\to 0$ we obtain from (10.3) and (10.4) that $\lim_{t\to 0}\mathrm{C}_p\big(E(t)\big)=\infty$ as required. □

We make the following assumptions:

(1) C is a compact set in X,
(2) v is a K-quasisuperminimizer in X such that $v\ge 1$ on C, and
(3) $v\ge 0$ on C_r, $r>0$.

We let $u_{E(r)}$ be the capacitary potential of the condenser $E(r)$.

10.5. Lemma. *If $\mathrm{C}_p(C) > 0$, then there exist t, $0 < t < r$, $r > 0$, and a point x_0 such that $d(x_0, C) = t$ with the property that any neighbourhood of x_0 contains a set A with $\mu(A) > 0$ where*

$$v(x) \geq \tfrac{1}{4} u_{E(r)}(x).$$

The number t depends on C and K and on the other data but not on the function v otherwise. The point x_0 and the set A depend on v.

Proof. Let $0 < t < r$, $0 < \varepsilon < 1$ and suppose that

$$\varepsilon u_{E(r)}(x) \geq v(x), \quad x \in \partial C_t.$$

Since $u_{E(r)} \leq 1$, this implies that $\varepsilon \geq v(x)$, $x \in \partial C_t$. Let $w = \min(v, u_{E(r)})$, then the function $(1-\varepsilon)^{-1}(w-\varepsilon)$ is admissible for the condenser $E(t)$. Thus

$$\begin{aligned} \mathrm{C}_p\bigl(E(t)\bigr) &\leq (1-\varepsilon)^{-p} \int_{C_t \setminus C} g_w^p \, d\mu \\ &= (1-\varepsilon)^{-p} \biggl(\int_{(C_t \setminus C) \cap \{u_{E(r)} < v\}} g_{u_{E(r)}}^p \, d\mu + \int_{(C_t \setminus C) \cap \{u_{E(r)} \geq v\}} g_v^p \, d\mu \biggr). \end{aligned}$$

The K-quasisuperminimizing property of v yields

$$\int_{(C_t \setminus C) \cap \{u_{E(r)} \geq v\}} g_v^p \, d\mu \leq K \int_{(C_t \setminus C) \cap \{u_{E(r)} \geq v\}} g_{u_{E(r)}}^p \, d\mu.$$

Note that $u_{E(r)} - v \in N_0^{1,p}\bigl((C_r \setminus C) \cap \{u_{E(r)} \geq v\}\bigr)$ because $v \geq 1 \geq u_{E(r)}$ on C, $v \geq 0$ and $u_{E(r)} = 0$ in $X \setminus C_r$. From this we conclude that

$$\begin{aligned} \mathrm{C}_p\bigl(E(t)\bigr) &\leq (1-\varepsilon)^{-p} \biggl(\int_{C_r \setminus C} g_{u_{E(r)}}^p \, d\mu + K \int_{C_r \setminus C} g_{u_{E(r)}}^p \, d\mu \biggr) \\ &\leq \frac{1+K}{(1-\varepsilon)^p} \int_{C_r \setminus C} g_{u_{E(r)}}^p \, d\mu = \frac{1+K}{(1-\varepsilon)^p} \, \mathrm{C}_p\bigl(E(r)\bigr). \end{aligned}$$

This implies that

$$(1-\varepsilon)^p \leq (1+K) \frac{\mathrm{C}_p\bigl(E(r)\bigr)}{\mathrm{C}_p\bigl(E(t)\bigr)},$$

or in other words,

$$\varepsilon \geq 1 - \biggl[(1+K) \frac{\mathrm{C}_p\bigl(E(r)\bigr)}{\mathrm{C}_p\bigl(E(t)\bigr)} \biggr]^{1/p}.$$

Since $\mathrm{C}_p(C) > 0$, by Lemma 10.1 we have

$$\lim_{t \to 0} \mathrm{C}_p\bigl(E(t)\bigr) = \infty$$

and we can choose t, independent of v, such that

$$\left[(1+K)\frac{C_p\big(E(r)\big)}{C_p\big(E(t)\big)}\right]^{1/p} \le \frac{1}{2}.$$

Then $\varepsilon \ge \frac{1}{2}$ and this means that there is a point $x_0 \in \partial C_t$ such that $v(x_0) > \frac{1}{4}$ and hence

$$\tfrac{1}{4}u_{E(r)}(x_0) \le \tfrac{1}{4} < v(x_0).$$

Since v is a K-quasisuperminimizer, by Lemma 5.3 we may assume that

$$\operatorname*{ess\,lim\,inf}_{x\to x_0} v(x) = v(x_0).$$

From this we conclude that there exists in any neighbourhood of x_0 a set A of positive measure such that

$$\tfrac{1}{4}u_{E(r)}(x) < v(x), \quad x \in A.$$

This completes the proof. □

We say that a set $C_0 \subset X$ is K*-polar*, if there is an open neighbourhood Ω of C_0 and a K-quasisuperharmonic function u in Ω such that $u(x) = \infty$ for every $x \in C_0$.

The following theorem is the main result of this section.

10.6. Theorem. *If* C_0 *is a* K*-polar set, then* $C_p(C_0) = 0$.

Proof. Let C_0 be a K-polar set in X. Then there is an open set Ω such that $C_0 \subset \Omega$ and a K-quasisuperharmonic function u_0 in Ω such that $u_0 = \infty$ on C_0. By Definition 7.1 there are open sets Ω_i and K-quasisuperminimizers v_i, $i = 1, 2, \ldots$, satisfying the properties (i)–(iv) in Definition 7.1. Because the statement of the theorem is local and the functions v_i are lower semicontinuous we may assume that $\Omega_i = \Omega$ and $v_i > 0$ for every $i = 1, 2, \ldots$.

We observe that

$$C_0 = \bigcap_{j=j_0}^{\infty} \bigcup_{i=i_0}^{\infty} C_{i,j},$$

where $C_{i,j} = \{x \in \Omega : v_i(x) > j\}$ for any integers j_0 and i_0. To see this let $x_0 \in C_0$. Then $v_i(x_0) \to \infty$ as $i \to \infty$ and hence for any j there are arbitrary large i such that $v_i(x_0) > j$. Consequently $x_0 \in \bigcap_{j=j_0}^{\infty} \bigcup_{i=i_0}^{\infty} C_{i,j}$. For the converse, let $x_0 \in \bigcap_{j=j_0}^{\infty} \bigcup_{i=i_0}^{\infty} C_{i,j}$. Then for every $j \ge j_0$ there is i such that $v_i(x_0) > j$. Since v_i is an increasing sequence, $v_{i'}(x_0) > j$ for $i' \ge i$. This means that $\lim_{i\to\infty} v_i(x_0) = \infty$ and hence $x_0 \in C_0$.

Note that the set $C_{i,j}$ is open by the lower semicontinuity of v_i and hence C_0 is a $\mathscr{G}_\delta$-set.

Suppose that $C_p(C_0) > 0$. Then there is a compact set $C \subset C_0$ such that $C_p(C) > 0$. This is due to the fact that the p-capacity is a Choquet capacity, see [KM1]. This means that if we choose $r > 0$ such that $r < \operatorname{dist}(C, X \setminus \Omega)$, then $C_p(C, C_t) > 0$. For every $j = 1, 2, \dots$, we have

$$C \subset \bigcup_{i=i_0}^{\infty} C_{i,j}$$

for arbitrarily large i_0. Now the sets $C_{i,j}$ are open and C is compact. Hence for any fixed j and i_0 there is $i' = i'(j, i_0)$ such that

$$C \subset \bigcup_{i=i_0}^{i'} C_{i,j}.$$

On the other hand, we have $C_{i,j} \subset C_{i+1,j}$ and hence $C \subset C_{i',j}$.

Let $E(r)$, $r > 0$, denote the condenser (C, C_r) and $u_{E(r)}$ the capacitary function of $E(r)$. We can now apply Lemma 10.5 to the K-quasiminimizer $v_{i'}/j$. By Lemma 10.5 there are t, $0 < t < r$, independent of i' and j, and a point x_0 such that $\operatorname{dist}(x_0, C) = t$ and any neighbourhood of x_0 contains a set A with $\mu(A) > 0$ and

$$\frac{v_{j'}(x)}{j} \geq \frac{1}{4} u_{E(r)}(x), \quad x \in A.$$

Since v_i is an increasing sequence, this means that

$$\frac{v_i(x)}{j} \geq \frac{1}{4} u_{E(r)}(x), \quad x \in A, \tag{10.7}$$

for all $i \geq i'$.

By Theorem 7.7 we have $u_0(x) < \infty$ for μ-almost every $x \in \Omega$. Since $v_i(x) \leq u_0(x)$ for all x and since (10.7) holds, we can pick $\tilde{x}_0 \in A$ such that $u(\tilde{x}_0) < \infty$, (10.7) holds at $\tilde{x}_0$ for $i \geq i'$ and

$$t - \varepsilon_1 < \operatorname{dist}(\tilde{x}_0, C) < t + \varepsilon_1,$$

where $\varepsilon_1 = \min(t, r - t)/10$.

By (10.7) this implies

$$\frac{u_0(\tilde{x}_0)}{j} \geq \frac{1}{4} u_{E(r)}(\tilde{x}_0)$$

for each j. Letting $j \to \infty$ we obtain that $u_{E(r)}(\tilde{x}_0) = 0$ and by the Harnack inequality $u_{E(r)} = 0$ in a component U of $C_t \setminus C$ containing $\tilde{x}_0$. Since the above reasoning can be applied to each such component, this means that $u_{E(r)} = 0$ in $C_t \setminus C$, but this implies that $C_p(C) = 0$, which is a contradiction. This completes the proof. □

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