Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 29, 2004, 3–20

MYRBERG'S NUMERICAL UNIFORMIZATION OF HYPERELLIPTIC CURVES

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Abstract. In this paper we derive an algorithm that computes, for a given algebraic hyperelliptic plane curve C of genus p, $p > 1$, defined by a polynomial $y^2 = (x-\lambda_1)\cdots(x-\lambda_{2p+2})$, an approximation of a Fuchsian group G acting in the unit disk D such that $C = D/G$. The method allows us also to approximate the projection mapping $\pi\colon D \to D/G = C$, thus giving a solution to the problem of numerical uniformization in the case of hyperelliptic curves. The method is based on work of P. J. Myrberg that appeared already in 1920 but did not get much attention at that time.

1. Introduction

The problem of *numerical uniformization* of an algebraic plane curve C defined by an affine polynomial equation $P(x,y) = 0$ consists of:

– Finding a domain Γ in the Riemann sphere and a discontinuous group G of Möbius transformations acting in Γ such that $\Gamma/G = C$.
– Finding an explicit form of the uniformizing mapping $\pi\colon \Gamma \to C$.

By the uniformization theorem, we know that such a presentation is possible. Explicit constructions for uniformization have been difficult to find in spite of huge efforts to solve this problem already for more than hundred years ago.

Burnside ([1], see also [18]) found perhaps the first explicit uniformization in a special case.

Later several examples of algebraic curves, for which it has been possible to determine a uniformizing group G, have been found by many persons. Hence, for these curves, the first part of the numerical uniformization problem has been solved. These curves are typically characterized by the fact that they have a large number of authomorphisms. Kulkarni has studied such curves (cf. [13]). More

2000 Mathematics Subject Classification: Primary 30F10; Secondary 14Q05.

This paper was written during my stay at EPF in Lausanne. I would like to thank Peter Buser, K.-D. Semmler and the Mathematics Department at EPFL for wonderful hospitality during that time. This paper is largely based on preliminary work done within the former European HCM-network "Computational Conformal Geometry" (ERBCHRX-CT93-0408).

examples have been found most recently by Rubí Rodríguez and Víctor González-Aguilera [19]. They found a 1-dimensional family of Fuchsian groups uniformizing a 1-dimensional pencil of algebraic curves (without giving explicit uniformizations for curves in the pencil). Common to these examples is that it has not been possible to find or even to approximate the uniformizing projection $\pi\colon \Gamma \to C$.

One approach to solving the numerical uniformization problem is afforded by so called *accessory parameters*. These are related to the fact that the inverse of the uniformizing projection satisfies a Schwarzian differential equation that depends on these parameters. The accessory parameters have been studied by Dennis Hejhal (cf. [6], [7], [8]) Linda Keen et al. ([10], [11]), Irwin Kra ([12]), P. G. Zograf and L. A. Takhtajan ([21], [24]), J. M. Whittaker ([23]) and others ([5], [20]) but relatively little is known about them and the uniformization problem has been solved in some very special cases only using this method.

In this paper we study a method of P. J. Myrberg ([15]) that can be used to approximate a Schottky uniformization of a given hyperelliptic curve. Myrberg's iterative method allows one to write down the projection map as a certain limit. This projection map is not of the type that Linda Keen meant in the above statement. Myrberg's method leads to a Schottky uniformization for which the projection is a mapping from an infinitely punctured sphere onto the given hyperelliptic curve.

Instead of Schottky uniformization, one would like to be able to approximate a Fuchsian uniformization of a given curve. In comparison with the Schottky uniformization, the Fuchsian uniformization has the advantage that it allows one to study the hyperbolic geometry of the curve in an explicit fashion. The hyperbolic metric of a disk (which is the domain of the action of a Fuchsian uniformizing group) is well understood and subject to explicit computations, while the hyperbolic metric of the domain of the action of a Schottky group is much more complicated and no explicit formulae for the metric exist. Hence it is desirable to find numerical approximations of Fuchsian uniformizations of algebraic curves. Here this is done, for the first time, for a non-trivial family of curves, namely for hyperelliptic algebraic curves.

Myrberg's work appeared in 1920. He credits Poincaré ([16]) for the main idea in the construction. Similar methods for uniformization, based on the ideas of Poincaré, have been presented by E. T. Whittaker ([23]) but Myrberg appears to have developed them further. Myrberg's work was also later discussed by Fernanda Esser in an unpublished manuscript [4].

In another direction, Peter Buser and Robert Silhol ([3]) have recently developed methods that allow one to find a Fuchsian group uniformizing a given real hyperelliptic curve. They are also able to compute equations for certain surfaces having given geometric properties. Buser has further studied differential geometric methods to calculate lengths of closed geodesic curves on given (hyperelliptic) curve. A Fuchsian group uniformizing the curve can be approximated in this way also. These methods are subject to generalizations. A former student of Buser, Matthias Wagner ([22]), has developed explicit differential-geometric methods to uniformize hyperelliptic algebraic curves.

2. Preliminaries

Myrberg's method for numerical uniformization [15] allows one to find a Möbius group G uniformizing a hyperelliptic curve C defined by the affine polynomial

$$y^2 = \prod_{j=1}^{2p+2} (x - \lambda_j), \quad \lambda_j \neq \lambda_i \text{ for } i \neq j. \tag{1}$$

More precisely, given the polynomial defining the curve C, Myrberg derived a numerical method to approximate the generators g_j of a Möbius group G such that

$$C = \Gamma(G)/G,$$

where $\Gamma(G)$ is the domain of discontinuity of the group G. Furthermore, the method allows one to approximate the uniformizing mapping $\Gamma \to \Gamma/G = C$.

In Myrberg's construction, the group G is freely generated by $p-1$ loxodromic Möbius transformations g_j. If all the λ_j's in the polynomial defining the curve C are real, then the generating transformations g_j are actually hyperbolic and they map the upper half-plane onto itself. In this case the limit set of the group G is a nowhere dense subset of the real line.

In this paper

(1) we recall Myrberg's construction,
(2) show that it actually converges (convergence is fast), and
(3) extend it to yield a way to approximate a Fuchsian uniformization of a given hyperelliptic curve.

The last step amounts to solving the following problem. Let G be a given finitely generated Schottky group with limit set in the real axis. Let Γ be the domain of discontinuity of G, and assume that Γ/G is a Riemann surface of genus $p > 1$. Find a Fuchsian group F which is of the first kind, acts in the upper half-plane U and for which $U/F = \Gamma/G$. This passing from a Schottky uniformization to Fuchsian uniformization uses quasiconformal mappings. Possible implementations depend on being able to approximate a μ-quasiconformal self-mapping of the upper half-plane onto itself for a given Beltrami differential μ. This clearly can be done using numerical methods. For the purposes of the implementations of this algorithm, it is important to find as good approximations of the μ-quasiconformal mappings as possible. This problem is not considered here.

3. Iteration of the opening of the slots

Assume now that C is a hyperelliptic plane algebraic curve defined by the affine equation (1). The projection $C \to \widehat{\mathbf{C}}$, $(x, y) \to x$, of the curve C onto the Riemann sphere $\widehat{\mathbf{C}}$ is ramified exactly over the points λ_j. Assume that all these points are real, and that $\lambda_1 < \lambda_2 < \cdots < \lambda_{2p+2}$. We will later get rid of this assumption.

From equation (1), one can solve y in terms of x locally uniquely off the points λ_j. If one continues such a solution along a closed curve going around an odd number of the points λ_j, then the sign of the solution changes. Hence there are no global solutions. This difficulty disappears, if one cuts the Riemann sphere open in such a way that it is not anymore possible to find closed loops going around an odd number of points λ_j. This limits the domain of the variable x. In this smaller domain a global solution (of y in terms of x) to (1) can be found, i.e., in this smaller domain

$$y = \sqrt{\prod_{j=1}^{2p+2} (x - \lambda_j)} \tag{2}$$

can be uniquely defined. Clearly there are two choices for the branch of the square root. Either of them will do.

One way to achieve this is to consider the complement S of the union of the intervals

$$I_j = [\lambda_{2j-1}, \lambda_{2j}], \quad j = 1, \ldots, p+1.$$

We view S as a domain in the Riemann sphere $\widehat{\mathbf{C}}$, and observe that any closed curve going around some points λ_j must go around an even number of such points. Hence, for x in S, y satisfying equation (1) can be represented as a function of x. There are two solutions, let $y = \pi(x)$ denote one of them. Then the other solution is $y = -\pi(x)$.

In the domain S, π is a well defined holomorphic function. Let γ_j be the curve on C going through the points λ_{2j-1} and λ_{2j}, $j = 1, \ldots, p+1$, and corresponding the values $x \in I_j$. Considering C as a Riemann surface with a hyperbolic metric, γ_j is a simple closed geodesic curve going through two hyperelliptic branch points. Observe that, on each γ_j, y takes purely imaginary values.

The complement of the curves γ_j consists of two planar Riemann surfaces with $p+1$ boundary curves. The mapping $\pi\colon S \to C$ maps S conformally onto one of these components, and the hyperelliptic involution exchanges these components.

Next we wish to extend the mapping π so that the image of π becomes the whole curve C. This is clearly not possible without modifying the domain S. The iterative modification of S, which we proceed to describe, is an idea of P. J. Myrberg ([15]). It is based on an auxiliary mapping that opens the slots I_j conformally. That mapping is introduced in the next subsection.

3.1. The opening mapping. Consider a mapping, called *opening mapping*, which maps the exterior of an interval in the complex plane onto the exterior of a disk. We will define it in the following way.

Let:

(1) α and β be two different points in the complex plane;
(2) I denote the closed interval having α and β as end-points;
(3) $m = \frac{1}{2}(\alpha+\beta)$ and $s = \frac{1}{4}(\beta-\alpha)$.

We assume that the real part of s is non-negative. In the applications considered here, the α and β will be real and we simply assume that $\alpha < \beta$.

Consider the equation

$$\frac{z-(m+2s)}{z-(m-2s)} = \left(\frac{w-(m+s)}{w-(m-s)}\right)^2. \tag{3}$$

A suitable choice of the square-root gives us, by equation (3), a mapping

$$\phi\colon \mathbf{C}\setminus I \to \mathbf{C}\setminus \overline{D}, \tag{4}$$

where D is a disk centered at m with radius $|s|$. This mapping extends to the end-points α and β of the interval I by setting $\phi(\alpha) = \frac{1}{4}(3\alpha+\beta)$ and $\phi(\beta) = \frac{1}{4}(\alpha+3\beta)$. So the mapping ϕ halves the distance of the points α and β.

This mapping ϕ plays a central role in our construction and it is, therefore, necessary to take a closer look at it.

We observe that the following lemmata hold:

Lemma 1. *The mapping ϕ determined by equation* (3) *is given by*

$$\phi(x) = x + \frac{x-m}{2}\left(\sqrt{1-4\frac{s^2}{(x-m)^2}}-1\right) \quad \text{for } |x-m| \ge 2|s|, \tag{5}$$

or

$$\phi(x) = x - s\bigl(t+t^3+t^5+O(t^7)\bigr), \tag{6}$$

where $t = s/(x-m)$ and $|x-m| \ge 2|s|$.

Lemma 2. *Assume that the α and β are real and $\alpha < \beta$. Then m is also real and $s > 0$. In this case the mapping ϕ moves points of the real axis outside of the interval $[m-2s, m+2s]$ towards the mid-point m. For $|x-m| \ge 2|s|$, let*

$$\delta(x) = |x-\phi(x)| \tag{7}$$

or

$$\delta(x) = |x-m|\bigl(t^2+t^4+2t^6+O(t^8)\bigr), \tag{8}$$

where $t = s/(x-m)$.

Then, for $x \geq m+2s$, $\delta(x)$ *is strictly decreasing, and, for* $x \leq m-2s$, $\delta(x)$ *is strictly increasing. The function* $\delta(x)$ *attains its maximum for* $x = m+2s$ *and for* $x = m-2s$. *The maximum value is*

$$\delta(m \pm 2s) = s. \tag{9}$$

Furthermore,

$$\lim_{x\to\pm\infty} \delta(x) = 0. \tag{10}$$

Both of the above lemmata are simple straightforward technical computations and proofs are left to the reader.

In the above, the mapping ϕ was determined by equation (3) and by requiring that ϕ maps the complement of the interval I onto the complement of the disk D with m as center and $|s|$ as radius.

The other choice of the sign of the square root (when solving w in terms of z from equation (3)) yields a mapping $\tilde{\phi}$: $\mathbf{C} \setminus I \to D$. The mapping

$$e(z) = \frac{mz-(m+s)(m-2)}{z-m} \tag{11}$$

is the elliptic rotation, with angle π, of the complex plane keeping the finite points $m-s$ and $m+s$ fixed. It maps the complement of D onto D and we have

$$\tilde{\phi} = e \circ \phi. \tag{12}$$

In the subsequent considerations we use this mapping ϕ with varying parameters α and β to perform the desired iteration.

3.2. Opening the slots. We now consider a curve C [1] given by equation (1) in which all the branch-points λ_j are real and arranged in an increasing order, i.e.,

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{2p+2}.$$

Then the closed intervals $I_j = [\lambda_{2j-1}, \lambda_{2j}]$, $j = 1, \ldots, p+1$, are disjoint. We call these intervals *slots* or, more precisely, *first generation slots.*

Let ϕ_1 be the mapping ϕ defined by equation (3) with $\alpha = \lambda_1$ and $\beta = \lambda_2$. The mapping ϕ_1 *opens up* the first slot I_1 replacing it by a disk D_1 and maps all other slots onto new slots $\phi_1(I_j)$ which are still intervals in the real axis.

The corresponding elliptic rotation e_1, defined by (11), maps the complement of $\overline{D}_1$ onto D_1. In particular, e_1 maps the deformed slots $\phi_1(I_2), \phi_1(I_3), \ldots$ onto

[1] $C : y^2 = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_{2p+2})$.

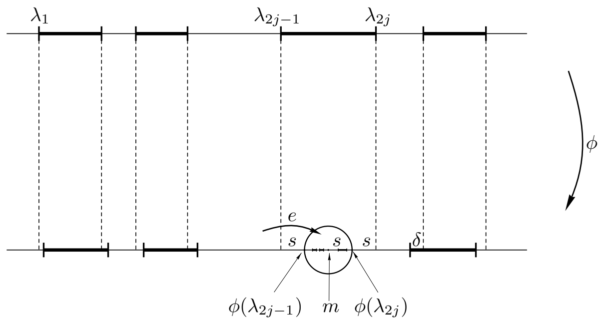

Figure 1. The mapping ϕ opens a slot of length $4s$ and replaces it by a disk D of radius s. The higher generation slots are obtained as images of the other slots under an elliptic Möbius transformation e mapping the complement of the closure of disk D onto D itself. Also the other slots move towards the mid-point m of the slot that is being opened. The function δ measures this movement. The points $\phi(\lambda_{2j-1})$ and $\phi(\lambda_{2j})$ are the fixed points of the elliptic transformation e_j corresponding to this opening of a slot.

new slots $e_1\big(\phi_1(I_j)\big)$, $j = 2, 3, \ldots$, in D_1. These are the first 2^{nd}*-generation slots.* Let us denote them by I_j^2.

The opening of a slot and the creation of 2^{nd} generation slots is illustrated in Figure 1.

Next, let ϕ_2 be the mapping ϕ defined by the equation (3) with $\alpha = \phi_1(\lambda_3)$ and $\beta = \phi_1(\lambda_4)$. The mapping ϕ_2

- opens up the slot $\phi_1(I_2)$ and replaces it by a disk D_2;
- maps the boundary of the disk D_1 onto a Jordan curve going through the points $\phi_2\big(\phi_1(\lambda_1)\big)$ and $\phi_2\big(\phi_1(\lambda_2)\big)$;
- maps the deformed slots $\phi_1(I_j)$ onto new slots $\phi_2\big(\phi_1(I_j)\big)$; these are still intervals in the real axis.

Let e_2 be the corresponding elliptic rotation, defined by (11) with

$$m = \tfrac{1}{2}\big(\phi_1(\lambda_3) + \phi_1(\lambda_4)\big) \quad \text{and} \quad s = \tfrac{1}{4}\big(\phi_1(\lambda_4) - \phi_1(\lambda_3)\big). \tag{13}$$

The rotation e_2 maps the slots $\phi_2\big(\phi_1(I_3)\big)\,\phi_2\big(\phi_1(I_4)\big)$, $\ldots$ onto new 2^{nd}-generation slots I_j^2, $j = p+1, \ldots, 2p$, inside the disk D_2. The images $e_2\big(\phi_2(I_j^2)\big)$ of the first 2^{nd}-generation slots I_j^2, $j = 1, \ldots, p$, inside D_2 are referred to as 3^{rd}-generation slots I_j^3.

Repeat this procedure until all the first generation slots have been opened and new higher generation slots have been formed.

In this way, when opening slots, one forms iteratively new slots which are divided into generations. Order these slots by generation and within a generation from left to right (these slots are always disjoint intervals in the real line). Later we will show that this ordering is actually irrelevant; the ordering of these slots is related to selecting a certain fundamental domain for the action of a group. For purposes of effectiveness of implementations, one should rather order the slots according to their size.

Continue with opening the slots in the order fixed above. Iterating this procedure we get a sequence of conformal mappings

$$\psi_j = \phi_j \circ \phi_{j-1} \circ \ldots \circ \phi_1 \colon S \to \widehat{\mathbf{C}}.$$

At this point we deliberately restrict the domain of definition of the mappings ϕ_k so that we view the composed mapping as being defined in S, the complement of the union of the slots.

Theorem 3. *The sequence (ψ_j) has a converging subsequence (ψ_{j_k}) and the limit mapping ψ_∞ is a conformal mapping of the domain S onto a domain in the Riemann sphere. Furthermore the limit mapping does not depend on the choice of the converging subsequence.*

Proof. Assume that all the first generation slots are contained in the interval $[-M, M]$, i.e., that $-M < \lambda_1 < \cdots < \lambda_{2p+2} < M$. Let $x < -M$ or $x > M$. By Lemma 2, we conclude that

- the sequence $\big(\psi_j(x)\big)$ (of real numbers) converges, and
- $\lim_{j\to\infty} \psi_j(x) \neq \lim_{j\to\infty} \psi_j(y)$ for $x \neq y$.

In fact, by Lemma 2, for $x < -M$ or $x > M$, the points $\psi_j(x)$, $j = 1, 2, 3, \ldots$, move towards the slots I_k which are contained in the interval $[-M, M]$. But these points cannot cross the initial slots, they always stay at the same side of them. Hence the convergence. The points closer to the slots move faster. This means that if x and y are both either $< -M$ or $> M$, then the distance of the iterated points $\psi_j(x)$ and $\psi_j(y)$ grows as $j \to \infty$. This means that the limit points cannot be the same, implying the second statement.

By [14, Theorem II.5.1] the sequence (ψ_j) is a normal family. This follows from the above-quoted result, since $\big(\psi_j(x)\big)$ converges for real x with sufficiently large $|x|$.

We conclude that (ψ_j) has a converging subsequence such that the limit mapping ψ_∞ is either a conformal homeomorphism of S, a constant mapping or a mapping of S onto two points. By the above remarks, the latter two possibilities are excluded. Hence we conclude that ψ_∞ is a conformal mapping of S onto a domain $\psi_\infty(S)$.

Assume now that ξ_∞ is the limit mapping of another converging subsequence of the sequence (ψ_j). Since the sequence $\big(\psi_j(x)\big)$ converges for $x < -M$ or $x > M$ (and x real), it follows that $\xi_\infty(x) = \psi_\infty(x)$ for $x < -M$ or $x > M$. Since the limit mappings are both conformal, they must agree everywhere.

A more detailed analysis of the sequence (ψ_j) implies that the whole sequence (and not only a subsequence) converges to the limit mapping ψ_∞. □

At this point, a normalization is useful. The limit mapping ψ_∞ is a conformal mapping of S into the Riemann sphere $\widehat{\mathbf{C}}$. The mapping ψ_∞ extends to the points λ_j, $j = 1, \dots, 2p+2$. Normalize the limit mapping ψ_∞ by requiring that

$$\psi_\infty(\lambda_1) = \infty, \quad \psi_\infty(\lambda_2) = 0 \quad \text{and} \quad \psi_\infty(\lambda_3) = 1. \tag{14}$$

This normalization can always be achieved replacing ψ_∞ by $g \circ \psi_\infty$ where g is a suitable Möbius transformation *mapping the upper half-plane onto itself*. When we later consider the limit mapping ψ_∞ we always assume that it is normalized in the above manner.

4. Schottky uniformization

It turns out that we can uniformize the curve C defined by equation (1) by a Schottky group G generated freely by p hyperbolic Möbius transformations mapping the real axis onto itself. The domain of discontinuity Γ of this group G contains both the upper and the lower half-planes, and the original curve C is the quotient Γ/G. Let proj: $\Gamma \to C$ be the projection.

To construct the group G, we actually construct the projection first; the group G is then the cover group of this projection. We construct the projection so that we first construct its inverse (locally), namely a mapping of a part of the algebraic curve C onto a domain in the Riemann sphere. That allows us to construct the complete projection and the group G.

Previously we observed that, for x in S, one can solve y in terms of x from the equation (1). The solutions were denoted by $y = \pm\pi(x)$. In particular,

$$\pi\colon S \to C, \quad x \mapsto \big(x, \pi(x)\big)$$

is a conformal mapping of S into C. The image of S in C is a component of the complement of the simple closed curves γ_j going through the hyperelliptic branch points. Hence the image of S under this (new) mapping π is half of the algebraic curve C. Use the notation $C_1 = \pi(S)$ for this half of C.

Consider the mapping

$$C_1 \to \widehat{\mathbf{C}}, \quad p \mapsto \psi_\infty \circ \pi^{-1}(p). \tag{15}$$

This is going to be a local inverse of the uniformizing projection proj: $\Gamma \to C$. Keep in mind that we assume the normalization conditions (14) be satisfied.

As the inverse of the mapping (15), the projection is now defined in the domain $\psi_\infty(S)$ which is the Riemann sphere minus $p+1$ disks $D_1, \dots, D_{p+1}$. These disks correspond to the initial slots $I_1, \dots, I_{p+1}$.

Consider the elliptic transformation e_1 that we formed after the opening of the first slot I_1. Consider the limit

$$\lim_{n\to\infty} g \circ \phi_n \circ \cdots \circ \phi_2 \circ e_1 \circ \phi_1 = \psi_\infty^1.$$

Here g is the hyperbolic Möbius transformation normalizing ψ_∞ (cf. (14)).

By the same argument as before, ψ^1_∞ is a conformal mapping of S onto a domain $\psi^1_\infty(S)$ which, this time, is contained in D_1, the topological disk corresponding to the first slot. The mapping

$$E_1 = \psi^1_\infty \circ (\psi_\infty)^{-1}$$

maps $\psi_\infty(S)$ onto $\psi^1_\infty(S)$, i.e., the outside of the disk D_1 (minus a number of other disks corresponding to the slots $I_2, \ldots, I_{p+1}$) gets mapped onto the inside of D_1. The image of E_1 in the disk D_1 is D_1 minus p other (topological) disks ${D^2}_2, \ldots, {D^2}_{p+1}$.

We iterate this process. All the mappings

$$\psi^k_\infty = \lim_{n\to\infty} g \circ \phi_n \circ \cdots \circ \phi_{k+1} \circ e_k \circ \phi_k \circ \ldots \circ \phi_1$$

are conformal mappings of S onto a domain in $\widehat{\mathbf{C}}$. For $k > p+1$, $\psi^k_\infty(S)$ is a topological disk minus p other disks inside (which disks contain $\psi^l_\infty(S)$ for some $l > k$). For each disk D_n that gets formed in this way, we associate the mappings E_n that map the outside of the disk D_n (minus a certain number of other disks) onto the inside (minus again the same number of other disks).

By their definition, the mappings E_n map a domain of $\widehat{\mathbf{C}}$ onto another one. These two domains do not intersect but they do have a common boundary arc. Requiring E_n to be an involution, we extend E_n first to the union of these two domains. The mapping E_n extends also to the common boundary arc. Let us assume that E_n has been extended in this way. Then the domain of E_n becomes the Riemann sphere minus $2p$ disks D_k.

We claim that E_n is actually an elliptic Möbius transformation. To see why this is true, we next extend E_n to (parts of) each one of the $2p$ disks D_k in the following way:

(1) Let E_k be the mapping associated to the disk D_k.
(2) Via the mapping E_n, the disk D_k corresponds to some other disk D_l, i.e., a small neighborhood of D_k gets mapped by E_n onto a small neighborhood of the disk D_l.
(3) Let E_l be the mapping associated to the disk D_l.
(4) In D_k (minus a number of small disks inside), extend the mapping E_n by

$$E_l \circ E_n \circ E_k.$$

Repeat this construction. At the limit one gets a mapping E^∞_n which is a conformal involution of a domain Γ in $\widehat{\mathbf{C}}$ such that the complement of the domain Γ is a nowhere dense subset of the real axis. This mapping can, furthermore, be extended to a homeomorphic mapping of the Riemann sphere onto itself. A nowhere dense subset of the real line is a removable singularity; hence we conclude that the mapping E^∞_n is a conformal involution of the Riemann sphere, i.e., that E^∞_n is an elliptic Möbius transformation.

5. Schottky uniformization

We summarize the preceding considerations in the following result.

Theorem 4. *Hyperbolic Möbius transformations* $g_j = E_j^\infty \circ E_{j+1}^\infty$ *generate a discontinuous group of Möbius transformations. The domain discontinuity of* G, Γ, *contains both the lower and the upper half-planes, and the limit-set of* G *is no-where dense in the real axis. The group* G *gives a Schottky uniformization of the hyperelliptic real algebraic* M*-curve* C, *i.e.,*

$$\Gamma/G = C.$$

Each mapping E_j^∞ *corresponds to the hyperelliptic involution of the algebraic curve* C. *Locally the projection* $\Gamma \to C$ *is the inverse of mappings of the type*

$$(\pm\pi)^{-1} \circ \psi_\infty^k. \tag{16}$$

This appears to have been known to E. T. Whittaker ([23], see also [17]). In the form presented here, the result comes from the 1920 paper of P. J. Myrberg ([15]). In that paper Myrberg credits Poincaré ([16]) for the idea of taking a limit of the opening mappings. Clearly this approach to numerical uniformization was known to the late masters of complex analysis and geometry. In those days, this method was, in most cases, too complicated for practical computations by hand.

Lemma 5. *The group* G *of Theorem* 4 *does not depend on the choice of ordering the slots when forming the mappings* ψ_∞^k.

Proof. Repeat the above construction by using two different orderings of the slots. Denote the respective groups obtained in this fashion by G_1 and by G_2. They both uniformize the same algebraic curve C. The identity mapping of C induces a Möbius transformation f such that $G_1 f = f G_2$. Furthermore this Möbius transformation f maps the points corresponding to λ_1, λ_2 and λ_3 in the domain of discontinuity of G_1 onto the respective points in the domain of discontinuity of G_2. In view of the imposed normalization (14), this means that f *is* the identity mapping, i.e., that $G_1 = G_2$. The corresponding normalized mappings ψ_∞ do not usually agree; the images $\psi_\infty(S)$ form two different pieces of a fundamental domain for the action of the group G. □

5.1. Myrberg's algorithm.

For purposes of implementations, one can compute the group G as follows.

Input: An integer p, $p > 1$, and a set of real numbers $\lambda_1, \ldots, \lambda_{2p+2}$; p is the genus of the curve in question, and the points λ_j are the parameters of the equation (1).

Approximation step: Let L denote the length of the largest slot, i.e., $L = \max\{|\lambda_{2j} - \lambda_{2j-1}| \mid j = 1, \ldots, p\}$. Approximate the mapping ψ_∞ of Theorem 3 by computing

$$\lfloor L \rfloor n(2p+2)$$

for $n = 4$ or $n = 5$ (increase this value if you want better accuracy) openings of the slots of various generations (open first generations slots first in some order (best would be to open the largest slots first), then proceed with higher order generations of slots). The sequence defining ψ_∞ converges very fast; the above number of iterations is sufficient for most purposes.

Fixed-points: Using the above computed approximation of the mapping ψ_∞ approximate the points $\lambda_j^\infty = \psi_\infty(\lambda_j)$; these are going to be the fixed-points of the generating elliptic transformations.

Elliptic elements: For $j = 1, \ldots, p+1$, let E_j be the elliptic rotation of the complex plane by angle π and with fixed-points λ_{2j-1}^∞ and λ_{2j}^∞. Use formula (11) to define these transformations.

Group generators: For $j = 1, \ldots, p$, compute $g_j = E_j E_{j+1}$. These are hyperbolic Möbius transformations which generate freely a Schottky group G. The group G acts in a domain Γ, and the original hyperelliptic algebraic curve C is now the quotient $C = \Gamma / G$.

The above algorithm produces approximations of the generators of the uniformizing Schottky group G and approximations of a number of elliptic Möbius transformations E_j^∞. Each one of these elliptic transformations approximates the hyperelliptic involution of the curve C, i.e., the mapping $(x, y) \mapsto (x, -y)$ (cf. (1)).

The convergence proof (Theorem 3) requires only qualitative analysis under the assumptions that all the points λ_j are real. Clearly, this is not a necessary condition for convergence. But the proof gets much more technical in the general case. The group G becomes also much more complicated if we drop the reality condition. Then the generators of G are loxodromic, and G is Kleinian.

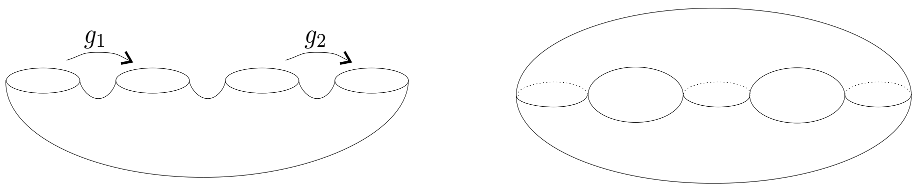

Figure 2. A topological side view of the Riemann sphere is shown on the left. The isometric circles of the hyperbolic Möbius transformations g_1 and g_2, together with their respective images are also shown. The group generated by g_1 and g_2 acts in such a way that a fundamental domain is the exterior of the disks shown on the left. The group simply identifies the isometric circle of g_j with its image, $j = 1, 2$, to form the genus 2 Riemann surface shown on the right.

6. Fuchsian uniformization of real hyperelliptic M-curves

Myrberg's algorithm produces an approximation of a Schottky uniformization of hyperelliptic algebraic curves whose all branch points are real. Such a curve is called hyperelliptic real algebraic M-curve. The 'M' here refers to the fact that such a curve has the maximal number, allowed by the genus, of real components.

Now let C be such a curve, and let G be an approximation of the Schottky group uniformizing C and obtained by Myrberg's algorithm.

While the Schottky uniformization already solves the classical problem of numerical uniformization, it is desirable to be able to approximate generators of a Fuchsian group of the first kind uniformizing C. This would be useful, for instance, in the study of the hyperbolic geometry of the algebraic curve in question. Also that is the usual way to consider uniformization.

6.1. Getting Fuchsian groups by deformations. The domain of discontinuity of G contains intervals of the real axis. These intervals project onto the real components of the hyperelliptic curve C, and the complement of the real part consists of two planar Riemann surfaces, one corresponding the upper half-plane and the other one corresponding the lower half-plane.

In this section our aim is to find a way to approximate generators of a Fuchsian group of the first kind that uniformizes a given hyperelliptic real algebraic M-curve. Later we extend these considerations to general hyperelliptic curves.

We do it by first performing some violence to the curve C.

Let $g_1, \dots, g_p$ be (approximations of) the free hyperbolic Möbius transformations generating the Schottky group G uniformizing C. Consider the axes A_{g_j} and the isometric circles IC_{g_j} of the transformations g_j, $j = 1, \dots, p$.

Let a_{g_j} and r_{g_j} be the attracting and the repelling fixed-points of the transformation g_j, $j = 1, \dots, p$. By the construction of Myrberg's algorithm and by the assumption that $\lambda_1 < \lambda_2 < \cdots < \lambda_{2p+2}$, it follows that the fixed-points satisfy

$$r_{g_1} < a_{g_1} < r_{g_2} < a_{g_2} < \cdots < r_{g_p} < a_{g_p}. \tag{17}$$

Furthermore, for all indices j, the isometric circles (considered as full circles in the Riemann sphere) IC_{g_j} and their images $g_j(IC_{g_j})$, are disjoint. The outside of all the isometric circles IC_{g_j} and their images $g_j(IC_{g_j})$, $j = 1, 2, \dots, p$, forms a fundamental domain for the action of the Schottky group G. Identifying IC_{g_j} with $g_j(IC_{g_j})$ one forms a 'handle' for the Riemann surface corresponding to the algebraic curve C. In the hyperbolic geometry of the domain of discontinuity of the group G, the interval of the real axis between IC_{g_j} and $g_j(IC_{g_j})$ corresponds to a simple closed geodesic curve going around this handle. This is illustrated in Figure 2.

On the original algebraic curve C viewn as the Riemann surface Γ/G, certain intervals on the real axis project onto geodesic curves going around handles.

The full axes (i.e. complete circles orthogonal to the real axis and going through the fixed points of the respective Möbius transformation) of the above-mentioned Möbius transformations project onto two symmetric curves also going around a handle in Γ/G.

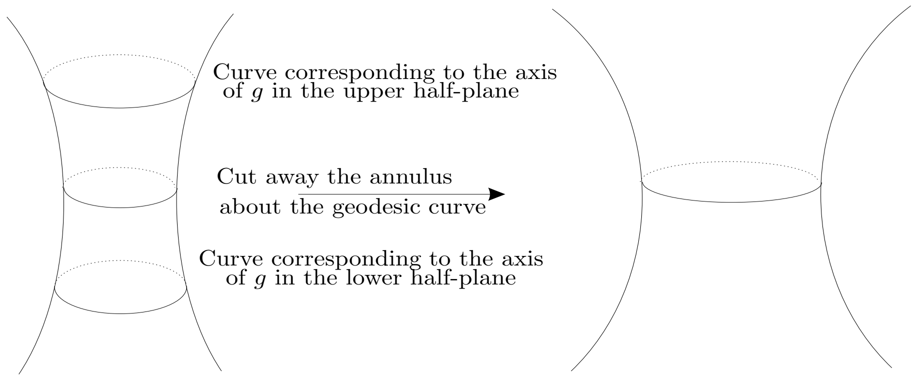

Figure 3. The original Schottky uniformized Riemann surface Γ/G is deformed by cutting away parts of the handles and identifying the closed curve corresponding to the axis of a hyperbolic Möbius transformation g in the upper half plane with that of the same axis in the lower half-plane. Handles get shorter and thicker as illustrated above. There is certain freedom here in the sense that the gluing angle of this gluing can be freely chosen. Let us choose it to be 0.

Next we delete, from the Riemann surface Γ/G, the annuli bounded by the projections of the axes. We get a deformed Riemann surface whose handles are now thicker. This deformed Riemann surface can now be uniformized by a Fuchsian group of the first kind in the following manner.

For convenience, observe that by conjugation we may assume that $r_{g_1} = \infty$ and $a_{g_1} = 0$, and that all other fixed points of the generating group elements are positive. Observe that the Schottky group G acts also in the upper half-plane H. The quotient H/G is a Riemann surface (of genus 0) with $p+1$ boundary components. The axes of the generating hyperbolic Möbius transformations $g_1, \ldots, g_p$ correspond to p of the boundary components. The remaining boundary component corresponds to the product $g_1 \circ g_2 \circ \cdots \circ g_p$. This correspondence means that the axes of these Möbius transformations project onto simple closed geodesic curves on the Riemann surface H/G which curves go around the respective boundary component.

Under the above normalization assumptions regarding the location of the fixed-points of the generating hyperbolic Möbius transformation, the deformation illustrated in Figure 3, can be performed as follows.

For $j = 1, \ldots, p$, let $\tilde{g}_j$ be the Möbius transformation defined by setting $\tilde{g}_j(z) = -g_j(-z)$. Observe that the fixed points of the mappings $\tilde{g}_j$ lie in the negative real axis, while the fixed points of the mappings g_j lie in the positive real axis.

Take, as generators of the Fuchsian group corresponding to the deformed surface, the following transformations $g_1, \dots, g_p$ plus the Möbius transformations h_j having the axis of $-g_j(-z)$ as its respective isometric circle with the property that the image of this isometric circle under the mapping h_j is the axis of g_j, $j = 1, 2, \dots, p$. Observe that then $\tilde{g}_j = h_j \circ g_j \circ h_j^{-1}$.

The mappings h_j perform the gluing as described in Figure 3. Observe that the mappings h_j are not uniquely defined by the above conditions. They become uniquely defined by requiring the gluing angle to be 0 (cf. Figure 3).

It follows from the standard theory of Fuchsian groups that the group F thus generated is a Fuchsian group of the first kind acting in the upper half-plane H, and that H/F is the Riemann surface obtained from our original Riemann surface Γ/G by the deformations described in Figure 3.

6.2. Approximating a Fuchsian uniformization. Our aim is to get a Fuchsian uniformization for the algebraic curve C defined by the equation (1). But what we got was a Fuchsian uniformization for a Riemann surface that is a deformation of the Riemann surface Γ/G of the algebraic curve C. We must now make up the violence we did to Γ/G. That is done by quasiconformal mappings.

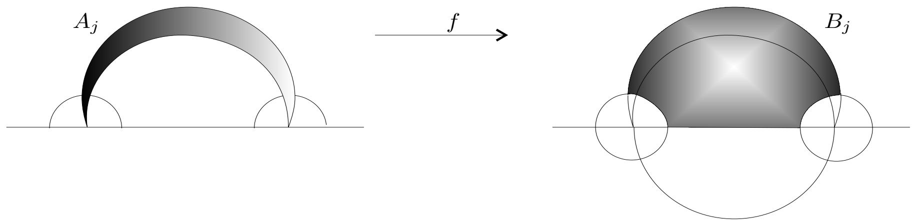

Figure 4. The quasiconformal mapping f that makes up the deformation we had to perform on Γ/G to get a Fuchsian uniformization stretches the collar on the left to cover the whole shaded collar on the right.

Let α be a simple closed geodesic curve on a compact Riemann surface of genus $p > 1$. Let l_α denote the length of α. Recall that by the Keen collar lemma (cf. [2], [9]) one can always find a collar of width

$$\log \coth l_\alpha/4$$

around α. Take now the largest disjoint collars allowed by the collar lemma around each curve corresponding to the axes of the elements g_j, $j = 1, \dots, p$.

For convenience, assume now that none of the fixed-points of the generators g_j lies at the infinity. This can always be achieved by a suitable conjugation.

The axes of the elements g_j are geodesic curves in the upper-half plane as illustrated in Figure 4 above. Consider the upper side A_j of the collar about the axis of g_j, i.e., the shaded domain on the left side of Figure 4. That domain corresponds also to a one sided collar on the original Riemann surface Γ/G of the algebraic curve C. On Γ/G, A_j is a one-sided collar about the projection of the axis of g_j. That collar can be continued to a one-sided collar B_j about the geodesic curve homotopic to the the projection of the axis of g_j. This geodesic curve corresponds to an interval in the real axis. The continuation of this collar corresponds to the larger shaded region on the right-hand side of Figure 4. Let f be a quasiconformal mapping which is the identity on the upper boundary of the collar A_j and maps the collar A_j onto B_j. Apart from the condition that f is the identity on the upper boundary of A_j, the mapping f can be freely chosen. It is to our advantage, however, to choose f so that it is as regular as possible, i.e., so that its maximal dilatation is as small as possible.

A good choice for f can be obtained as follows. From each point of the axis of g_j drop a geodesic (in the hyperbolic metric of the upper half-plane) curve onto the real axis. Continue this curve so that it crosses through the one sided collar A_j. Consider the interval of this geodesic curve contained in the collar A_j. Map this interval by a constant Euclidean stretching so that the image of the interval becomes the arc of the geodesic that is contained in the larger shaded area of Figure 4.

Continue the mapping f to the lower sides of the collars about the axes of the Möbius transformations g_j by symmetry, i.e., if σ_j denotes the reflection in the axis of g_j then set

$$f(z) = \overline{f(\sigma(z))}$$

for points in the lower half on the collar around the axes of g_j. Outside of the collars about the axes of the elements g_j, set f to be the identity.

All of the above can be explicitly computed. Let μ be the Beltrami differential of the mapping f, and let f_μ be a μ-quasiconformal mapping of the upper half-plane onto itself. The following is then obvious by the construction.

Theorem 6. *The group*

$$F_\mu = f_\mu F f_\mu^{-1}$$

is a Fuchsian group of the first kind uniformizing the algeraic curve C. The uniformizing projection is given locally in $\psi_\infty^k(S)$ by

$$\pm\pi \circ (\psi_\infty^k)^{-1} \circ f \circ f_\mu^{-1}.$$

7. Uniformization of general hyperelliptic curves

In this section we extend the above considerations to cover a general hyperelliptic curve C defined by equation (1) *without* the assumption that all the parameters λ_j are real. We will do this by quasiconformal mappings.

The starting data is the genus p of the curve, $p > 1$, and a set of $2p+2$ hyperelliptic branch points $\lambda_1, \dots, \lambda_{2p+2}$. Without loss of generality we may assume that all the parameters λ_j are finite.

Let $h\colon \mathbf{C} \to \mathbf{C}$ be a quasiconformal mapping such that $h(\lambda_j) \in \mathbf{R}$ for all $j = 1, \dots, 2p+2$. There are infinitely many such mappings, any one will do. For our purposes it would be good to find one with as small maximal dilatation as possible. For implementations one can, for instance, do the following:

(1) Find first a direction $e^{i\theta}$ such that the minimum distance between the lines going through the points λ_j in the direction $e^{i\theta}$ is as large as possible.

(2) Then let the points λ_j flow to real line along these lines.

Use the notation $\lambda'_j = h(\lambda_j)$, $j = 1, \dots, 2p+2$, and let C' be the curve defined by

$$y^2 = (x - \lambda'_1) \cdots (x - \lambda'_{2p+2}).$$

7.1. Uniformization of general hyperelliptic curves. The curve C' is now a hyperelliptic real algebraic M-curve. To approximate Myrberg's uniformization of C do the following:

(1) Compute the complex dilatation μ of the quasiconformal mapping $h^{-1}\colon C' \to C$.

(2) Approximate Myrberg's uniformization for the curve C'; let Γ be the domain of discontinuity of the uniformizing group G and let $\varrho\colon \Gamma \to C'$ be the projection (cf. (16)).

(3) Approximate the lifting of the Beltrami differential μ to a Beltrami differential of the group G. Extend this differential by 0 to all of $\mathbf{C}$.

(4) Approximate a μ-quasiconfomal mapping $H_\mu\colon \mathbf{C} \to \mathbf{C}$.

(5) Approximate the generators of $G_\mu = f_\mu G f_\mu^{-1}$.

Then G_μ is a Möbius group acting discontinuously in $\Gamma_\mu = h_\mu(\Gamma)$, $\Gamma_\mu/G_\mu = C$ and the projection is given by

$$\Gamma_\mu \to C, \quad z \mapsto h^{-1} \circ \varrho \circ H_\mu^{-1}(z). \tag{18}$$

Fuchsian uniformization of general hyperelliptic curves is achieved by a further quasiconformal deformation.

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Received 14 September 2001