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POINTWISE MULTIPLIERS FROM WEIGHTED BERGMAN SPACES AND HARDY SPACES TO WEIGHTED BERGMAN SPACES

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Abstract. Pointwise multipiers from weighted Bergman spaces and Hardy spaces to weighted Bergman spaces are characterized by using Bloch type spaces, BMOA type spaces, weighted Bergman spaces and tent spaces.

1. Introduction

Let $D=\{z:|z|<1\}$ be the unit disk in the complex plane, and let $\partial D=\{z:|z|=1\}$ be the unit circle. Let $H(D)$ be the space of all analytic functions on the unit disk D. For $0<p<\infty$, let H^p denote the Hardy space which contains $f\in H(D)$ such that

$$\|f\|_{H^p}^p=\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta<\infty.$$

For $0<p<\infty$ and $-1<\alpha<\infty$, let $L^{p,\alpha}$ denote the weighted Lebesgue spaces which contain measurable functions f on D such that

$$\|f\|_{p,\alpha}^p=\int_D|f(z)|^p\,dA_\alpha(z)<\infty,$$

where $dA_\alpha(z)=(1-|z|^2)^\alpha\,dA(z)=(1-|z|^2)^\alpha\,dx\,dy/\pi$. We also denote by $L_a^{p,\alpha}=L^{p,\alpha}\cap H(D)$, the weighted Bergman space on D, with the same norm as above. If $\alpha=0$, we simply write them as L^p and L_a^p, respectively.

Let g be an analytic function on D, let X and Y be two spaces of analytic functions. We say that g is a *pointwise multiplier* from X into Y if $gf\in Y$ for any $f\in X$. The space of all pointwise multipliers from the space X into the space Y will be denoted by $M(X,Y)$. In this paper we will give complete criteria of the pointwise multipliers between two weighted Bergman spaces and between a Hardy space and a weighted Bergman space. Let M_g be the multiplication operator defined by $M_gf=fg$. A simple application of the closed graph theorem shows

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that g is a pointwise multiplier between two weighted Bergman spaces or between a Hardy space and a weighted Bergman space if and only if M_g is a bounded operator between the same spaces.

Pointwise multipliers are closely related to Toeplitz operators and Hankel operators. They have been studied by many authors. See [Ax1], [Ax2], [At], [F] and [Vu] for a few examples. In [At], the pointwise multipliers between unweighted Bergman spaces were characterized. In [F], the pointwise multipliers between the Hardy space H^2 and the unweighted Bergman space L_a^2 were characterized by using the Carleson measure. Our results generalize their results.

In order to state our results, we need notation of various other function spaces.

First, for $0<\alpha<\infty$, we say an analytic function f on D is in the α-Bloch space B^α, if

$$\sup_{z\in D}|f'(z)|(1-|z|^2)^\alpha<\infty.$$

As $\alpha=1$, $B^1=B$, the well-known Bloch space. As $0<\alpha<1$, the space $B^\alpha=\mathrm{Lip}_{1-\alpha}$, the analytic Lipshitz space which contains analytic functions f on D satisfying

$$|f(z)-f(w)|\le C|z-w|^{1-\alpha},$$

for any z and w in D (see [D2]). If $\alpha>1$, it is known that $f\in B^\alpha$ if and only if

$$\sup_{z\in D}|f(z)|(1-|z|^2)^{\alpha-1}<\infty,$$

or the antiderivative of f is in $B^{\alpha-1}$.

Next, we define a general family of function spaces. We will use a special Möbius transformation $\varphi_a(z)=(a-z)/(1-\bar{a}z)$, which exchange 0 and a, and has derivative $\varphi_a'(z)=-(1-|a|^2)/(1-\bar{a}z)^2$. Let p, q and s be real numbers such that $0<p<\infty$, $-2<q<\infty$ and $0<s<\infty$. We say that an analytic function f on D belongs to the space $F(p,q,s)$, if

$$\|f\|_{F(p,q,s)}^p=\sup_{a\in D}\int_D|f'(z)|^p(1-|z|^2)^q\left(1-|\varphi_a(z)^2|\right)^s dA(z)<\infty.$$

The spaces $F(p,q,s)$ were introduced in [Z2]. They contain, as special cases, many classical function spaces. See [Z2] for the details. It was proved in [Z1] that, for $-1<\alpha<\infty$, $F(p,p\alpha-2,s)=B^\alpha$ for any $p>0$ and any $s>1$ (see also [Z2, Theorem 1.3]). When $s=1$, we define BMOA type spaces as follows: $\mathrm{BMOA}_p^\alpha=F(p,p\alpha-2,1)$. Unlike the α-Bloch spaces, the spaces BMOA_p^α are different for different values of p ([Z2, Theorem 6.5]). It is known that, $\mathrm{BMOA}_2^1=\mathrm{BMOA}$, the classical space of analytic functions of bounded mean oscillation.

We also need a version of tent spaces. Let μ be a Borel measure on D; we say that an analytic function f is in the tent space $T_q^p(d\mu)$ if

$$\|f\|_{T_p^q(d\mu)} = \left(\int_0^{2\pi}\left(\int_{\Gamma(\theta)} |f(z)|^q \frac{d\mu(z)}{(1-|z|^2)}\right)^{p/q} d\theta\right)^{1/p} < \infty,$$

where $\Gamma(\theta)$ is the Stolz angle at θ, which is defined for real θ as the convex hull of the set $\{e^{i\theta}\} \cup \{z : |z| < \sqrt{1/2}\}$. The tent spaces were introduced in [CMS]. The above version of tent spaces was introduced in [L4].

Our main results are the following two theorems.

Theorem 1. *Let g be an analytic function on D, let $-1 < \alpha, \beta < \infty$ and let $\gamma = (\beta+2)/q - (\alpha+2)/p$.*
(i) *If $0 < p \le q < \infty$ and $\gamma > 0$ then $M(L_a^{p,\alpha}, L_a^{q,\beta}) = B^{1+\gamma}$.*
(ii) *If $0 < p \le q < \infty$ and $\gamma = 0$ then $M(L_a^{p,\alpha}, L_a^{q,\beta}) = H^\infty$.*
(iii) *If $0 < p \le q < \infty$, and $\gamma < 0$ then $M(L_a^{p,\alpha}, L_a^{q,\beta}) = \{0\}$.*
(iv) *If $0 < q < p < \infty$, then $M(L_a^{p,\alpha}, L_a^{q,\beta}) = L_a^{s,\delta}$, where $1/s = 1/q - 1/p$ and $\delta/s = \beta/q - \alpha/p$.*

Theorem 2. *Let g be an analytic function on D, let $-1 < \beta < \infty$ and $\gamma = (\beta+2)/q - 1/p$.*
(i) *If $0 < p < q < \infty$, and $\gamma > 0$, then $M(H^p, L_a^{q,\beta}) = B^{1+\gamma}$.*
(ii) *If $0 < p < q < \infty$, and $\gamma = 0$, then $M(H^p, L_a^{q,\beta}) = H^\infty$.*
(iii) *If $0 < p < q < \infty$, and $\gamma < 0$, then $M(H^p, L_a^{q,\beta}) = \{0\}$.*
(iv) *If $0 < q < p < \infty$, then $M(H^p, L_a^{q,\beta}) = T_s^q(dA_\beta)$, where $1/s = 1/q - 1/p$.*
(v) *If $0 < p = q < \infty$ then $M(H^p, L_a^{q,\beta}) = \mathrm{BMOA}_p^{1+(\beta+1)/p}$.*

Remark. The results of Theorem 1 for the unweighted cases (i.e., $\alpha = \beta = 0$) were obtained by Attele in [At]. Note that, when $\alpha = \beta = 0$, the case (i) in Theorem 1 will never happen since two restrictions about p and q there contradict to each other. However, if α and β are not zeros, then the case (i) in Theorem 1 may happen if $\alpha < \beta$.

2. Carleson type measures

Carleson type measures are the main tools of our investigation. Let X be a space of analytic functions on D. Following the notations in [AFP], we say a Borel measure $d\mu$ on D is an (X,q)-*Carleson measure* if

$$\int_D |f|^q \, d\mu(z) \le C\|f\|_X^q$$

for any function $f \in X$.

Let $I \subset \partial D$ be an arc. Denote by $|I|$ the normalized arc length of I so that $|\partial D| = 1$. Let $S(I)$ be the Carleson box defined by

$$S(I) = \{z : 1 - |I| < |z| < 1,\ z/|z| \in I\}.$$

There are many different versions of Carleson type theorems. Here we collect those results we need later.

The first result is the classical result due to L. Carleson [C] for the case $p = q$ and P. Duren [D1] for the case $p < q$. A proof of the equivalence of (ii) and (iii) can be found in [ASX].

Theorem A. *For μ a positive Borel measure on D and $0 < p \le q < \infty$, the following statements are equivalent*:

(i) *The measure μ is an (H^p, q)-Carleson measure.*

(ii) *There is a constant $C_1 > 0$ such that, for any arc $I \subset \partial D$,*

$$\mu\big(S(I)\big) \le C_1 |I|^{q/p}.$$

(iii) *There is a constant $C_2 > 0$ such that, for every $a \in D$,*

$$\int_D |\varphi_a'(z)|^{q/p}\, d\mu(z) \le C_2.$$

For the case $0 < q < p < \infty$, the following result is due to I. V. Videnskii ([Vi]) and D. Leucking ([L3]).

Theorem B. *For μ a positive Borel measure on D and $0 < q < p < \infty$, the following statements are equivalent*:

(i) *The measure μ is an (H^p, q)-Carleson measure.*

(ii) *The function $\theta \to \int_{\Gamma(\theta)} d\mu/(1-|z|^2)$ belongs to $L^{p/(p-q)}$, where $\Gamma(\theta)$ is the Stolz angle at θ.*

For the weighted Bergman spaces $L_a^{p,\alpha}$, the following result was obtained by several authors and can be found in [L1]. The equivalence of (ii) and (iii) is the same as the equivalence of (ii) and (iii) in Theorem A.

Theorem C. *For μ a positive Borel measure on D, $0 < p \le q < \infty$, and $-1 < \alpha < \infty$, the following statements are equivalent*:

(i) *The measure μ is an $(L_a^{p,\alpha}, q)$-Carleson measure.*

(ii) *There is a constant $C_1 > 0$ such that, for any arc $I \subset \partial D$,*

$$\mu\big(S(I)\big) \le C_1 |I|^{(2+\alpha)q/p}.$$

(iii) *There is a constant $C_2 > 0$ such that, for every $a \in D$,*

$$\int_D |\varphi_a'(z)|^{(2+\alpha)q/p}\, d\mu(z) \le C_2.$$

We denote by $D(z) = D\big(z, \frac{1}{4}\big) = \big\{w : |\varphi_z(w)| < \frac{1}{4}\big\}$. For the case $0 < q < p < \infty$, the following result is due to D. Luecking ([L2] and [L4]), for the case $\alpha = 0$. For $-1 < \alpha < \infty$, the result can be similarly proved as in [L4].

Theorem D. *For μ a positive Borel measure on D, $0 < q < p < \infty$ and $-1 < \alpha < \infty$, the following statements are equivalent*:

(i) *The measure μ is an $(L_a^{p,\alpha}, q)$-Carleson measure.*

(ii) *The function $z \to \mu\big(D(z)\big)(1-|z|^2)^{-2-\alpha} \in L^{p/(p-q),\alpha}$.*

3. Proofs of the theorems

In order to give a unified proof of (i), (ii) and (iii) of Theorem 1, we first give a simple integral criterion for H^∞ which seems not to be seen in literature.

Lemma 1. *Let $p > 0$ and let $f \in H(D)$. Then the following conditions are equivalent*:

(i) $f \in H^\infty$.

(ii) $\{f \circ \varphi_a\}$ *is a bounded subset of $L_a^{p,\alpha}$ for some $\alpha > -1$.*

(iii) $\{f \circ \varphi_a\}$ *is a bounded subset of $L_a^{p,\alpha}$ for all $\alpha > -1$.*

(iv) $\sup_{a\in D} \int_D |f(z)|^p (1-|z|)^{-2}\big(1-|\varphi_a(z)|^2\big)^s \, dA(z) < \infty$ *for some $s > 1$.*

(v) $\sup_{a\in D} \int_D |f(z)|^p (1-|z|)^{-2}\big(1-|\varphi_a(z)|^2\big)^s \, dA(z) < \infty$ *for all $s > 1$.*

Proof. Let $f \in H^\infty$. Then

$$\sup_{a\in D} \int_D |f\circ\varphi_a(z)|^p (1-|z|)^\alpha \, dA(z) \le \|f\|_{H^\infty}^p \int_D (1-|z|^2)^\alpha \, dA(z) < \infty$$

for any $\alpha > -1$. Thus (i) implies (iii). It is trivial that (iii) implies (ii).

Let $\{f \circ \varphi_a\}$ be a bounded subset of $L_a^{p,\alpha}$ for $\alpha > -1$. If $\alpha \ge 0$, we fix an $r \in (0,1)$. By subharmonicity of $|f\circ\varphi_a|^p$, we get

$$
\begin{aligned}
|f(a)|^p = |f\circ\varphi_a(0)|^p &\le \frac{1}{r^2} \int_{D(0,r)} |f\circ\varphi_a(z)|^p \, dA(z) \\
&\le \frac{1}{r^2(1-r^2)^\alpha} \int_{D(0,r)} |f\circ\varphi_a(z)|^p (1-|z|^2)^\alpha \, dA(z).
\end{aligned}
\tag{1}
$$

Thus

$$\sup_{a\in D} |f(a)|^p \le c(r) \sup_{a\in D} \int_D |f\circ\varphi_a(z)|^p (1-|z|^2)^\alpha \, dA(z) < \infty.$$

So $f \in H^\infty$. For the case $-1 < \alpha < 0$, we notice that

$$\int_D |f\circ\varphi_a(z)|^p \, dA(z) \le \int_D |f\circ\varphi_a(z)|^p (1-|z|^2)^\alpha \, dA(z).$$

Thus this reduces the problem to the case $\alpha = 0$. Thus (ii) implies (i).

If we change the variable $\varphi_a(z)$ by w and let $s = \alpha + 2$, then it is easy to see that (iv) is equivalent to (ii), and (v) is equivalent to (iii). The proof is complete. ▫

Replacing f by f', we immediately have an integral criterion for the space $B^0 = \{f \in H(D),\ f' \in H^\infty\}$.

Lemma 2. *Let $p > 0$ and let $f \in H(D)$. Then the following conditions are equivalent*:

(i) $f \in B^0$.
(ii) $\{f' \circ \varphi_a\}$ *is a bounded subset of* $L_a^{p,\alpha}$ *for some* $\alpha > -1$.
(iii) $\{f' \circ \varphi_a\}$ *is a bounded subset of* $L_a^{p,\alpha}$ *for all* $\alpha > -1$.
(iv) $f \in F(p,-2,s)$ *for some* $s > 1$.
(v) $f \in F(p,-2,s)$ *for all* $s > 1$.

We also need the following lemma.

Lemma 3. *Let $0 < p < \infty$, $q < -2$ and $s > 0$. Let $f \in H(D)$. If*

$$\sup_{a\in D} \int_D |f(z)|^p (1-|z|)^q \big(1-|\varphi_a(z)|^2\big)^s \, dA(z) < \infty, \tag{2}$$

then $f = 0$.

Proof. Let $0 < p < \infty$, $q < -2$ and $s > 0$. Let $f \in H(D)$ and satisfy (2). Fix $r \in (0,1)$. Similarly as in the proof of Lemma 1, by subharmonicity of $|f \circ \varphi_a|^p$, we get

$$\begin{aligned} |f(a)|^p = |f\circ\varphi_a(0)|^p &\le \frac{1}{r^2} \int_{D(0,r)} |f\circ\varphi_a(w)|^p \, dA(w) \\ &= \frac{1}{r^2} \int_{D(a,r)} |f(z)|^p |\varphi_a'(z)|^2 \, dA(z) \\ &\le \frac{16}{r^2(1-|a|^2)^2} \int_{D(a,r)} |f(z)|^p \, dA(z), \end{aligned}$$

where $D(a,r) = \{z : |\varphi_a(z)| < r\}$. It is known that, for $z \in D(a,r)$, $1-|z|^2 \sim 1-|a|^2$ (see [Zhu, p. 61]). Thus

$$\begin{aligned} |f(a)|^p (1-|a|^2)^{q+2} &\le \frac{16C}{r^2} \int_{D(a,r)} |f(z)|^p (1-|z|^2)^q \, dA(z) \\ &\le \frac{16C}{r^2(1-r^2)^s} \int_{D(a,r)} |f(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s \, dA(z). \end{aligned}$$

Thus, if (2) holds then

$$\sup_{a\in D} |f(a)|(1-|a|^2)^{q+2} \le M < \infty,$$

where M is an absolute constant. Thus $|f(a)| \le M(1-|a|^2)^{-q-2}$. When $q < -2$, $-q-2 > 0$. Letting $|a| \to 1$ we see that $\lim_{|a|\to1} |f(a)| = 0$. By the maximal principle, we get that $f(z) = 0$ for any $z \in D$. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. By definition, an analytic function $g \in M(L_a^{p,\alpha}, L_a^{q,\beta})$ if and only if, for any $f \in L_a^{p,\alpha}$,

$$\int_D |f(z)g(z)|^q \, dA_\beta(z) \le C\|f\|_{p,\alpha}^q. \tag{3}$$

Let $d\mu_g(z) = |g(z)|^q \, dA_\beta(z)$. Then (3) means that $d\mu_g$ is an $(L_a^{p,\alpha}, q)$-Carleson measure.

Now we will prove (i), (ii) and (iii) at the same time. By Theorem C, if $0 < p \le q < \infty$, (3) is equivalent to the fact that

$$\sup_{a\in D} \int_D |\varphi_a'(z)|^{(2+\alpha)q/p} \, d\mu_g(z) < \infty,$$

which is the same as

$$\sup_{a\in D} \int_D |g(z)|^q (1-|z|^2)^{\beta-(2+\alpha)q/p} \big(1 - |\varphi_a(z)|^2\big)^{(2+\alpha)q/p} \, dA(z) < \infty. \tag{4}$$

Notice that, as $q \ge p$, $(2+\alpha)q/p > 1$. Let G be an antiderivative of g.

If $(\beta+2)/q - (\alpha+2)/p > 0$, then $\beta - (2+\alpha)q/p > -2$. By Theorem 1 of [Z1] (see also Theorem 1.3 of [Z2]), (4) means $G \in B^{(\beta-(2+\alpha)q/p+2)/q} = B^{(\beta+2)/q-(\alpha+2)/p}$, which is equivalent to the fact that $g = G' \in B^{1+(\beta+2)/q-(\alpha+2)/p}$. Thus (i) is proved.

If $(\beta+2)/q - (\alpha+2)/p = 0$ then $\beta - (\alpha+2)q/p = -2$. By Lemma 1, (4) is equivalent to that $g \in H^\infty$, which proves (ii).

If $(\beta+2)/q - (\alpha+2)/p < 0$, then $\beta - (\alpha+2)q/p < -2$. By Lemma 3, (4) implies $g = 0$, which proves (iii).

For proving (iv), we use Theorem D. Let $0 < q < p < \infty$. By Theorem D, (3) is equivalent to the fact that

$$\int_D \Big(\mu_g\big(D(z, \tfrac14)\big)(1-|z|^2)^{-2-\alpha}\Big)^{p/(p-q)} \, dA_\alpha(z) < \infty,$$

where $d\mu_g$ is given above. Thus

$$\int_D \left(\frac{1}{(1-|z|^2)^{2+\alpha}} \int_{D(z,1/4)} |g(w)|^q \, dA_\beta(w) \right)^{p/(p-q)} dA_\alpha(z) < \infty. \tag{5}$$

By subharmonicity of $|g|^q$, it is easy to see that (see the proof of Lemma 3 above),

$$|g(z)|^q (1-|z|^2)^{\beta+2} \le C \int_{D(z,1/4)} |g(w)|^q \, dA_\beta(w).$$

Thus (5) implies that

$$\begin{aligned}(6)\qquad &\int_D \big(|g(z)|^q(1-|z|^2)^{\beta-\alpha}\big)^{p/(p-q)}\,dA_\alpha(z)\\ &\qquad = \int_D |g(z)|^{pq/(p-q)}(1-|z|^2)^{(\beta p-\alpha q)/(p-q)}\,dA(z)<\infty.\end{aligned}$$

Let $1/s = 1/q - 1/p$ and $\delta/s = \beta/q - \alpha/p$. Then $s = pq/(p-q)$ and $\delta = (\beta p - \alpha q)/(p-q)$. Thus (6) means $g \in L_a^{s,\delta}$.

Conversely, if $g \in L_a^{s,\delta}$, then an easy application of Hölder's inequality shows that $g \in M(L_a^{p,\alpha}, L_a^{q,\beta})$. The proof is complete. □

We need some preliminary results for proving Theorem 2(v).

Proposition 1. *Let $f \in H(D)$ and let $0 < p < \infty$. Then $f \in L_a^{p,\alpha}$ if and only if $f^{(n)}(z)(1-|z|^2)^n \in L^{p,\alpha}$, and $\|f\|_{p,\alpha}$ is comparable to*

$$\sum_{k=1}^{n-1} |f^{(k)}(0)| + \|f^{(n)}(z)(1-|z|^2)^n\|_{p,\alpha}.$$

For the case $1 \le p < \infty$, a proof is given in [HKZ, pp. 12–13]. When $0 < p < 1$, the unweighted case ($\alpha = 0$) was proved by J. Shi in [S, Theorem 3] (in fact, Shi's proof was given for the unit ball of $\mathbf{C}^n$). The proof of the weighted case is similar to that in [S]. We sketch the proof here for completion.

Denote by $T_n f(z) = f^{(n)}(z)(1-|z|^2)^n$ and

$$M_p^p(r,f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\,d\theta.$$

We need the following lemma.

Lemma 4. *Let $f \in H(D)$ and $0 < p < \infty$. Then, for any integer $n > 0$,*

(i) *if $T_n f \in L^{p,\alpha}$ then $\int_0^1 M_p^p(r, T_n f)\,dr \le K\|T_n f\|_{p,\alpha}^p$;*

(ii) *if $\int_0^1 M_p^p(r, T_n f)(1-r^2)^\alpha\,dr < \infty$ then $T_n f \in L^{p,\alpha}$ and*

$$\|T_n f\|_{p,\alpha}^p \le K\int_0^1 M_p^p(r, T_n f)(1-r^2)^\alpha\,dr.$$

The proof is the same as the proof of Lemma 9 in [S], and so is omitted here.

Proof of Proposition 1. Let $T_n f(z) = f^{(n)}(z)(1-|z|^2)^n$. Let $f \in L_a^{p,\alpha}$. Then by [S, Theorem 1] and Lemma 4,

$$\begin{aligned}\|T_n f\|_{p,\alpha}^p &\le K\int_0^1 M_p^p(r, T_n f)(1-r^2)^\alpha\,dr = K\int_0^1 M_p^p(r, f^{(n)})(1-r^2)^{np+\alpha}\,dr\\ &\le K\int_0^1 M_p^p(r, f)(1-r^2)^\alpha\,dr \le K\|f\|_{p,\alpha}^p.\end{aligned}$$

This proved that $T_n f \in L^{p,\alpha}$ and $\|T_n f\|_{p,\alpha} \le K\|f\|_{p,\alpha}$. On the other hand, by Proposition 1.1 in [HKZ, p. 2], we see that

$$|f^{(n)}(0)| \le K\|f\|_{p,\alpha}.$$

Thus

$$\sum_{k=1}^{n-1} |f^{(k)}(0)| + \|T_n f\|_{p,\alpha} \le K\|f\|_{p,\alpha}.$$

Conversely, let $T_n f \in L^{p,\alpha}$. Then by [S, Theorem 2] and Lemma 4, we get

$$\begin{aligned}\|f\|_{p,\alpha}^p &\le K\int_0^1 M_p^p(r, T_n f)(1-r^2)^\alpha\,dr\\ &\le K\bigg(\sum_{k=1}^{n-1} |f^{(n)}(0)|^p + \int_0^1 M_p^p(r, T_n f)(1-r^2)^\alpha\,dr\bigg)\\ &\le K\bigg(\sum_{k=1}^{n-1} |f^{(n)}(0)|^p + \|T_n f\|_{p,\alpha}^p\bigg),\end{aligned}$$

which implies that

$$\|f\|_{p,\alpha} \le K\bigg(\sum_{k=1}^{n-1} |f^{(k)}(0)| + \|T_n f\|_{p,\alpha}\bigg).$$

The proof is complete. □

Proposition 2. *Let* $f \in H(D)$. *Let* $0 < p < \infty$, $-2 < q < \infty$ *and* $n \in \mathbf{N}$. *Then* $f \in F(p,q,1)$ *if and only if*

$$\sup_{a\in D}\int_D |f^{(n)}(z)|^p(1-|z|^2)^{(n-1)p+q}\big(1-|\varphi_a(z)|^2\big)\,dA(z) < \infty.$$

Remark. Since $\mathrm{BMOA}_p^\alpha = F(p, p\alpha-2, 1)$, Proposition 2 says that, for $0 < p < \infty$ and $0 < \alpha < \infty$, $f \in \mathrm{BMOA}_p^\alpha$ if and only if

$$\sup_{a\in D}\int_D |f^{(n)}(z)|^p(1-|z|^2)^{(n-1+\alpha)p-2}\big(1-|\varphi_a(z)|^2\big)\,dA(z) < \infty.$$

Using Proposition 1, the proof of Proposition 2 is exactly the same as the proof of Theorem 4.2.1 in [R], and so is omitted here. Note that, however, the proof cannot go through for the general space $F(p,q,s)$ when $0 < s < 1$ and $0 < p < 1$, even with Proposition 1.

Proof of Theorem 2. We will prove (i), (ii), (iii) and (v) at the same time, by using Theorem A. The proof is similar to the proof of Theorem 1. Let $g \in M(H^p, L_a^{q,\beta})$. This means, for any $f \in H^p$,

$$\int_D |f(z)g(z)|^q \, dA_\beta(z) \le C\|f\|_{H^p}^q. \tag{7}$$

Let $d\mu_g(z) = |g(z)|^q \, dA_\beta(z)$. Then (7) says that μ_g is an (H^p, q)-Carleson measure. If $0 < p \le q < \infty$, by Theorem A, this is equivalent to the fact that

$$\sup_{a\in D} \int_D |\varphi_a'(z)|^{q/p} \, d\mu_g(z) < \infty,$$

which is the same as

$$\sup_{a\in D} \int_D |g(z)|^q (1-|z|^2)^{\beta-q/p} \bigl(1-|\varphi_a(z)|^2\bigr)^{q/p} \, dA(z) < \infty. \tag{8}$$

If $q > p$ then $q/p > 1$. Let G be an antiderivative of g. By Theorem 1 of [Z1], if $(\beta+2)/q - 1/p > 0$, then $\beta - q/p > -2$ and so (8) means $G \in B^{(\beta-q/p+2)/q} = B^{(\beta+2)/q-1/p}$, which is equivalent to the fact that $g = G' \in B^{1+(\beta+2)/q-1/p}$. Thus (i) is proved.

If $(\beta+2)/q - 1/p = 0$ then $\beta - q/p = -2$. By Lemma 1, (8) is equivalent to that $g \in H^\infty$, which proves (ii).

If $(\beta+2)/q - 1/p < 0$, then $\beta - q/p < -2$, by Lemma 3, (8) implies $g = 0$, which proves (iii).

If $q = p$, then (8) is the same as

$$\sup_{a\in D} \int_D |g(z)|^p (1-|z|^2)^{\beta-1} \bigl(1-|\varphi_a(z)|^2\bigr) \, dA(z) < \infty. \tag{9}$$

Applying Proposition 2 to the antiderivative G of g with $n = 2$ and $q = \beta - 1 > -2$, we see that (9) is equivalent to

$$\sup_{a\in D} \int_D |g'(z)|^p (1-|z|^2)^{p+\beta-1} \bigl(1-|\varphi_a(z)|^2\bigr) \, dA(z) < \infty.$$

Thus, $g \in F(p, p+\beta-1, 1) = F\bigl(p, p\bigl(1+(\beta+1)/p\bigr) - 2, 1\bigr) = \mathrm{BMOA}_p^{1+(\beta+1)/p}$. This proves (v).

For proving (iv), we use Theorem B. By Theorem B, the fact that μ_g is an (H^p, q)-Carleson measure is equivalent to that the function

$$\theta \to \int_{\Gamma(\theta)} \frac{d\mu_g(z)}{1-|z|^2}$$

belongs to $L^{p/(p-q)}$, where $\Gamma(\theta)$ is the Stolz angle at θ, and $d\mu_g$ is given above. Thus

$$\int_0^{2\pi}\left(\int_{\Gamma(\theta)}\frac{d\mu_g(z)}{1-|z|^2}\right)^{p/(p-q)}d\theta<\infty,$$

or

$$\int_0^{2\pi}\left(\int_{\Gamma(\theta)}\frac{|g(z)|^q\,dA_\beta(z)}{1-|z|^2}\right)^{p/(p-q)}d\theta<\infty,$$

which means $g\in T_s^q(dA_\beta)$, where $1/s=1/q-1/p$. Thus (iv) holds and the proof is completed. □

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