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PATHWISE CONNECTIVITY IN UNIFORM DOMAINS WITH SMALL EXCEPTIONAL SETS

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Abstract. Let D be a uniform domain of the euclidean space $\mathbf{R}^n$, $n \geq 2$, and suppose that $E \subset D$ is small compared to D. We establish a simple inequality in terms of Hausdorff content to measure the size of the exceptional set of pathwise connectivity in $D \setminus E$. Similar inequalities are given in the special cases where D is a bounded convex domain of $\mathbf{R}^n$, $n \geq 2$, or the boundary of the unit ball $\mathbf{B}^n$, $n \geq 3$. Also, a problem on the connectivity properties of a conformal boundary, where such an inequality has applications, is briefly described.

1. Introduction

It is a general problem to study what kind of sets can be thrown out of a domain so that the connectivity of it is not essentially destroyed. It is well known that a set of topological dimension less than $n-1$ does not separate a domain of $\mathbf{R}^n$, $n \geq 2$. Also, using the Hausdorff measure H^α we can say that a set A does not destroy the simple connectivity of a domain in $\mathbf{R}^n$ if $H^\alpha(A) = 0$ for some $0 \leq \alpha \leq n-2$.

However, the Hausdorff measure is not always the most useful tool in applications. One reason for this is the fact that Hausdorff measure is sensitive to the topological dimension of the set to be measured. Indeed, $H^\alpha(A)$ is not finite if the dimension of A is more than α. This also holds in the cases where A is a compact subset of $\mathbf{R}^n$.

On the other hand, when measuring sets we do not necessarily need to use all the properties of the inner measure but we are satisfied with those of the outer measure. Then the Hausdorff measure can be replaced with the *Hausdorff content* H_α^∞ (*of a set* A), by which we mean the number

$$H_\alpha^\infty(A) = \inf\biggl\{\sum_{k=1}^\infty r_k^\alpha : A \subset \bigcup_{k=1}^\infty B(x_k, r_k)\biggr\}.$$

Here it is convenient to take $0 < \alpha \leq n$ for sets in $\mathbf{R}^n$. To verify that H_α^∞ is an outer measure in $\mathbf{R}^n$, cf. [14].

Trivially, for any bounded $A \subset \mathbf{R}^n$, we have $H_\alpha^\infty(A) < \infty$. Especially for a ball $B(x,r)$ it holds that $H_\alpha^\infty\bigl(B(x,r)\bigr) = r^\alpha$.

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Before stating our results we record a few definitions. Assume from now on that $n \geq 2$ if not stated otherwise.

Let $\mathbf{B}^n$ be the unit ball of $\mathbf{R}^n$. Following Alestalo's definition in [1], a domain $D \subset \mathbf{R}^n$ is (p,C)*-uniform*, where $C \geq 1$ is constant and $0 \leq p < n$ is an integer, if every continuous function $f\colon \partial\mathbf{B}^{p+1} \to D$ has a continuous extension $g\colon \mathbf{B}^{p+1} \to D$ such that

$$\operatorname{diam}\bigl(g(\mathbf{B}^{p+1})\bigr) \leq C\operatorname{diam}\bigl(f(\partial\mathbf{B}^{p+1})\bigr)$$

and

$$d\bigl(y, f(\partial\mathbf{B}^{p+1})\bigr) \leq Cd(y,\partial D) \qquad \text{for every } y \in g(\mathbf{B}^{p+1}).$$

The first inequality above is called the *turning condition* while the second one is the *lens condition*, and together they form the *uniformity conditions.* $(0,C)$-uniformity can be considered as a quantitative version of path-connectedness, and similarly, (p,C)-uniformity of a domain D is a quantitative version of the property $\pi_p(D) = 0$ in algebraic topology; cf. [1].

Theorem A. *Let $0 < \alpha \leq 1$, and let $D \subset \mathbf{R}^n$ be $(0,C)$-uniform. Assume that $E \subset D$ is such that $H_\alpha^\infty(E) \leq \delta$, where $\delta < \operatorname{diam}(D)/5C$ if D is bounded and otherwise a fixed number. Then there is a set F such that*

$$H_\alpha^\infty(F) \leq (C^\alpha + 2)\delta,$$

and for every $x,y \in D\backslash F$, there exists a path γ in $D\backslash E$ joining x and y such that

$$\operatorname{diam}(\gamma) \leq \inf_{\gamma_{x,y}} \operatorname{diam}(\gamma_{x,y}) + 5\delta,$$

where the infimum is taken over all paths $\gamma_{x,y}$ joining x and y in D.

If D is a bounded convex domain, we can give another upper bound for the size of the bad set of the pathwise connectivity that also depends on the shape of D.

To that end, let $D \subset \mathbf{R}^n$ be a bounded convex domain. Define for every $x \in D$,

$$R(D,x) = \frac{\inf_{y\in\mathbf{R}^n\backslash D}|x-y|}{\sup_{y\in D}|x-y|}$$

and

$$R(D) = \sup_{x\in D} R(D,x).$$

Clearly, $0 \leq R(D) \leq 1$, and the number $R(D)$ tells us how roundish D is. For example, $R(B) = 1$ for any ball B, and for $D = \operatorname{Cone}(x,y,\lambda,\varepsilon)$, we have $R(D) = \lambda$ for every $0 < \varepsilon \leq 1$. Here $\operatorname{Cone}(x,y,\lambda,\varepsilon)$ is a finite cone about $[x,y] = \{tx + (1-t)y : 0 \leq t \leq 1\}$ with vertex at x, or more precisely, the union of balls

$$\bigcup\bigl\{B(tx + (1-t)y, (1-t)\lambda|x-y|) : 1-\varepsilon \leq t < 1\bigr\},$$

where $\varepsilon, \lambda \in (0,1]$ are fixed.

Theorem B. *Let $0 < \alpha \le 1$, and $D \subset \mathbf{R}^n$ be bounded and convex with $R(D) = \rho > 0$. Assume that $E \subset D$ is such that*

$$H_\alpha^\infty(E) \le \delta < \frac{\rho}{10}\operatorname{diam}(D).$$

Then there is a set F such that, for every $x, y \in D\backslash F$, there exists a path γ in $D\backslash E$ joining x and y so that

$$\operatorname{length}(\gamma) \le |x-y| + 5\delta.$$

Moreover,

$$H_\alpha^\infty(F) \le \left(2 + \max\left\{\sqrt{\left(\frac{\rho^2+1}{2\rho^2}\right)^\alpha}, \sqrt{2^\alpha}\right\}\right)\delta. \tag{1.1}$$

For $n \ge 3$, Theorem B can be modified so that $D = \partial\mathbf{B}^n$, if we reset $[x, y]$ to be a geodesic arc of $\partial\mathbf{B}^n$ joining x and y. Then we have the following conclusions.

Theorem C. *Let $0 < \alpha \le 1$, and assume that $E \subset \partial\mathbf{B}^n$ is such that $H_\alpha^\infty(E) \le \delta$, where δ is small. Then there exists a set F such that $H_\alpha^\infty(F) \le 2\delta$ and, for every $x, y \in \partial\mathbf{B}^n\backslash F$, there is a path γ in $\partial\mathbf{B}^n\backslash E$ joining them so that*

$$\operatorname{length}(\gamma) \le \operatorname{length}([x, y]) + 5\delta.$$

Notice that in all theorems we restrict $\alpha \le 1$. This is due to the fact that it is easy to separate, for example, a domain in $\mathbf{R}^2$ with $\bigcup_j B(x_j, r_j)$ so that $\sum_j r_j^\alpha < \delta$ for any fixed $\alpha > 1$ and $\delta > 0$. If the dimension n of the domain D is greater than 2, a more natural upper bound for α is likely to be $n-1$. However, for simplicity, we leave this question for later considerations.

Another interesting question that so far remains somewhat open is how the deletion of a small set E of a (p, C)-uniform domain D, where $p > 0$, affects the higher uniformity of $D\backslash E$. It is easy to see that a set E, for which $H_\alpha^\infty(E) < \varepsilon$ for any fixed $0 < \alpha \le 1$ and $\varepsilon > 0$, may destroy the $(n-1, C)$-uniformity of $D\backslash E$, where $D \subset \mathbf{R}^n$ is such a domain. On the other hand, there are no obvious reasons why $D\backslash E$ could not be (p', C')-uniform for some $0 < p' < p$ whenever E is small enough. However, we seem to lack the appropriate tools to verify this. For example, consider how to give explicitly the cone construction of the $(1, C)$-uniformity of $\mathbf{R}^n\backslash E$, where $n \ge 3$ and $H_\alpha^\infty(E) \le \varepsilon$ for given $0 < \alpha \le 1$ and $\varepsilon > 0$.

The idea to apply the Hausdorff measure or content to the theory of uniform or convex sets is, of course, not new. Recently the Hausdorff content has appeared in several areas of mathematical analysis; cf. [2], [4], [10] and [11], but our results seem to be new in this kind of a setting. Only a version of Theorem C was originally proved in [13]. In the last section, we shall briefly describe an application of this kind of inequalities in the theory of conformal boundaries.

2. The proof of Theorem A

In order to prove Theorem A, fix $0 < \varepsilon < 1$, and notice that E can be covered with countably many open balls $B(x_j, r_j)$ such that

$$\sum_{j=1}^{\infty} r_j^{\alpha} < (1+\varepsilon)\delta.$$

We shall prove the theorem by showing that the H_α^∞-measure of the complement of the largest path component of $D \setminus \cup B(x_j, r_j)$ does not exceed $(C^\alpha + 2)\delta$.

Replace the open balls of the cover of E with corresponding closed balls and consider the complementary path components of the union of the first k closed balls. To simplify our language, we say that every path component of $D \setminus \bigcup_{j=1}^{k} B(x_j, r_j)$ except the largest (with respect to, for example, H_α^∞) of them is *blocked*.

If a path component is blocked, it is already blocked, i.e. separated from the largest path component, by a component of the union of these balls and perhaps a subset of ∂D. This component can be taken to be a compact and connected set, i.e. a continuum. Label the continua arising this way by $E_{j,k}$. Observe that a single continuum can block several path components. Let $F_{j,k} \subset D$ be the union of the path components that are blocked by $E_{j,k}$.

We consider two cases. Assume first that $\bigcup_j E_{j,k} \cap \partial D = \emptyset$ with every k. In this case we set

$$F = \Big(\bigcup_{k=1}^{\infty} \bigcup_{j=1}^{j(k)} F_{j,k} \Big) \cup \Big(\bigcup_{j=1}^{\infty} B(x_j, r_j) \Big).$$

Then $D \setminus F$ is path connected.

By the classical result of Jung [7], every $F_{j,k}$ can be covered by a ball with radius $\operatorname{diam}(E_{j,k})/\sqrt{2}$. Now, by elementary geometry,

$$H_\alpha^\infty(F_{j,k}) \le \left(\frac{\operatorname{diam}(E_{j,k})}{\sqrt{2}} \right)^{\alpha} \le \left(\frac{\sum_{i \in I_{j,k}} r_i}{\sqrt{2}} \right)^{\alpha} \le \frac{\sum_{i \in I_{j,k}} r_i^{\alpha}}{2^{\alpha/2}},$$

where $I_{j,k}$ is the set of those indices i such that $B(x_i, r_i)$ is in $E_{j,k}$. Furthermore, for every k,

$$H_\alpha^\infty \Big(\bigcup_{j=1}^{j(k)} F_{j,k} \Big) \le \sum_{j=1}^{j(k)} \left(\frac{\operatorname{diam}(E_{j,k})}{\sqrt{2}} \right)^{\alpha} \le \frac{\sum_{j=1}^{k} r_j^{\alpha}}{2^{\alpha/2}} < \frac{(1+\varepsilon)\delta}{2^{\alpha/2}}.$$

For all k and $l \ge 0$, we also have

$$\Big(\bigcup_{j=1}^{j(k)} F_{j,k} \Big) \cup \Big(\bigcup_{j=1}^{k} B(x_j, r_j) \Big) \subset \Big(\bigcup_{j=1}^{j(k+l)} F_{j,k+l} \Big) \cup \Big(\bigcup_{j=1}^{k+l} B(x_j, r_j) \Big).$$

Thus, by Theorem 47 in [12],

$$H_\alpha^\infty(F) = \lim_{k\to\infty} H_\alpha^\infty\Big(\bigcup_{j=1}^{j(k)} F_{j,k} \cup \bigcup_{j=1}^{k} B(x_j,r_j)\Big) \le \frac{(1+\varepsilon)\delta}{2^{\alpha/2}} + (1+\varepsilon)\delta. \tag{2.1}$$

Consider next the case $\bigcup_j E_{j,k} \cap \partial D \neq \emptyset$. Then, by the lens condition of $(0,C)$-uniformity of the domain D, a single ball $B(x_j,r_j)$ together with ∂D can block only a subset F_j of D that can be covered with a ball of radius Cr_j. So, in the worst case, every ball $B(x_j,r_j)$ blocks with ∂D a path component F_j for which

$$H_\alpha^\infty\Big(\bigcup_j F_j\Big) \le \sum_j (Cr_j)^\alpha < C^\alpha(1+\varepsilon)\delta. \tag{2.2}$$

Observe also that in this case the upper bound for δ guarantees that the largest path component of $D\backslash\bigcup_j B(x_j,r_j)$ exists uniquely.

Define now

$$F = \Big(\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{j(k)} F_{j,k}\Big) \cup \Big(\bigcup_{j=1}^{\infty} B(x_j,r_j)\Big) \cup \Big(\bigcup_j F_j\Big).$$

Then $D\backslash F$ is path connected and we have, by (2.1) and (2.2),

$$H_\alpha^\infty(F) \le (C^\alpha + 2)\delta$$

whenever $\varepsilon > 0$ is small enough.

In order to verify the last assertion of the theorem, let γ_0 be any bounded curve that joins x and y in D. Let $d_0 = \operatorname{diam}(\gamma_0)$. If, for every k, we have $\gamma_0 \cap \big(\bigcup_j E_{j,k}\big) = \emptyset$, we are done. Suppose this is not the case. Then, for any k, we may pass by each $E_{j,k}$ on γ_0 by going around along the boundaries of $E_{j,k}$. This gives us a new curve γ_k with

$$\operatorname{diam}(\gamma_k) \le d_0 + \sum_{i=1}^{k} \pi r_i \le d_0 + \pi(1+\varepsilon)\delta.$$

Since $\operatorname{diam}(\gamma_k)$ is bounded for every k, there exists a subsequence of (γ_k) that converges to a curve γ, which joins x and y so that $\gamma \cap E = \emptyset$ and

$$\operatorname{diam}(\gamma) \le d_0 + 5\delta.$$

This completes the proof of Theorem A.

3. The proofs of Theorems B and C

The setting in Theorem B is quite similar to the one in Theorem A, so we use here the same notation as in the previous section as far as possible. Moreover, the case $\bigcup_j E_{j,k} \cap \partial D = \emptyset$ with every k can be handled by repeating the argument of the respective case in the proof of Theorem A. Therefore, assume that $\bigcup_j E_{j,k} \cap \partial D \neq \emptyset$.

By the convexity of D and the upper bound for δ, for every $k \geq 1$, the worst that can happen is that $\bigcup_{j=1}^k B(x_j, r_j)$ with ∂D blocks a union of cones (or subsets of them). Therefore, fix any $k \geq 1$ and assume that $F_i \subset \mathrm{Cone}(x_i, y_i, \lambda_i, \varepsilon_i)$ is blocked by $E_i = \bigcup_{j \in I_i} B(x_j, r_j) \cup \partial D$. Then the convexity of D implies $\lambda_i \geq \rho$, and hence if $H_\alpha^\infty\bigl(\bigcup_{j \in I_i} B(x_j, r_j)\bigr) = r_i^\alpha$, it follows, by elementary geometry, that

$$\mathrm{diam}(F_i) \leq \max\left\{\sqrt{\left(\frac{r_i}{\rho}\right)^2 + r_i^2}\,, 2r_i\right\}. \tag{3.1}$$

Again, by the result of Jung, the set F_i can be covered by a ball with radius $\mathrm{diam}(F_i)/\sqrt{2}$. It follows from (3.1) that, for every i,

$$\begin{aligned} H_\alpha^\infty(F_i) &\leq \left(\max\left\{\sqrt{\left(\frac{\rho^2+1}{2\rho^2}\right)}, \sqrt{2}\right\} r_i\right)^\alpha \\ &= \max\left\{\sqrt{\left(\frac{\rho^2+1}{2\rho^2}\right)^\alpha}, \sqrt{2^\alpha}\right\} H_\alpha^\infty\left(\bigcup_{j\in I_i} B(x_j, r_j)\right). \end{aligned}$$

Letting $k \to \infty$, this implies, as in the proof of Theorem A, that

$$\begin{aligned} H_\alpha^\infty\left(\bigcup_i F_i\right) &\leq \sum_i \max\left\{\sqrt{\left(\frac{\rho^2+1}{2\rho^2}\right)^\alpha}, \sqrt{2^\alpha}\right\} H_\alpha^\infty\left(\bigcup_{j\in I_i} B(x_j, r_j)\right) \\ &< \max\left\{\sqrt{\left(\frac{\rho^2+1}{2\rho^2}\right)^\alpha}, \sqrt{2^\alpha}\right\}(1+\varepsilon)\delta. \end{aligned} \tag{3.2}$$

Set now

$$F = \left(\bigcup_{k=1}^\infty \bigcup_{j=1}^{j(k)} F_{j,k}\right) \cup \left(\bigcup_{j=1}^\infty B(x_j, r_j)\right) \cup \left(\bigcup_i F_i\right).$$

Then $D \backslash F$ is path connected and we have, by (3.2) and (2.1), as in the proof of Theorem A,

$$H_\alpha^\infty(F) \leq \left(2 + \max\left\{\sqrt{\left(\frac{\rho^2+1}{2\rho^2}\right)^\alpha}, \sqrt{2^\alpha}\right\}\right)\delta$$

whenever $\varepsilon > 0$ is small enough.

To complete the proof of Theorem B, for any $x, y \in D \backslash F$, we choose $\gamma_0 = [x, y]$ and then repeat the argument of the last part of the proof of Theorem A to show the existence of the suitable path γ between x and y.

To verify Theorem C, it is enough to notice that if $D = \partial \mathbf{B}^n$ for $n \geq 3$, and $[x, y]$ is a geodesic arc of $\partial \mathbf{B}^n$ joining x and y, then the argument of the first case of the proof of Theorem A and the same argument as used in the last part of the proof of Theorem B still apply.

4. An application

We believe that Theorems A–C are of some interest on their own but, perhaps, they serve better as applicative tools together with, say, the Besicovitch covering theorem. As a matter of fact, the last theorem, or more precisely, a version of it, has already been applied to the theory of conformal boundaries in this way.

In recent years, the theory of quasiconformal mappings of subdomains of $\mathbf{R}^n$, $n \geq 2$, has been generalized in many ways. Following [2], we consider a quasiconformal map $f \colon \mathbf{B}^n \to \mathbf{R}^n$, $n \geq 3$, by interpreting the average of the Jacobian of f as a strictly positive continuous density ρ in $\mathbf{B}^n$. In [2] it is shown that a remarkable part of the properties of f follows from two simple geometric conditions. There are also other classes of functions that can be studied similarly by using suitable densities that satisfy the same conditions.

Let ρ be a suitable density. Then it induces a metric d_ρ in $\mathbf{B}^n$. If ρ arises from the quasiconformal mapping f, then $d_\rho(x, y)$ is the internal euclidean distance of $f(x)$ and $f(y)$ in $f(\mathbf{B}^n)$.

The rho-boundary of the unit ball, $\partial_\rho \mathbf{B}^n$ is defined as $\overline{(\mathbf{B}^n, d_\rho)} \backslash (\mathbf{B}^n, d_\rho)$, where we take the abstract completion of the metric space $(\mathbf{B}^n, d_\rho)$. The metric d_ρ extends in a natural way also to this boundary. Then $\partial_\rho \mathbf{B}^n$ is a metric space and it can be characterized as the set of those $\varsigma \in \partial \mathbf{B}^n$ for which $d_\rho(0, \varsigma) < \infty$.

Let $n \geq 3$. One can show that there exists a finite constant $M > 0$ depending on $0 < \alpha \leq 1$ and $\delta > 0$, such that the H_α^∞-measure of the points $\varsigma \in \partial_\rho \mathbf{B}^n$, for which $d_\rho(0, \varsigma) > M$, is less than δ. There are also some other useful properties, for which it is possible to find a set E such that the properties hold on $\partial_\rho \mathbf{B}^n \backslash E$ and $H_\alpha^\infty(E) < \delta$ for any fixed $\delta > 0$. One of the most important of these is that the identity function

$$\text{id} \colon (\partial_\rho \mathbf{B}^n \backslash E, |\cdot|) \to (\partial_\rho \mathbf{B}^n \backslash E, d_\rho) \tag{4.1}$$

is Hölder continuous.

Theorem C is now very useful, for example, in proving the d_ρ-pathwise connectivity of $\partial_\rho \mathbf{B}^n \backslash F$, where $H_\alpha^\infty(F) = 0$ and $0 < \alpha \leq 1$, in the following way. Write $\delta_j = 1/j$. Next, for $j = 10, 11, 12, \ldots$, find the exceptional set E_j of both the euclidean pathwise connectivity and the identity function of (4.1), for which $H_\alpha^\infty(E_j) < \delta_j$. For every j, use Theorem C to get the corresponding set F_j.

Finally, let $F = \bigcap_j F_j$. Now it is easy to see that every x and y outside F can be joined with a curve along which the identity function is continuous. For the details, see [13] and [2].

Very recently, it was shown in [9] that the whole $\partial_\rho \mathbf{B}^n$ is pathwise connected in the sense of d_ρ. The proof of this result relies on different arguments than the use of Theorem C.

However, by the Hölder continuity of the identity function in (4.1) in the major part of $\partial_\rho \mathbf{B}^n$ and the arguments used in [9], there are reasons to believe that when $n \geq 4$, the stronger connectivity properties with respect to d_ρ might hold on the whole $\partial_\rho \mathbf{B}^n$ or at least on the major part of it. As mentioned in the end of Introduction, some technical problems have prevented us from verifying this so far. If one could find out whether the results of this paper can be extended to cover also the uniformity of higher order, we would also be closer to the answer to the question of the stronger connectivity properties of $\partial_\rho \mathbf{B}^n$.

References

[1] Alestalo, P.: Uniform domains of higher order. - Ann. Acad. Sci. Fenn. Math. Diss. 94, 1994.

[2] Bonk, M., P. Koskela, and S. Rohde: Conformal metrics on the unit ball in euclidean space. - Proc. London Math. Soc. 3, 77, 1998, 635–664.

[3] Evans, L. C., and R. F. Gariepy: Measure Theory and Fine Properties of Functions. - CRC Press, 1992.

[4] O'Farrell, A. G.: Continuity properties of Hausdorff content. - J. London Math. Soc. 13, 1976, 403–410.

[5] Federer, H.: Geometric Measure Theory. - Springer-Verlag, 1969.

[6] Heinonen, J., and P. Koskela: Quasiconformal maps in metric spaces with controlled geometry. - Acta Math. 181, 1998, 1–61.

[7] Jung, H. W. E.: Über die kleinste Kugel, die eine räumliche Figur einschliesst. - J. Reine Angew. Math. 123, 1901, 241–257.

[8] Koskela, P., and S. Rohde: Hausdorff dimension and mean porosity. - Math. Ann. 4, 309, 1998, 593–609.

[9] Koskela, P., and T. Tossavainen: Pathwise connectivity of a conformal boundary. - Bull. London Math. Soc. 35, 2003, 645–650.

[10] Mattila, P., and J. Orobitg: On some properties of Hausdorff content related to instability. - Ann. Acad. Sci. Fenn. Ser. A I Math. 19, 1994, 393–398.

[11] Orobitg, J., and J. Verdera: Choquet integrals, Hausdorff content and the Hardy–Littlewood maximal operator. - Bull. London Math. Soc. 30, 1998, 145–150.

[12] Rogers, C. A.: Hausdorff Measures. - Cambridge University Press, 1970.

[13] Tossavainen, T.: On the connectivity properties of the ρ-boundary of the unit ball. - Ann. Acad. Sci. Fenn. Math. Diss. 123, 2000.

[14] Väisälä, J.: Lectures on n-dimensional Quasiconformal Mappings. - Springer-Verlag, 1971.

[15] Väisälä, J.: A survey of nearisometries. - In: Papers on Analysis: A Volume Dedicated to Olli Martio on the Occasion of his 60th Birthday, Univ. of Jyväskylä, 2001, 305–315.

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