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HAMILTON SEQUENCES AND EXTREMALITY FOR CERTAIN TEICHMÜLLER MAPPINGS

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Abstract. Suppose f is a Teichmüller mapping with complex dilatation

$$\mu(z) = k\frac{\overline{\varphi(z)}}{|\varphi(z)|},$$

where $\varphi(z)$ is holomorphic in the unit disk. If

$$m(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} |\varphi(re^{i\theta})|\,d\theta \asymp \log^s\left(\frac{1}{1-r}\right)/(1-r), \quad \text{as } r \to 1,$$

for some given $s > 0$, then the putative sequence $\varphi(Rz)/\|\varphi(Rz)\|$, $R \uparrow 1$ is a Hamilton sequence of μ and hence f is extremal.

1. Introduction

Let Δ be the unit disk $\{|z| < 1\}$ in the complex plane $\mathbf{C}$. Suppose f is a quasiconformal self-mapping of Δ. We denote by $Q(f)$ the class of all quasiconformal self-mappings of Δ which agree with f on the boundary $\partial\Delta$. A quasiconformal mapping f is said to be extremal in $Q(f)$ if it minimizes the maximal dilatations of $Q(f)$, i.e.

$$K[f] = \inf\{K[g] : g \in Q(f)\},$$

where $K[g]$ is the maximal dilatation of g. The mapping f is uniquely extremal if it is extremal and if there are no other extremal mappings for its boundary values.

Let $B(\Delta) = \{\phi(z)$ holomorphic on $\Delta : \|\phi\| = \iint_\Delta |\phi(z)|\,dx\,dy < \infty\}$. A necessary and sufficient condition that f is extremal in $Q(f)$ is that [8] its Beltrami differential μ has a so-called Hamilton sequence, namely, a sequence

$$\{\phi_n(z) \in B(\Delta) : \|\phi_n\| = 1,\ n \in \mathbf{N}\},$$

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such that

$$\lim_{n\to\infty} \iint\limits_{\Delta} \mu(z)\phi_n(z)\,dx\,dy = \|\mu\|_\infty. \tag{1.1}$$

In this paper, unless otherwise specified, a Teichmüller mapping f is said to be a quasiconformal mapping of the unit disk onto itself, which has the complex dilatation

$$\mu(z) = \frac{f_{\bar z}}{f_z} = k\frac{\overline{\varphi(z)}}{|\varphi(z)|}, \qquad z \in \Delta, \tag{1.2}$$

where $\varphi \not\equiv 0$ is holomorphic on Δ and $k \in [0,1)$ is a constant. It is of interest to know whether f is extremal or, in particular, uniquely extremal among $Q(f)$.

From now on, we call a holomorphic function φ in Δ satisfying the condition of a global Hamilton sequence or call φ GHS if the putative sequence $\{\phi_R(z) = \varphi(Rz)/\|\varphi(Rz)\|,\ R \uparrow 1\}$ is a Hamilton sequence of f; in other words,

$$\lim_{R\to 1} \iint_\Delta \frac{\overline{\varphi(z)}}{|\varphi(z)|}|\phi_R(z)|\,dx\,dy = 1. \tag{1.3}$$

In some papers such as [5], [7] and [8], the following possibility is investigated: If $\{R_n\}$ is a sequence of numbers, $R_n \in (0,1)$, $\lim_{n\to\infty} R_n = 1$, does $\{\varphi(R_n z)/\|\varphi(R_n z)\|\}$ constitute a Hamilton sequence?

In 1974, Reich and Strebel proved

Theorem A ([8]). *Suppose $\varphi(z)$ is holomorphic on Δ and satisfies the growth condition*

$$m(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} |\varphi(re^{i\theta})|\,d\theta = O\Big(\frac{1}{1-r}\Big), \qquad r \to 1. \tag{1.4}$$

Then φ is GHS and hence f is extremal. Moreover, the extremality of f is no longer implied if $O\big((1-r)^{-1}\big)$ is replaced by $O\big((1-r)^{-s}\big)$ for any $s > 1$.

Hayman and Reich [4] proved that f is also uniquely extremal if $\varphi(z)$ satisfies the growth condition (1.4).

Sethares solved the unique extremality of Teichmüller mappings for certain holomorphic functions φ:

Theorem B ([9]). *Suppose $\varphi(z)$ is holomorphic on Δ and meromorphic in $\bar\Delta$. Then f is uniquely extremal if and only if all poles of $\varphi(z)$ are of order not exceeding two.*

Furthermore, in 1988, Reich considered the relation between the unique extremality and the construction of the Hamilton sequence for certain Teichmüller mappings.

Theorem C ([7]). *Suppose $\varphi(z)$ is holomorphic on Δ and meromorphic in $\bar{\Delta}$. Then the corresponding Teichmüller mapping f is uniquely extremal if and only if φ is GHS.*

But, for more general φ, we do not know too much about such a relation. It is interesting for us to consider

Problem 1. *If φ is GHS, is the corresponding Teichmüller mapping f uniquely extremal in $Q(f)$? Conversely, if a Teichmüller mapping f is uniquely extremal in $Q(f)$, is the corresponding φ GHS?*

Problem 2. *When is φ GHS?*

To answer these problems, we have the following two theorems as our main results:

Theorem 1. *Given $s > 0$. Suppose $\varphi(z)$ is holomorphic on Δ and satisfies the growth condition*

$$m(r,\varphi) \asymp \left(\frac{1}{1-r}\log^s\frac{1}{1-r}\right), \qquad r \to 1. \tag{1.5}$$

Then φ is GHS.

As an application of the result in [6] (or see Theorem D in Section 3) and Theorem 1, we answer the first part of Problem 1 negatively. But we notice that the second part of Problem 1 is still open.

Theorem 2. *For any given real number $s > 1$, there exists a holomorphic function $\varphi(z)$ in Δ satisfying the growth condition (1.5), and hence φ is GHS while the corresponding Teichmüller mapping f is extremal instead of being uniquely extremal.*

2. Some lemmas

In order to prove our main results, we need two lemmas. Such results are related to the theories of H^p spaces and of univalent functions. We refer the readers to Duren's book [1].

The following lemma is a counterpart of the results in [2] and [3] by Hardy and Littlewood.

Lemma 1. *Suppose $\varphi(z)$ is holomorphic on Δ. Given $s \in \mathbf{R}$, $\alpha > 0$ and $1 \le p < \infty$. Then for any positive integer n and with $r \to 1$, the following two conditions are equivalent*:

$$\text{(i)} \qquad m_p(r,\varphi) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |\varphi(re^{i\theta})|^p \, d\theta \right\}^{1/p} = O\left(\frac{1}{(1-r)^\alpha} \log^s \frac{1}{1-r} \right),$$

$$\text{(ii)} \; m_p(r,\varphi^{(n)}) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |\varphi^{(n)}(re^{i\theta})|^p \, d\theta \right\}^{1/p} = O\left(\frac{1}{(1-r)^{\alpha+n}} \log^s \frac{1}{1-r} \right).$$

Proof. By induction, it suffices to show that when $n = 1$, this lemma holds. Assume that (i) holds. Set $R = \frac{1}{2}(1+r)$. By the Cauchy formula

$$\varphi'(re^{i\theta}) = \frac{1}{2\pi i} \int_{|z|=R} \frac{\varphi(z)}{(z - re^{i\theta})^2} \, dz = \frac{R}{2\pi} \int_0^{2\pi} \frac{\varphi(Re^{i(t+\theta)}) e^{i(t-\theta)}}{(Re^{it} - r)^2} \, dt.$$

Minkowski's inequality then gives

$$\begin{aligned} m_p(r,\varphi') &\le \frac{1}{2\pi} \int_0^{2\pi} \frac{R m_p(R,\varphi)}{R^2 - 2Rr\cos t + r^2} \, dt \\ &= \frac{R\, m_p(R,\varphi)}{R^2 - r^2} = O\left(\frac{1}{(1-r)^{\alpha+1}} \log^s \frac{1}{1-r} \right), \quad r \to 1. \end{aligned}$$

Conversely, if (ii) holds, we apply Minkowski's inequality to the relation

$$|\varphi(re^{i\theta})| \le |\varphi(0)| + \int_0^r |\varphi'(te^{i\theta})| \, dt$$

and obtain

$$m_p(r,\varphi) \le |\varphi(0)| + \int_0^r m_p(t,\varphi') \, dt = O\left(\frac{1}{(1-r)^\alpha} \log^s \frac{1}{1-r} \right), \quad r \to 1. \ \square$$

Lemma 2. *Set $\Delta_r = \{z \in \Delta : |z| < r < 1\}$. Suppose $s \ge -1$. If φ satisfies the growth condition*

$$m(r,\varphi) \asymp \left(\frac{1}{1-r} \log^s \frac{1}{1-r} \right), \qquad r \to 1, \tag{2.1}$$

then as $r \to 1$

$$A(r,\varphi) = \iint_{\Delta_r} |\varphi(z)| \, dx\, dy = \int_0^r t\, dt \int_0^{2\pi} |\varphi(te^{i\theta})| \, d\theta$$

$$\asymp \begin{cases} \left(\log^{s+1} \dfrac{1}{1-r} \right) & s > -1, \\ \left(\log\log \dfrac{1}{1-r} \right) & s = -1, \end{cases} \tag{2.2}$$

and hence

$$\lim_{r\to 1}\frac{\log^s \dfrac{1}{1-r}}{A(r,\varphi)} = 0, \tag{2.3}$$

where we prescribe that $\log^s\big((1/(1-r)\big) = 1$ *as* $s = 0$.

Proof. First let $s > -1$. By equation (2.1), there exist $C_1 > 0$, $C_2 > 0$ and $r_0 \ge 0$ such that, when $r \ge r_0$,

$$\frac{C_2}{1-r}\log^s \frac{1}{1-r} \ge m(r,\varphi) \ge \frac{C_1}{1-r}\log^s \frac{1}{1-r}.$$

Therefore,

$$\begin{aligned} A(r,\varphi) &\ge \int_{r_0}^{r} t\,dt\, m(t,\varphi) \ge C_1\int_{r_0}^{r}\frac{t}{1-t}\log^s\frac{1}{1-t}\,dt \\ &\ge C_1 r_0\int_{r_0}^{r}\frac{1}{1-t}\log^s\frac{1}{1-t}\,dt \\ &= \frac{C_1 r_0}{1+s}\log^{s+1}\frac{1}{1-t}\,\Big|_{r_0}^{r} \\ &\asymp \left(\log^{s+1}\frac{1}{1-r}\right), \qquad r\to 1. \end{aligned}$$

Further,

$$\begin{aligned} A(r,\varphi) &= \int_0^r t\,dt\, m(t,\varphi) \le \int_0^r dt\, m(t,\varphi) \\ &\le C_2\int_0^r \frac{1}{1-t}\log^s\frac{1}{1-t}\,dt \\ &= \frac{C_2}{1+s}\log^{s+1}\frac{1}{1-t}\,\Big|_0^r \\ &\asymp \left(\log^{s+1}\frac{1}{1-r}\right), \qquad r\to 1. \end{aligned}$$

Thus, we obtain (2.2). When $s = -1$, we omit the proof, because it is similar to the above. Equation (2.3) is obvious. □

3. Proofs of the main results

The proof of Theorem 1 is somewhat similar to that of the "only if" part of Theorem C. As usual, we write

$$\frac{\iint_\Delta (\overline{\varphi(z)}/|\varphi(z)|)\varphi(Rz)\,dx\,dy}{\iint_\Delta |\varphi(Rz)|} = R^2 + R^2\frac{\alpha(R)+\beta(R)}{A(R,\varphi)}, \tag{3.1}$$

where

$$\alpha(R) = \iint\limits_{\Delta_R} \frac{\overline{\varphi(z)}}{|\varphi(z)|}\big[\varphi(Rz)-\varphi(z)\big]\,dx\,dy, \tag{3.2}$$

$$\beta(R) = \iint\limits_{U_R} \frac{\overline{\varphi(z)}}{|\varphi(z)|}\varphi(Rz)\,dx\,dy, \tag{3.3}$$

and $U_R = \{z\in\Delta : R<|z|<1\}$.

It is sufficient to show

$$\lim_{R\to 1}\frac{\alpha(R)}{A(R,\varphi)} = 0, \tag{3.4}$$

$$\lim_{R\to 1}\frac{\beta(R)}{A(R,\varphi)} = 0. \tag{3.5}$$

For one thing,

$$\begin{aligned}
|\alpha(R)| &= \left|\iint\limits_{\Delta_R} \frac{\overline{\varphi(z)}}{|\varphi(z)|}\big[\varphi(Rz)-\varphi(z)\big]\,dx\,dy\right| \\
&\le \iint\limits_{\Delta_R} |\varphi(Rz)-\varphi(z)|\,dx\,dy \\
&\le \int_0^R r\,dr\int_0^{2\pi} d\theta\int_{Rr}^{r} |\varphi'(te^{i\theta})|\,dt \\
&= 2\pi\int_0^R r\,dr\int_{Rr}^{r} m(t,\varphi')\,dt.
\end{aligned}$$

By equation (1.5) and Lemma 1, we have

$$\begin{aligned}
|\alpha(R)| &\le 2C\pi \int_0^R r\,dr \int_{Rr}^r \left(\frac{1}{(1-r)^2}\log^s\frac{1}{1-r}\right)dt \\
&\le 2C\pi \int_0^R \frac{1}{(1-r)^2}\log^s\frac{1}{1-r}\left(\int_{Rr}^r dt\right) r\,dr \\
&= 2C\pi(1-R)\int_0^R \frac{r^2}{(1-r)^2}\log^s\frac{1}{1-r}\,dr \\
&\le 2C\pi(1-R)\int_0^R \log^s\frac{1}{1-r}\,d\frac{1}{1-r} \\
&= 2C\pi(1-R)\left[\frac{1}{1-r}\log^s\frac{1}{1-r}\Big|_0^R - s\int_0^R \frac{1}{(1-r)^2}\log^{s-1}\frac{1}{1-r}\,dr\right] \\
&\le 2C\pi(1-R)\left(\frac{1}{1-r}\log^s\frac{1}{1-r}\Big|_0^R\right).
\end{aligned}$$

So we obtain

$$|\alpha(R)| \le 2C\pi \log^s\frac{1}{1-R}, \tag{3.6}$$

where C is a suitable constant.

Next, choose R sufficiently close to 1 such that $\log\big(1/(1-R)\big) > 1$,

$$\begin{aligned}
|\beta(R)| &= \left|\iint_{U_R} \frac{\overline{\varphi(z)}}{|\varphi(z)|}\varphi(Rz)\,dx\,dy\right| \\
&\le \int_0^{2\pi} d\theta \int_R^1 |\varphi(Rte^{i\theta})|t\,dt \\
&= \frac{1}{R^2}\int_0^{2\pi}\int_{R^2}^R |\varphi(ue^{i\theta})|u\,du \\
&= \frac{2\pi}{R^2}\int_{R^2}^R rm(r,\varphi)\,dr \\
&\le \frac{2\pi}{R^2}\int_{R^2}^R m(r,\varphi)\,dr \\
&\le B\int_{R^2}^R \frac{1}{1-r}\log^s\frac{1}{1-r}\,dr
\end{aligned}$$

$$
\begin{aligned}
&= \frac{B}{1+s}\log^{s+1}\frac{1}{1-r}\Big|_{R^2}^{R} \\
&= \frac{B}{1+s}\left[\log^{s+1}\frac{1}{1-R} - \log^{s+1}\frac{1}{1-R^2}\right] \\
&= \frac{B}{1+s}\left[\log^{s+1}\frac{1}{1-R} - \left(\log\frac{1}{1-R} + \log\frac{1}{1+R}\right)^{s+1}\right] \\
&\le \frac{B}{1+s}\left[\log^{s+1}\frac{1}{1-R} - \left(\log\frac{1}{1-R} - 1\right)^{s+1}\right],
\end{aligned}
$$

where B is a constant. Notice that when $x > 1$, $s \ge 0$,

$$
\lim_{x\to+\infty}\frac{x^{s+1}-(x-1)^{s+1}}{x^{s+1}} = 0 \qquad \text{or} \qquad \lim_{x\to+\infty}\frac{x^{s+1}-(x-1)^{s+1}}{x^{s}} = s+1.
$$

Let $x = \log\big(1/(1-R)\big)$. By Lemma 2 we derive

$$
\lim_{R\to1}\frac{|\beta(R)|}{A(R,\varphi)} = \lim_{R\to1}\frac{\log^{s+1}\frac{1}{1-R} - \left(\log\frac{1}{1-R} - 1\right)^{s+1}}{\log^{s+1}\frac{1}{1-R}} = 0.
$$

That is (3.5). Inequality (3.6) and Lemma 2 evidently provide (3.4). This completes the proof of Theorem 1.

In 1995, Lai and Wu obtained the following theorem as an improvement of Theorem 5 in [9] by Sethares.

Theorem D ([6]). *Let $z_1,\dots,z_m$ be points of $\partial\Delta$ such that removing an arbitrary neighborhood D_i of z_i from Δ results in a region of a finite φ-area. Let $\alpha_1,\dots,\alpha_m$ be non-zero complex numbers and let $t_1,\dots,t_m$ be real numbers such that φ satisfies, for each $i=1,\dots,m$, the growth condition*

$$
\left|\frac{\sqrt{\varphi(z)}}{\log^{t_i}(z_i-z)} - \frac{\alpha_i}{z_i-z}\right| = O(1), \qquad z\to z_i. \tag{3.7}
$$

Then f is uniquely extremal if and only if $t_i \le \frac{1}{2}$, $i=1,\dots,m$.

It is easy to see that φ with the condition (3.7) satisfies the growth estimate

$$
m(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi}|\varphi(re^{i\theta})|\,d\theta < C\frac{1}{1-r}\log\frac{1}{1-r}, \tag{3.8}
$$

where C is a positive constant.

We continue to show that

$$\varphi(z) = \frac{\log^s(1-z)}{(1-z)^2}, \qquad s > 0,$$

satisfies the growth condition (1.5).

On the one hand, by virtue of

$$\lim_{r\to 1} \frac{|1 - re^{i(1-r)}|}{1-r} = \sqrt{2}\,, \tag{3.9}$$

we have

$$\begin{aligned} \int_0^{2\pi} |\varphi(re^{i\theta})|\,d\theta &\geq \int_0^{1-r} \frac{\left|\log \dfrac{1}{1-re^{i\theta}}\right|^s}{|1-re^{i\theta}|^2}\,d\theta \\ &\geq C_1 \frac{1}{1-r} \log^s \frac{1}{1-r}, \qquad r \to 1. \end{aligned} \tag{3.10}$$

On the other hand,

$$\begin{aligned} m(r,\varphi) &= \frac{1}{2\pi} \int_0^{2\pi} |\varphi(re^{i\theta})|\,d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\left|\log \dfrac{1}{1-re^{i\theta}}\right|^s}{|1-re^{i\theta}|^2}\,d\theta \\ &\leq C_2 \frac{1}{1-r} \log^s \frac{1}{1-r}, \qquad r \to 1, \end{aligned}$$

where C_1 and C_2 are two positive constants. Therefore, φ assumes the growth condition (1.5).

Thus, combining Theorem 1 and Theorem D, we have

Corollary 3.1. *Suppose $s > 1$ is a real number and*

$$\varphi(z) = \frac{\log^s(1-z)}{(1-z)^2}.$$

Then $\varphi(z)$ is GHS and f is extremal instead of being uniquely extremal. Here some suitable univalent branch is chosen for φ in Δ.

Now, Theorem 2 is evident.

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