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ANALYTIC CLIFFORDIAN FUNCTIONS

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Abstract. In classical function theory, a function is holomorphic if and only if it is complex analytic. For higher dimensional spaces it is natural to work in the context of Clifford algebras. The structures of these algebras depend on the parity of the dimension n of the underlying vector space. The theory of holomorphic Cliffordian functions reflects this dependence. In the case of odd n the space of functions is defined by an operator (the Cauchy–Riemann equation) but not in the case of even n. For all dimensions the powers of identity (z^n, x^n) are the foundation of function theory.

I. Introduction

A complex analytic function $f(z)$ may be defined as being locally the sum of a convergent power series $f(z) = \sum_{N=1}^{\infty} a_N z^{N-1}$ or as being holomorphic, that is such that

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f(x+iy) = 0.$$

A real analytic function $u(x)$ may be defined as being locally the sum of a convergent power series or by $u(x) = f(x+i0)$ where f is a complex holomorphic function such that $f(\bar{z}) = \overline{f(z)}$. The main difference between the complex and the real case is the existence or non-existence of a differential relation characterizing holomorphy.

We extend the definitions of analyticity and holomorphy to functions defined on Clifford algebras $\mathbf{R}_{0,n}$ distinguishing between the case of odd n, $n = 2m+1$, and the case of even n, $n = 2m$. We show that the equivalence between analyticity and holomorphy still holds. The cases of odd n and even n interrelate in a way that reflects the difference between the structures of the algebras $\mathbf{R}_{0,2m}$ and $\mathbf{R}_{0,2m+1}$. In particular the center of $\mathbf{R}_{0,2m}$ is $\mathbf{R}$ although the center of $\mathbf{R}_{0,2m+1}$ is $\mathbf{R} \oplus \mathbf{R}e_{12\cdots 2m+1}$, where $e_{12\cdots 2m+1}$ is a pseudoscalar.

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II. Notation

Let V_n be an anti-Euclidean vector space of dimension n. For any orthonormal basis $e_1, \dots, e_n$ of V_n we have for all distinct i and j in $\{1, \dots, n\}$

$$e_i^2 = -1 \qquad \text{and} \qquad e_i e_j = -e_j e_i.$$

If $I \subset \{1, \dots, n\}$ and $I = \{i_1, \dots, i_k\}$ with $i_1 < \cdots < i_k$ we set $e_I = e_{i_1} e_{i_2} \cdots e_{i_k}$. For $I = \emptyset$, we set $e_\emptyset = e_0 = 1$. Then $(e_I)_{I \subset \{1,\dots,n\}}$ is a basis of the Clifford algebra $\mathbf{R}_{0,n}$ seen as a real vector space. If $A = \sum_{I \subset \{1,\dots,n\}} A_I e_I$, with $A_I \in \mathbf{R}$, is an element of $\mathbf{R}_{0,n}$ we call $A_0 = A_\emptyset$ the scalar part of A and denote it by $A_0 = S(A)$. Following the $\mathbf{R} - \mathbf{C}$ case and also Leutwiler and Eriksson-Bique [EL2], we introduce the decomposition

$$\mathbf{R}_{0,n} = \mathbf{R}_{0,n-1} \oplus e_n \mathbf{R}_{0,n-1}.$$

(For convenience in our computations we have chosen $e_n \mathbf{R}_{0,n-1}$ instead of $\mathbf{R}_{0,n-1} e_n$.) This decomposition means that given a vector e_n there are two maps from $\mathbf{R}_{0,n}$ to $\mathbf{R}_{0,n-1}$, denoted $\mathscr{R}$ and $\mathscr{J}$, such that for any A in $\mathbf{R}_{0,n}$ we have

$$A = \mathscr{R} A + e_n \mathscr{J} A.$$

We have chosen the notation $\mathscr{R}$ and $\mathscr{J}$ and the following notation for conjugation to stress the fact that for $n = 1$, we have $\mathbf{R}_{0,1} = \mathbf{C}$, $\mathbf{R}_{0,0} = \mathbf{R}$ which yield the usual relations between $\mathbf{C}$ and $\mathbf{R}$. If $A \in \mathbf{R}_{0,n}$, we call the conjugate of A and denote by $\bar{A}$ the element of $\mathbf{R}_{0,n}$ defined by

$$\bar{A} = \mathscr{R} A - e_n \mathscr{J} A.$$

If z is a paravector, that is an element of $\mathbf{R} \oplus V_n$, we have

$$z = z_0 + z_1 e_1 + \cdots + z_n e_n \quad \text{with } z_0 \in \mathbf{R},\ z_1 \in \mathbf{R}, \dots, z_n \in \mathbf{R}.$$

We denote by $|z|$ the positive real number such that $|z|^2 = z_0^2 + z_1^2 + \cdots + z_n^2$ and by $x = \mathscr{R} z = z_0 + z_1 e_1 + \cdots + z_{n-1} e_{n-1}$ the paravector in $\mathbf{R} \oplus V_{n-1}$ such that

$$z = x + z_n e_n \qquad \text{and} \qquad \bar{z} = x - z_n e_n.$$

We define z_* by $z_* = z_0 - z_1 e_1 - \cdots - z_n e_n$. Then $|z|^2 = z z_* = z_* z$.

We introduce the differential operators

$$\begin{aligned}
D &= \frac{\partial}{\partial z_0} + e_1 \frac{\partial}{\partial z_1} + \cdots + e_n \frac{\partial}{\partial z_n}, \\
D_* &= \frac{\partial}{\partial z_0} - e_1 \frac{\partial}{\partial z_1} - \cdots - e_n \frac{\partial}{\partial z_n} \qquad \text{and} \\
\Delta &= D D_* = D_* D.
\end{aligned}$$

Note that Δ is the usual Laplacian.

If $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in (\mathbf{N} \cup \{0\})^{n+1}$ is a multi-index we denote its length by $|\alpha| = \alpha_0 + \alpha_1 + \cdots + \alpha_n$. The elementary multi-index ε_k is defined by $\varepsilon_k = (\delta_{k0}, \delta_{k1}, \ldots, \delta_{kn})$ where δ_{ij} is the Kronecker symbol equal to 1 if $i = j$ and to 0 if $i \neq j$.

The order on the set of multi-indexes is the lexicographical order.

III. Analytic Cliffordian polynomials

Laville and Ramadanoff [LR1] have defined holomorphic Cliffordian polynomials for odd n. The same definitions can also be used for even n. We will see that these homogeneous polynomials are the building blocks of analytic Cliffordian functions in both cases. Therefore we call them analytic Cliffordian polynomials. Polynomials with the same structure were introduced by Heinz Leutwiler in [Le1] and also in [EL1]. Monogenic polynomials are particular cases of analytic Cliffordian polynomials [BDS] and [DSS].

III.1. *Three classes of analytic Cliffordian polynomials.*

Definition 1.a. Let a be a paravector and $N \in \mathbf{N}$. We call elementary analytic monomial and denote by $M_N^a(z)$ the homogeneous monomial function of degree $N-1$ defined by

$$M_N^a(z) = (az)^{N-1}a = a(za)^{N-1}.$$

Remark 1. We may also define $M_N^a(z)$ for all integers by $M_0^a(z) = z^{-1}$ and for $N < 0$, $M_N^a(z) = M_{1-N}^{z^{-1}}(a^{-1})$. It is often convenient to write $a(za)^{N-1}$ instead of $M_N^a(z)$. We have then for $N \in \mathbf{Z}$ and $\sqrt{a}$ a paravector such that $\left(\sqrt{a}\right)^2 = a$

$$a(za)^N = \sqrt{a}\left(\sqrt{a}\, z\sqrt{a}\,\right)^N \sqrt{a}\,.$$

These polynomials are close to similar ones in [Le2].

Remark 2. The monomial $M_N^a(z) = |a|^N M_N^{a/|a|}(z) = |z|^{N-1} M_N^a(z/|z|)$.

Proposition 1. *We have $M_N^a(z) \in \mathbf{R} \oplus V_n$ and $|M_N^a(z)| = |a|^N |z|^{N-1}$.*

Proof. Computing aza explicitly, we get $aza \in \mathbf{R} \oplus V_n$ and $|aza| = |a|^2|z|$. Note that

$$(*) \qquad M_N^a(z) = a M_{N-1}^z(a) a. \ \square$$

Proposition 2. *We have*

$$M_N^a(z) = D_* \frac{1}{N} S\big((az)^N\big).$$

Proof. Let us define θ by $|a|\,|z|\cos\theta = S(az)$, $0 \le \theta \le \pi$. Then $2S(az) = az + z_*a_*$ and $aza = (az + z_*a_*)a - z_*a_*a = 2S(az)a - |a|^2 z_* = 2|a|\,|z|\cos\theta a - |a|^2|z|^2 z^{-1}$. A simple recursion using $(*)$ yields

$$M_N^a(z) = |a|^{N-1}|z|^{N-1}\frac{\sin N\theta}{\sin\theta}a - |a|^N|z|^N\frac{\sin(N-1)\theta}{\sin\theta}z^{-1}.$$

Since $2S\big((az)^N\big) = M_N^a(z)z + z_*M_N^a(z)_*$, we get

$$S\big((az)^N\big) = |a|^N|z|^N\cos(N\theta).$$

From the definitions of $|z|$ and θ, we get $D_*|z| = |z|z^{-1}$ and

$$D_*\theta = \frac{\cos\theta}{\sin\theta}z^{-1} - |a|^{-1}|z|^{-1}\frac{1}{\sin\theta}a.$$

Finally

$$D_*S(az)^N = N|a|^N|z|^N z^{-1}\cos(N\theta) - N|a|^N|z|^N\sin N\theta D_*\theta = NM_N^a(z). \square$$

Corollary. *We have*

$$z^N = D_*\frac{1}{N+1}S(z^{N+1}).$$

Proof. Choose $a = e_0$ and replace N by $N+1$ in Proposition 2. $\square$

Remark. Let $T_N(x)$ and $U_N(x)$, $x \in \mathbf{R}$, be the classical Tchebycheff polynomials of the first and second kind. Recall that

$$T_N(x) = \cos(N\operatorname{Arc}\cos x),$$
$$U_N(x) = \frac{\sin\big((N+1)\operatorname{Arc}\cos x\big)}{\sin(\operatorname{Arc}\cos x)}.$$

Thus, when $|a| = 1$, $|z| = 1$, we get

$$S\big((az)^N\big) = T_N\big(S(az)\big) \qquad \text{and} \qquad M_N^a(z) = U_{N-1}\big(S(az)\big)a - U_{N-2}\big(S(az)\big)z^{-1}.$$

Definition 1.b. Let $a_1,\dots,a_k$ be paravectors and let $N_1,\dots,N_k$ be integers belonging to $\mathbf{N}\cup\{0\}$. We set $N = N_1+\cdots+N_k$ and denote by $\mathscr{P}_{N_1,\dots,N_k}$ the set of partitions I of $\{1,\dots,N\}$ into a union of disjoint subsets $I = (I_1,\dots,I_k)$ such that $\operatorname{Card} I_1 = N_1,\dots,\operatorname{Card} I_k = N_k$. For $I \in \mathscr{P}_{N_1,\dots,N_k}$ and $\nu \in \{1,\dots,N\}$, we define b_ν^I by $b_\nu^I = a_j$ where j is the element of $\{1,\dots,k\}$ such that $\nu \in I_j$. We define the $a_1,\dots,a_k$ symmetrical analytic homogeneous polynomial of degree $N-1$ in z and N_j in a_j by

$$S_{N_1,\dots,N_k}^{a_1,\dots,a_k}(z) = \sum_{I\in\mathscr{P}_{N_1,\dots,N_k}}\left(\prod_{\nu=1}^{N-1}(b_\nu^I z)\right)b_N^I.$$

Proposition 3. *The polynomial $S_{N_1,\dots,N_k}^{a_1,\dots,a_k}$ is a real linear combination of elementary analytic Cliffordian monomials of degree $N_1+\cdots+N_k-1$.*

Proof. For any real λ, we have

$$M_N^{a+\lambda b}(z) = \sum_{p+q=N} \lambda^q S_{p,q}^{a,b}(z).$$

Choose $N+1$ different values for λ; one gets a Van der Monde matrix which is invertible. This shows the result for $k=2$. We can iterate the same argument noting that for any real λ

$$S_{N_1,\ldots,N_{k-1},N_k}^{a_1,\ldots,a_{k-1},a_k+\lambda a_{k+1}}(z) = \sum_{p+q=N_k} \lambda^q S_{N_1,\ldots,N_{k-1},p,q}^{a_1,\ldots,a_{k-1},a_k,a_{k+1}}(z). \ \square$$

Definition 2. For each multi-index $\alpha \in (\mathbf{N} \cup \{0\})^{n+1}$ we define Q_α as the homogeneous polynomial in $z_0, \ldots, z_n$ of degree $|\alpha|-1$, by

$$Q_\alpha(z) = \partial_\alpha z^{2|\alpha|-1}$$

where ∂_α is the differential operator of order $|\alpha|$: $\partial_\alpha = \partial^{|\alpha|}/\partial z_0^{\alpha_0} \partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}$.

Definition 3. For each multi-index $\alpha \neq (0,0,\ldots,0)$ we define the analytic Cliffordian polynomial P_α by

$$P_\alpha(z) = \sum_{\sigma \in S_\alpha} \left(\prod_{\nu=1}^{|\alpha|-1} (e_{\sigma(\nu)} z) \right) e_{\sigma(|\alpha|)}$$

where S_α is the set of maps σ from $\{1,\ldots,|\alpha|\}$ to $\{0,1,\ldots,n\}$ such that

$$\operatorname{Card}\big(\sigma^{-1}(\{k\})\big) = \alpha_k \qquad \text{for all } k \text{ in } \{0,1,\ldots,n\}.$$

III.2. *Relations among the $M_N^a(z)$, the $Q_\alpha(z)$ and the $P_\alpha(z)$.* For any N in $\mathbf{N}$, the real linear space generated by the elementary analytic monomials M_N^a, the real linear space generated by the Q_α with $|\alpha| = N$ and the real linear space generated by the P_α with $|\alpha| = N$ are identical.

Proposition 4. *The polynomial $Q_\alpha(z) = \partial_\alpha M^1_{2|\alpha|}(z)$.*

Corollary. *For any multi-index α, $Q_\alpha(z) \in \mathbf{R} \oplus V_n$ and there exists a scalar polynomial $q_\alpha(z)$ homogeneous of degree $|\alpha|$ such that $Q_\alpha(z) = D_* q_\alpha(z)$.*

Proof. ∂_α and D_* commute and $M^1_{2|\alpha|}|z| = z^{2|\alpha|-1}$. Use Proposition 1 and corollary of Proposition 2. $\square$

Proposition 5. *We have*

$$Q_\alpha(z) = k_\alpha P_\alpha(z) + \sum_{\substack{\alpha' > \alpha \\ |\alpha'| = |\alpha|}} \lambda_{\alpha\alpha'} P_{\alpha'}(z)$$

where $k_\alpha = \binom{2|\alpha|-1}{\alpha_0} \alpha_0! \alpha_1! \cdots \alpha_n!$ *and* $\lambda_{\alpha\alpha'} \in \mathbf{Z}$.

Proof. Let $\beta = (\alpha_1, \ldots, \alpha_n) \in (\mathbf{N} - \{0\})^n$ be the multi-index such that $\alpha = (\alpha_0, \beta)$. From the definition of Q_α follows:

$$Q_\alpha(z) = \binom{2|\alpha|-1}{\alpha_0} \alpha_0! \frac{\partial^{|\beta|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} z^{2|\beta|-1+\alpha_0}.$$

Consider $z^{2|\beta|-1+\alpha_0}$ as an explicit product of $2|\beta|-1+\alpha_0$ factors each equal to z, that is: $z^{2|\beta|-1+\alpha_0} = z \cdot z \cdot z \cdots z$. To apply $\partial/\partial x_k$ is equivalent to replace each z once by e_k and to add all the products obtained. If we never derivate two successive z then we get $\alpha_1! \cdots \alpha_n! P_\alpha(z)$. If we derivate two successive z, we get factors $e_k e_h$ which anihilate if $k \neq h$ because $e_k e_h = -e_h e_k$ and else -1 if $k = h$. Since $1 = e_0$ we get the terms of $P_{\alpha'}$ with $\alpha' > \alpha$ in the lexicographical order.

Corollary 1. *The polynomial* $P_\alpha(z) = k_\alpha^{-1} Q_\alpha(z) + \sum_{\substack{\alpha' > \alpha \\ |\alpha'| = |\alpha|}} \mu_{\alpha\alpha'} Q_{\alpha'}(z)$ *where* $\mu_{\alpha\alpha'} \in Q$.

Corollary 2. *For any multi-index* α, $P_\alpha(z) \in \mathbf{R} \oplus V_n$ *and there exists a scalar polynomial* $p_\alpha(z)$ *homogeneous of degree* $|\alpha|$ *such that* $P_\alpha(z) = D_* p_\alpha(z)$.

Proposition 6. *The monomial function* $M_N^a(z) = \sum_{|\alpha|=N} a^\alpha P_\alpha(z)$, *where* $a^\alpha := a_0^{\alpha_0} a_1^{\alpha_1} \cdots a_n^{\alpha_n}$.

Proposition 7. *For* $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$, *we have* $P_\alpha(z) = S^{e_0, e_1, \ldots, e_n}_{\alpha_0, \alpha_1, \ldots, \alpha_n}(z)$.

Proof. The definition of P_α is the definition of S in which the k paravectors $a_1 \cdots a_k$ are the $(n+1)$ elements of a basis of $S \oplus V : e_0, e_1, \ldots, e_n$.

Corollary. *The real linear space generated by the* P_α *with* $|\alpha| = N$ *is independent of the basis* $e_1, \ldots, e_n$ *of* V_n.

Remarks. In the case of odd n, this is already known.

Note that generally the polynomials $P_\alpha(z)$ with $|\alpha| = N$ are not $\mathbf{R}$-linearly independent. For example, if $n = 3$ and $N = 4$ we have

$$3P_{4000} + 3P_{0400} + 3P_{0040} + 3P_{0004} + P_{2200} + P_{2020} + P_{2002} + P_{0220} + P_{0202} + P_{0022} = 0.$$

III.3. *Some properties of the polynomials* P_α.

Proposition 8. *For* $\alpha = (\alpha_0, \ldots, \alpha_n)$, *the polynomial* $p_\alpha(z)$ *is even in* z_k *if* α_k *is even and odd in* z_k *if* α_k *is odd.*

Proof. We know that $P_\alpha(z) \in \mathbf{R} \oplus V_n$. Then we can write

$$P_\alpha(z) = A_0(z) + A_1(z)e_1 + \cdots + A_n(z)e_n.$$

Suppose α_k even. $P_\alpha(z)$ is a sum of terms like

$$e_{i_1} z e_{i_2} z \cdots z e_{i_{|\alpha|}}$$

in which e_k occurs α_k times, that is an even number of times. Write $z = z_0 + z_1e_1+\cdots+z_ne_n$ and develop all the terms. The terms which contribute to $A_k(z)e_k$ must contain an odd number of times the vector e_k and then z_ke_k has to appear an odd number of times. $A_k(z)$ is then a sum of terms which are all odd in z_k and $A_k(z)$ is odd in z_k. Since $P_\alpha(z) = D_*p_\alpha(z)$, we have $A_k(z) = -(\partial/\partial z_k)p_\alpha(z)$. Since $p_\alpha(z)$ is homogeneous, we can conclude that $p_\alpha(z)$ is even in z_k.

If α_k is odd the proof is the same. □

Examples. $p_{(2,0,1,0)}(z) = (3z_0^2 - z_1^2 - z_2^2 - z_3^2)z_2$ is even in z_0, z_1 and z_3 and odd in z_2. $p_{(1,1,1,0)}(z) = 8z_0z_1z_2$ is odd in z_0, z_1 and z_2 and even in z_3.

Corollary. *If α_n is even, then $P_\alpha(\bar z) = \overline{P_\alpha(z)}$; if α_n is odd then $P_\alpha(\bar z) = -\overline{P_\alpha(z)}$.*

Proof. Let us write

$$P_\alpha(z) = A_0(z) + A_1(z)e_1 + \cdots + A_{n-1}(z)e_{n-1} + A_n(z)e_n.$$

If α_n is even, then $p_\alpha(z)$ is even in z_n and the polynomials

$$A_k = \pm\frac{\partial}{\partial z_k}p_\alpha(z)$$

for $k \neq n$, are all even in z_n, but A_n is odd in z_n. Thus

$$P_\alpha(\bar z) = A_0(z) + A_1(z)e_1 + \cdots + A_{n-1}(z)e_{n-1} - A_n(z)e_n = \overline{P_\alpha(z)}.$$

If α_n is odd, $A_0, A_1, \ldots, A_{n-1}$ are odd in z_n and A_n is even in z_n, so that

$$P_\alpha(\bar z) = -A_0(z) - A_1(z)e_1 - \cdots - A_{n-1}(z)e_{n-1} + A_n(z)e_n = -\overline{P_\alpha(z)}. \ \square$$

Remark. We have

$$\frac{\partial}{\partial z_i}P_\alpha(z) = e_i^2\left\{2|\alpha|P_{\alpha-\varepsilon_i}(z) - (\alpha_i+1)\sum_{k=0}^{n}P_{\alpha+\varepsilon_i-2\varepsilon_k}(z)\right\}.$$

IV. Analytic Cliffordian functions and holomorphic Cliffordian functions

Definition 1. Let Ω be a domain of $\mathbf{R}\oplus V_n$ and $f\colon \Omega \to \mathbf{R}_{0,n}$. We say that f is a left analytic Cliffordian function if any ω in $\mathbf{R}\oplus V_n$ has a neighbourhood Ω_ω in Ω such that for any z in Ω_ω, $f(z)$ is the sum of a convergent series

$$f(z) = \sum_{N=1}^{\infty}\sum_{a\in A_N} M_N^a(z-\omega)C_a$$

where for each N in $\mathbf{N}$, A_N is a finite subset of $\mathbf{R}\oplus V_n$, for each a in A_N, $C_a \in \mathbf{R}_{0,n}$ and $\sum_{N=1}^{\infty}\sum_{a\in A_N}|a|^N|z-\omega|^{N-1}|C_a|$ is convergent in Ω_ω.

Remark 1. The relation $M_N^a(z-\omega)=\sum_{p+q=N}(-1)^q S_{p,q}^{a,a\omega a}(z)$ and Proposition 3 prove the consistency of the definition with respect to translations. Consequentely we will restrict ourselves to the case $\omega=0$.

Remark 2. The above definition is obviously intrinsic, but we get an equivalent definition if we replace the monomials M_N^a by the polynomials P_α; the function $f\colon \mathbf{R}\oplus V_n\to\mathbf{R}_{0,n}$ is left analytic Cliffordian in a neighbourhood Ω of 0 if for every z in Ω, $f(z)$ is the sum of a convergent series

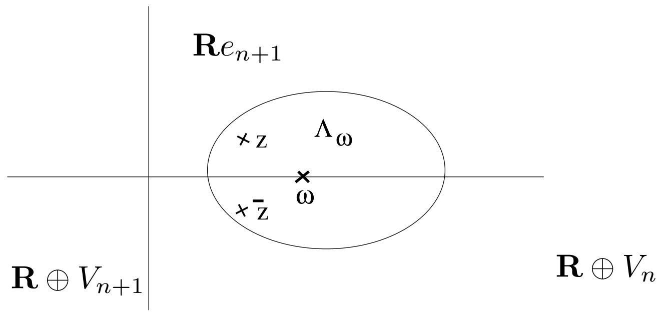
$$f(z)=\sum_{N=1}^{\infty}\sum_{|\alpha|=N}P_\alpha(z)c_\alpha$$

where α are multi-indexes belonging to $(\{0\}\cup\mathbf{N})^{1+n}$, and for each α we have $c_\alpha\in\mathbf{R}_{0,n}$ and $\sum_{N=1}^{\infty}\sum_{|\alpha|=N}|P_\alpha(z)|\,|c_\alpha|$ is convergent.

Definition 2. Let Ω be a domain in $\mathbf{R}\oplus V_n$. A function $u\colon\mathbf{R}\oplus V_n\to\mathbf{R}_{0,n}$ is called a left holomorphic Cliffordian function

(i) for odd n, if $D\Delta^m u=0$ where $m=\frac{1}{2}(n-1)$,

(ii) for even n, if for any $\omega\in\Omega$ a neighbourhood Λ_ω in $\mathbf{R}\oplus V_{n+1}$ and a left holomorphic Cliffordian function f defined on Λ_ω exist such that
   - for all z in Λ_ω: $\bar z$ is in Λ_ω and $f(\bar z)=\overline{f(z)}$,
   - for all x in $\Lambda_\omega\cap(\mathbf{R}\oplus V_n)$, $u(x)=f(x)$.



Theorem. *Let Ω be a domain of $\mathbf{R}\oplus V_n$. A function $f\colon\Omega\to\mathbf{R}_{0,n}$ is left analytic Cliffordian if and only if it is left holomorphic Cliffordian.*

For odd n, the theorem has already been proven in [LR1]. Let n be even, $n=2m$. Let u be a left analytic Cliffordian function on a neighbourhood Ω of 0 and let $S_n(r)$ be a sphere of center 0 and radius $r>0$ included in Ω. For $|x|<r$ we have

$$u(x)=\sum_{N=1}^{\infty}\sum_{a\in A_N}M_N^a(x)C_a$$

where $\sum_{N=1}^{\infty}\sum_{a\in A_N}|a|^N|x|^{N-1}|C_a|$ is convergent. Choose $\Lambda_0 = S_{n+1}(r)$ the interior of the sphere of center 0 and radius r in $\mathbf{R}\oplus V_{n+1}$ and let $f\colon S_{n+1}(r)\to \mathbf{R}_{0,n+1}$ be defined by

$$f(z)=\sum_{N=1}^{\infty}\sum_{a\in A_N} M_N^a(z)C_a.$$

Then f is left holomorphic Cliffordian on $S_{n+1}(r)$ and $f(x)=u(x)$. Since $a\in\mathbf{R}\oplus V_n$, one gets $M_N^a(\bar z)=\overline{M_N^a(z)}$. And since $C_a\in\mathbf{R}\oplus V_n$, we have $f(\bar z)=\overline{f(z)}$.

Conversely, let f be a left holomorphic Cliffordian function defined on a neighbourhood Λ_0 of 0 in $\mathbf{R}\oplus V_{n+1}$ such that $f(\bar z)=\overline{f(z)}$. We want to show that $u\colon \Lambda_0\cap\mathbf{R}\oplus V_n\to\mathbf{R}_{0,n}$, $x\longmapsto u(x)=f(x)$ is left analytic Cliffordian.

Since $n+1$ is odd, f is analytic Cliffordian and we can write

$$f(z)=\sum_{N=1}^{\infty}\sum_{|\beta|=N}P_\beta(z)c_\beta = D_*\sum_{N=1}^{\infty}\sum_{|\beta|=N}p_\beta(z)c_\beta.$$

Let H_N be the real linear space of scalar homogeneous polynomials in $z_0,\dots,z_n,z_{n+1}$ of total degree N and $(m+1)$-harmonic generated by $(p_\beta)_{|\beta|=N}$ (the dimension of H_N is $C^n_{N+n}-C^n_N$). We can extract a subset B_N of $\big\{\beta\in(\{0\}\cup\mathbf{N})^{n+2}/|\beta|=N\big\}$ such that $(p_\beta)_{\beta\in B_N}$ is a basis of H_N. D_* is a linear map from H_N to $\mathbf{R}\otimes V_{n+1}$ and $\operatorname{Ker} D_*$ is a subspace of H_N. Let $(\psi_j)_{j\in\mathscr{J}}$ be a basis of $\operatorname{Ker} D_*$; since $(p_\beta)_{\beta\in B_N}$ is a basis of H_N there is a subset B_N° of B_N such that $\big((\psi_j)_{j\in\mathscr{J}},(p_\beta)_{\beta\in B_N^\circ}\big)$ is a basis of H_N. Then $(\psi_j\otimes e_I)_{j\in\mathscr{J},I\subset\{1,\dots,n+1\}},(p_\beta\otimes e_I)_{\beta\in B_N^\circ,I\subset\{1,\dots,n+1\}}$ is a basis of the real linear space $H_N\otimes\mathbf{R}_{0,n+1}$, and there are unique real numbers $\theta_{j,I}$ and $d_{\beta,I}$ such that

$$\sum_{|\beta|=N}p_\beta(z)c_\beta=\sum_{j\in\mathscr{J}}\sum_{I\subset\{1,\dots,n+1\}}\theta_{j,I}\psi_j(z)e_I+\sum_{\beta\in B_N^\circ}\sum_{I\subset\{1,\dots,n+1\}}d_{\beta,I}p_\beta(z)e_I.$$

Let us write $d_\beta=\sum_I d_{\beta,I}e_I$, using homogeneity we get for any analytic Cliffordian function f the existence and unicity of the coefficients d_β in $\mathbf{R}_{0,n+1}$ such that

$$f(z)=\sum_{N=1}^{\infty}\sum_{\beta\in B_N^\circ}P_\beta(z)d_\beta.$$

Let $B_N^{\circ+}$ be the set of multi-indexes $\beta=(\beta_0,\dots,\beta_n,\beta_{n+1})$ where $\beta\in B_N^\circ$ and β_{n+1} is even and $B_N^{\circ-}=B_N^\circ\text{-}B_N^{\circ+}$. If $\beta\in B_N^{\circ+}$, the corollary of Proposition 8 implies $P_\beta(\bar z)=\overline{P_\beta(z)}$ and if $\beta\in B_N^{\circ-}$ we have $P_\beta(\bar z)=-\overline{P_\beta(z)}$. The relation $f(\bar z)=\overline{f(z)}$ becomes then

$$\sum_{N=1}^{\infty}\Big\{\sum_{\beta\in B_N^{\circ+}}\overline{P_\beta(z)}d_\beta-\sum_{\beta\in B_N^{\circ-}}\overline{P_\beta(z)}d_\beta\Big\}=\sum_{N=1}^{\infty}\Big\{\sum_{\beta\in B_N^{\circ+}}\overline{P_\beta(z)}\bar d_\beta+\sum_{\beta\in B_N^{\circ-}}\overline{P_\beta(g)}\bar d_\beta\Big\}.$$

By homogeneity we get

$$\sum_{\beta\in B_N^{\circ+}} \overline{P_\beta(z)}(\bar d_\beta - d_\beta) + \sum_{\beta\in B_N^{\circ-}} \overline{P_\beta(z)}(\bar d_\beta + d_\beta) = 0$$

and by conjugation

$$\sum_{\beta\in B_N^{\circ+}} P_\beta(z)(d_\beta - \bar d_\beta) + \sum_{\beta\in B_N^{\circ-}} P_\beta(z)(d_\beta + \bar d_\beta) = 0.$$

By unicity of the coefficients of the P_β for $\beta \in B_N^\circ$, we get $d_\beta = \bar d_\beta$ if $\beta \in B_N^{\circ+}$ and $\bar d_\beta = -d_\beta$ if $\beta \in B_N^{\circ-}$. Let us write $a_\beta = d_\beta = d_{\bar\beta}$ if $\beta \in B_N^{\circ+}$ and $e_{n+1}b_\beta = d_\beta = -\bar d_\beta$ if $\beta \in B_N^{\circ-}$, we get

$$f(z) = \sum_{N=1}^{\infty}\Big\{\sum_{\beta\in B_N^{\circ+}} P_\beta(z)a_\beta + \sum_{\beta\in B_N^{\circ-}} P_\beta(z)e_{n+1}b_\beta\Big\},$$

where $a_\beta \in \mathbf{R}_{0,n}$ and $b_\beta \in \mathbf{R}_{0,n}$. The two following lemmas will then prove the theorem. □

Lemma 1. *Let $n = 2m$ and $\beta = (\beta_0,\dots,\beta_n,\beta_{n+1})$. If β_{n+1} is even then the restriction of P_β to $\mathbf{R}\oplus V_n$ is analytic Cliffordian from $\mathbf{R}\oplus V_n$ to $\mathbf{R}_{0,n}$.*

Proof. First we show that if $x \in \mathbf{R}\oplus V_n$ then $P_\beta(x) \in \mathbf{R}_{0,n}$ or better $P_\beta(x) \in \mathbf{R}\oplus V_n$. We know that $P_\beta(z) = D_* p_\beta(z)$ in $\mathbf{R}_{0,n+1}$ and $p_\beta(z)$ is even in z_{n+1} since β_{n+1} is even. So

$$\left[\frac{\partial}{\partial z_{n+1}} p_\beta(z)\right]_{z_{n+1}=0} = 0$$

and $P_\beta(x) \in \mathbf{R}\oplus V_n$ for $x = \mathscr{R}z$.

Secondly we show that $P_\beta(x)$ is analytic Cliffordian. We know that

$$P_\beta(x) = \sum_{\sigma\in S_\beta}\left(\prod_{\nu=1}^{n}\big(e_{\sigma(\nu)}x\big)\right)e_{\sigma(n+1)},$$

which means that $P_\beta(x)$ is the sum of all different polynomials deduced from

$$(*)\qquad \underbrace{e_0xe_0\cdots e_0}_{\beta_0 \text{ times } e_0}\,x\,\underbrace{e_1xe_1\cdots e_1}_{\beta_1 \text{ times } e_1}\,xe_2xe_2\cdots\cdots e_nx\,\underbrace{e_{n+1}xe_{n+1}x\cdots x_ne_{n+1}}_{\beta_{n+1} \text{ times } e_{n+1}}$$

by permutations of the $e_i's$. Note that

$$e_{n+1} = (-1)^m e_{1\,2\cdots n}e_{1\,2\cdots n+1}$$

and that the pseudo scalar $e_{1\,2\cdots 2m\,2m+1}$ belongs to the center of $\mathbf{R}_{0,2m+1}$. Since β_{n+1} is even and since $(e_{1\,2\cdots n+1})^2 \in \{1,-1\}$, we deduce that up to a sign we can replace $(*)$ by:

$$(**)\qquad \underbrace{e_0xe_0\cdots e_0}_{\beta_0\text{ times }e_0}\,x\,\underbrace{e_1xe_1\cdots e_1}_{\beta_1\text{ times }e_1}\,xe_2xe_2\cdots\cdots e_nx\,\underbrace{e_{1\,2\cdots n}xe_{1\,2\cdots n}\cdots xe_{1\,2\cdots n}}_{\beta_{n+1}\text{ times }e_{1\,2\cdots n}}.$$

Let us choose a basis in V_n such that $x = x_0e_0 + x_1e_1$. Then e_1 commutes with x and for $i \geq 2$, e_ie_{i+1} commutes with e_1, x and of course e_0. Suppose $\beta_2 > 0$, for each term of the form

$$Ae_2Be_{1\,2\cdots n}C$$

we have another term equal to $Ae_{1\,2\cdots n}Be_nC$. But then the commutation rules give, since $n = 2m$,

$$Ae_2Be_{1\,2\cdots n}C + Ae_{1\,e\cdots n}Be_2C = 0.$$

Thus $\beta_2 = 0$. And similarly $\beta_3 = \cdots = \beta_n = 0$. We get:

$$P_\beta(x) = \sum_{\sigma\in S_\beta}\left(\prod_{\nu=1}^{|\beta|-1} E_{\sigma(\nu)}x\right)E_{\sigma(|\beta|)}$$

where $E_0 = e_0$, $E_1 = e_1$ and $E_{n+1} = e_{1\,2\cdots n}$. Commuting systematically e_1 and $e_3e_4, e_5e_6, \ldots, e_{n-1}e_n$ from E_{n+1} to the right

$$P_\beta(x) = \left(\sum_{\sigma\in S_\alpha}\left(\prod_{\nu=1}^{|\alpha|-1} e_{\sigma(\nu)}x\right)e_{\sigma(|\alpha|)}\right)(e_1)^{\beta_{n+1}}(e_{3\,4\cdots n})^{\beta_{n+1}}$$

where $\alpha = (\beta_0, \beta_1, \beta_{n+1}, 0, \ldots, 0) \in (\{0\}\cup\mathbf{N})^{n+1}$. Thus $P_\beta(x) = \pm P_\alpha(x)$ with P_α analytic Cliffordian. ▫

Lemma 2. *Let $n = 2m$ and $\beta = (\beta_0, \ldots, \beta_n, \beta_{n+1})$. If β_{n+1} is odd then the restriction of $P_\beta e_{n+1}$ to $\mathbf{R}\oplus V_n$ is analytic Cliffordian from $\mathbf{R}\oplus V_n$ to $\mathbf{R}_{0,n}$.*

Proof. Now $p_\beta(z)$ is odd in z_{n+1} so

$$\left[\frac{\partial}{\partial z_i}p_\beta(z)\right]_{z_{n+1}=0} = 0 \quad \text{for } i = 0, \ldots, n, \quad \text{and}$$

$$P_\beta(x) = -\left[\frac{\partial}{\partial z_{n+1}}p_\beta(z)\right]_{z_{n+1}=0} e_{n+1}$$

and $P_\beta(x)e_{n+1}$ is a scalar. For the second part of the proof we reduce the sum

$$P_\beta(x)e_{n+1} = \sum_{\sigma\in S_\beta}\left(\prod_{\nu=1}^{|\beta|-1} e_{\sigma(\nu)}x\right)e_{\sigma(|\beta|)}e_{n+1}$$

as in Lemma 1. ▫

Corollary 1. *The space of left analytic Cliffordian functions is an* $\mathbf{R}$*-vector space and an* $\mathbf{R}_{0,n}$*-right module, closed relatively to scalar derivations.*

To deduce Corollary 2 and 3 from the above theorem, the following lemma is convenient.

Lemma 3. *If* $v \in \mathbf{R} \oplus V_{2m}$*, then* $\sum_{i=0}^{2m} e_i v e_i = (1-2m)v_*$.

Corollary 2. *If* f *is left analytic Cliffordian, then* Df *is also left analytic Cliffordian.*

Proof. If n is odd let $n = 2m+1$. Since f is left analytic Cliffordian it is left holomorphic Cliffordian and $D\Delta^m f = 0$. But since D commutes with D and Δ we have

$$D\Delta^m(Df) = D(D\Delta^m f) = 0.$$

Thus Df is left holomorphic Cliffordian or left analytic Cliffordian.

If n is even let $n = 2m$. Let us write: $x = z_0 + z_1 e_1 + \cdots + z_{2m} e_{2m}$ and $D' = \sum_{i=0}^{2m} e_i \partial/\partial z_i$. Then we have $D = D' + e_{2m+1}\partial/\partial z_{2m+1}$.

We want to show that if u is left analytic Cliffordian then $D'u$ is also left analytic Cliffordian. In fact, we need only to show this for $u(x) = M_N^a(x)$ for any $a \in \mathbf{R} \oplus V_{2m}$ and any $N \in \mathbf{N}$. A straightforward computation gives

$$D' M_N^a(x) = \sum_{k=1}^{N-1} \sum_{i=0}^{2m} e_i M_k^a(x) e_i M_{N-k}^a(x)$$

and since $M_k^a(x) in \mathbf{R} \oplus V_{2m}$, Lemma 3 gives us

$$D' M_N^a(x) = -(2m-1) \sum_{k=1}^{N-1} \left[M_k^a(x)\right]_* M_{N-k}^a(x).$$

If N is odd, let $N = 2M+1$; we have

$$D' M_{2M+1}^a(x) = -2(2m-1) \sum_{k=1}^{M} |a|^{2k} |x|^{2k-2} S\big((xa)^{2M-2k+1}\big).$$

If N is even, let $N = 2M$; we have

$$D' M_{2M}^a(x) = -(2m-1)\left\{|a|^{2M}|x|^{2M-2} + 2\sum_{k=1}^{M} |a|^{2k}|x|^{2k-2} S\big((xa)^{2M-2k}\big)\right\}.$$

The same computations in $\mathbf{R}_{0,2m+1}$ give

$$DM_{2M+1}^a(z) = -2(2m) \sum_{k=1}^{M} |a|^{2k} |z|^{2k-2} S\big((za)^{2M-2k+1}\big)$$

and

$$DM^a_{2M}(z) = -(2m)\Big\{|a|^{2M}|z|^{2M-2} + 2\sum_{k=1}^{M} |a|^{2k}|z|^{2k-2}S\big((za)^{2M-2k}\big)\Big\}.$$

Then we have for any N

$$D'M^a_N(x) = \frac{2m-1}{2m}\big[DM^a_N(z)\big]_{z=x}.$$

Since $a \in \mathbf{R}_{0,2m}$, we have $S\big((za)^{N-2k}\big) = S\big((xa)^{N-2k}\big)$, and since $|\bar{z}| = |z|$, we have $DM^a_N(\bar{z}) = DM^a_N(z) = \overline{DM^a_N(z)}$. The theorem then proves that $D'M^a_N(x)$ is left analytic Cliffordian. □

Corollary 3. *If f is left analytic Cliffordian, then $D_* f_*$ is also left analytic Cliffordian.*

Proof. Let us write f as

$$f(z) = \sum_{N=1}^{\infty}\sum_{|\alpha|=N} P_\alpha(z)c_\alpha = \sum_{N=1}^{\infty}\sum_{|\alpha|=N} D_* p_\alpha(z)c_\alpha$$

and define the left analytic Cliffordian function $\tilde{f}$ by

$$\tilde{f}(z) = \sum_{N=1}^{\infty}\sum_{|\alpha|=N} P_\alpha(z)c_{\alpha*}.$$

Then we have

$$D_* f_*(z) = \big(Df(z)\big)_* = \sum_{N=1}^{\infty}\sum_{|\alpha|=N} \Delta p_\alpha(z)c_{\alpha*} = D\tilde{f}(z).$$

By Corollary 1, $D\tilde{f}$ is left analytic Cliffordian, thus $D_* f_*$ is also left analytic Cliffordian. □

V. Cauchy's problem and boundary data

V.1. *Our aim.* We intend to generalize to Clifford algebras of any dimensions the method of extending real analytic functions into complex holomorphic functions. Let Ω be a domain of $\mathbf{R} \oplus V_{2m}$ and $u\colon \Omega \to \mathbf{R}_{0,2m}$. The Cauchy problem

$$\begin{cases} D\Delta^m f = 0, \\ f|_\Omega = u \end{cases}$$

where the unknown function f has to be defined on an open set Λ containing Ω, seems not well defined since the partial differential equations are of order $2m+1$. The Cauchy–Kowalewski theorem tells us that we need the normal derivatives of f to $\mathbf{R}\oplus V_{2m}$ up to the order $2m$. We will see that the algebraic and analytic properties of f and u enable us to compute these derivatives uniquely given the function u.

We will denote by $\mathscr{A}_{2m}$ the linear space of left analytic Cliffordian functions defined on Ω and taking their values in $\mathbf{R}_{0,2m}$. Similarly, $\mathscr{A}_{2m+1}$ is the linear space of left analytic Cliffordian functions defined on Λ and with values in $\mathbf{R}_{0,2m+1}$.

V.2. *The operator* $(A \mid \nabla_n)$.

Definition. Let $A\in\mathbf{R}_{0,n}$. We define $(A\mid\nabla_n)\colon \mathscr{A}_n \longrightarrow \mathscr{A}_n$ by:

(i) for all $a\in\mathbf{R}\oplus V_n$ and for all $N\in\mathbf{N}$ we have

$$(A\mid\nabla_n)M_N^a(z)=\sum_{k=1}^{N-1}M_k^a(z)AM_{N-k}^a(z),$$

(ii) for all $f\in\mathscr{A}_n$ and for all $K\in\mathbf{R}_{0,n}$, $(A\mid\nabla_n)\big(f(z)K\big)=\big(A\mid\nabla_n)f(z)\big)K$,

(iii) $(A\mid\nabla_n)$ is $\mathbf{R}$-linear and continuous.

Consequence. If $f(z)=\sum_{N=1}^{\infty}\sum_{a\in A_n}M_N^a(z)C_a$, then

$$(A\mid\nabla_n)f(z)=\sum_{N=1}^{\infty}\sum_{a\in A_N}\big((A\mid\nabla_n)M_N^a(z)\big)C_a.$$

Proposition 9. *If* $f\in\mathscr{A}_n$, *then* $(\partial/\partial z_k)f(z)=(e_k\mid\nabla_n)f(z)$.

Proof. We need only to verify the proposition for $f(z)=M_N^a(z)$. We have

$$\begin{aligned}\frac{\partial}{\partial z_k}azaz\cdots za&=ae_kaz\cdots za+azae_k\cdots za+\cdots+azaz\cdots e_ka\\&=(e_k\mid\nabla_n)azaz\cdots za.\ \square\end{aligned}$$

Proposition 10. *If* $f\in\mathscr{A}_n$ *and if* λ *belongs to the center of* $\mathbf{R}_{0,n}$ *then*

$$(\lambda A\mid\nabla_n)f=\lambda(A\mid\nabla_n)f.$$

Lemma. *If* $v\in\mathbf{R}\oplus V_{2m}$, *then*

$$\sum_{i=0}^{2m}e_ive_i=(-1)^m(1-2m)e_{1\,2\cdots2m}ve_{1\,2\cdots2m}.$$

Proof. Both sides of the equality are equal to $(1-2m)v_*$. $\square$

Proposition 11. *If* $u \in \mathscr{A}_{2m}$, *then*

$$e_{1\,2\cdots 2m}(e_{1\,2\cdots 2m} \mid \nabla_{2m})u = \frac{(-1)^{m+1}}{2m-1} Du.$$

Proof. We need only to verify the relation for $u(x) = M_N^a(x)$. Using the definition and the lemma, we get

$$\begin{aligned}
(-1)^m(1-2m)e_{1\,2\cdots 2m}(e_{1\,2\cdots 2m} \mid \nabla_{2m})u &= \sum_{k=1}^{N-1}\sum_{i=0}^{2m} e_i M_k^a(x) e_i M_{N-k}^a(x) \\
&= \sum_{i=0}^{2m} e_i \sum_{k=1}^{N-1} M_k^a(x) e_i M_{N-k}^a(x) \\
&= \sum_{i=0}^{2m} e_i \frac{\partial}{\partial x_i} M_N^a(x) = DM_N^a(x). \;\square
\end{aligned}$$

Proposition 12. *For* $\mu \in \{0\} \cup \mathbf{N}$, *we have*

$$\begin{aligned}
(e_{1\,2\cdots 2m} \mid \nabla_{2m})^{2\mu} u &= \frac{(-1)^{m\mu}}{(2m-1)^{2\mu}} \Delta^\mu u, \\
(e_{1\,2\cdots 2m} \mid \nabla_{2m})^{2\mu+1} u &= \frac{(-1)^{1+m\mu}}{(2m-1)^{2\mu+1}} e_{1\,2\cdots 2m} D \Delta^\mu u.
\end{aligned}$$

Proof. From Proposition 3, we get

$$(e_{1\,2\cdots 2m} \mid \nabla_{2m})u = \frac{-1}{2m-1} e_{1\,2\cdots 2m} Du = \frac{-1}{2m-1} D_* u_* e_{1\,2\cdots 2m}.$$

Corollary 2 of Section IV shows that $D_* u_* e_{1\,2\cdots 2m}$ is left analytic Cliffordian. Thus we may apply the operator as many times as we want. We get

$$\begin{aligned}
\big((e_{1\,2\cdots 2m} \mid \nabla_{2m}\big)^2 u &= \frac{-1}{2m-1} e_{1\,2\cdots 2m} D \left\{ \frac{-1}{2m-1} D_* u_* e_{1\,2\cdots 2m} \right\} \\
&= \frac{1}{(2m-1)^2} e_{1\,2\cdots 2m} \Delta u_* e_{1\,2\cdots 2m} \\
&= \frac{1}{(2m-1)^2} (e_{1\,2\cdots 2m})^2 \Delta u = \frac{(-1)^m}{(2m-1)^2} \Delta u.
\end{aligned}$$

A simple recursion gives then the general formulae. $\square$

V.3. *Analytic extension.*

Theorem. *Let Ω be a domain of $\mathbf{R}\oplus V_{2m}$ and let $u\colon \Omega\to\mathbf{R}_{0,2m}$ be left analytic Cliffordian. There exist a domain Λ in $\mathbf{R}\oplus V_{2m+1}$ with $\Omega\subset\Lambda$ and a unique left holomorphic Cliffordian function f defined on $\Lambda\subset\mathbf{R}\oplus V_{2m+1}$, such that $f|_\Omega = u$. That function f is such that $f(\bar z)=\overline{f(z)}$ and if we denote by $\partial/\partial n$ the normal derivative to Ω, we have for any μ in $\{0\}\cup\mathbf{N}$*

$$\left(\frac{\partial}{\partial n}\right)^{2\mu} f\,\Big|_\Omega = \frac{(-1)^\mu}{(2m-1)^{2\mu}}\Delta^\mu u,\quad \left(\frac{\partial}{\partial n}\right)^{2\mu+1}\Big|_\Omega = \frac{(-1)^{\mu+1}}{(2m-1)^{2\mu+1}} e_{2m+1} D\Delta^\mu\ u.$$

Proof. 1° Suppose f is a solution. We use the usual notation

$$z = z_0 + z_1 e_1 + \cdots + z_{2m}e_{2m} + z_{2m+1}e_{2m+1} = x + z_{2m+1}e_{2m+1}.$$

Thus we have

$$\left(\frac{\partial}{\partial n}\right)^j f = \left(\frac{\partial}{\partial z_{2m+1}}\right)^j f.$$

Since f is left analytic Cliffordian, we have

$$\frac{\partial}{\partial z_{2m+1}} f = \big(e_{2m+1}\mid \nabla_{2m+1}\big) f$$

and thus

$$\left(\frac{\partial}{\partial n}\right)^j f = \big(e_{2m+1}\mid \nabla_{2m+1}\big)^j f.$$

Note that $e_{2m+1} = (-1)^m e_{1\,2\cdots 2m2m+1} e_{1\,2\cdots 2m}$ and that $(-1)^m e_{1\,2\cdots 2m2m+1}$ belongs to the center of $\mathbf{R}_{0,2m+1}$. Using Proposition 10 we get

$$\left(\frac{\partial}{\partial n}\right)^j f = \big((-1)^m e_{1,2\cdots 2m2m+1}\big)^j (e_{1\,2\cdots 2m}\mid \nabla_{2m+1})^j f.$$

Taking the restriction to $z=x\in\Omega$, we get

$$\left(\frac{\partial}{\partial n}\right)^j f|_\Omega(x) = (-1)^{mj}(e_{1\,2\cdots 2m2m+1})^j (e_{1\,2\cdots 2m}\mid \nabla_{2m})^j u(x).$$

Proposition 12 gives us then for $j=2\mu$ and $j=2\mu+1$

$$\left(\frac{\partial}{\partial n}\right)^{2\mu} f|_\Omega(x) = \frac{(-1)^\mu}{(2m-1)^{2\mu}}\Delta^\mu u(x)$$

and

$$\left(\frac{\partial}{\partial n}\right)^{2\mu+1} f|_\Omega(x) = \frac{(-1)^{\mu+1}}{(2m-1)^{2\mu+1}} e_{2m+1} D\Delta^\mu u(x).$$

2° Using the above conditions for $j\le 2m$, the theorem of Cauchy–Kowalewski proves the existence of $\Lambda'\subset\mathbf{R}\oplus V_{2m+1}$ with $\Lambda'\cap(\mathbf{R}\oplus V_{2m})=\Omega$ and the existence and unicity of f in Λ'.

3° Knowing the existence of f, the usual Taylor formula gives us

$$f(z) = \sum_{\mu=0}^{\infty} \frac{(-1)^\mu}{(2m-1)^{2\mu}(2\mu)!}(z_{2m+1})^{2\mu}\Delta^\mu u(x)$$
$$+ e_{2m+1}\sum_{\mu=0}^{\infty} \frac{(-1)^{\mu+1}}{(2m-1)^{2\mu+1}(2\mu+1)!}(z_{2m+1})^{2\mu+1}D\Delta^\mu u(x).$$

This formula shows, since $Du(x) \in \mathbf{R}_{0,2m}$, that $f(\bar{z}) = \overline{f(z)}$ in a subdomain Λ of $\mathbf{R} \oplus V_{2m+1}$ such that $\Omega \subset \Lambda \subset \Lambda'$. □

VI. Fueter's method

Fueter's method is well known and widely used to construct functions connected to monogenic functions [Fu], [De], [Qi]. It is known to be effective to construct holomorphic Cliffordian functions in the case of odd n. We show that it is still valid for our definition in the case of even n.

Theorem. *Let φ be a complex holomorphic function defined on D_φ an open subset of the upper half-plane and let p and q be the real functions of two variables defined by*

$$\forall\, \zeta = xi + i\eta \in D_\varphi, \qquad \varphi(\zeta) = p(\xi,\eta) + iq(\xi,\eta).$$

Let $\vec{z} = z_1e_1 + \cdots + z_ne_n$, for $z_0 + i|\vec{z}\,| \in D_\varphi$ we define $u(z_0 + \vec{z}\,)$ by

$$u(z_0+\vec{z}\,) = p(z_0, |\vec{z}|\,) + \frac{\vec{z}}{|\vec{z}\,|}q(z_0, |\vec{z}\,|).$$

Then u is a (left and right) holomorphic Cliffordian function.

Proof. For odd n, this result is already known [LR1].
If n is even, let $n = 2m$. Let $x = z_0 + \vec{z}$ and $z = x + z_{2m+1}e_{2m+1}$. Define f by

$$f(z) = p(z_0, |\vec{z} + z_{2m+1}e_{2m+1}|) + \frac{\vec{z} + z_{2m+1}e_{2m+1}}{|\vec{z} + z_{2m+1}e_{2m+1}|}q(z_0, |\vec{z} + z_{2m+1}e_{2m+1}|).$$

From the case of odd n, we know that f is a left and right holomorphic Cliffordian function. The theorem of Section IV shows then that u is a left and right holomorphic Cliffordian function, since we have $f(\bar{z}) = \overline{f(z)}$ and $u(x) = f(x)$. □

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