Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 29, 2004, 307–328

HALF-TWISTS AND EQUATIONS IN GENUS 2

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Abstract. The uniformization problem is to find equations for the algebraic curve associated to a given hyperbolic surface. If one can describe corresponding group actions both on the spaces of algebraic curves and hyperbolic surfaces, the whole orbits can be uniformized at the same time. We study here the action of a group generated by half-twists on the space of hyperbolic surfaces of genus 2 with a non-trivial involution and describe the corresponding action on the equations for the corresponding algebraic curves.

1. Introduction

The uniformization theorem of Poincaré and Koebe allows to assert that any compact connected Riemann surface of genus $g > 1$ is conformally equivalent to a quotient of the upper half-plane $\mathbf{H}$ by a Fuchsian group, i.e. a discrete subgroup of $\mathrm{PSL}_2(\mathbf{R})$. On the other hand, a Riemann surface is also an algebraic curve defined by an equation. The classical uniformization problem is to relate explicitly the two descriptions.

In this context, a new approach to tackle this question was initiated in [6], and developed in [2], [8] and [1].

It consists first in working inside families of surfaces with the idea that surfaces tiled with the same pattern by the same type of polygon must have equations of the same form. Then, in defining groups actions on those families. The two groups, one acting on the hyperbolic surfaces and the other on the algebraic curves, are not necessarily the same but there exists a correspondence between their actions. With this approach, whole orbits can be uniformized at the same time.

In this article, we generalize the D_5-action on the space of real genus 2 M-curves with a real involution $\mathscr{M}_{\mathbf{R}}^{(2,3,0)}(\mathbf{Z}/2\times\mathbf{Z}/2)$ described by Buser and Silhol in [6] to the complex family F_2 of those Riemann surfaces having a non-trivial involution.

There are, nevertheless, two major differences in the way we tackle the problem in this paper. The first difference is in the method: we use a quotient, namely the Riemann sphere ramified over 5 points, while the authors considered coverings to define the D_5-action in [6]. Secondly, the complex situation is less rigid than the

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2000 Mathematics Subject Classification: Primary 30F10; Secondary 14H15, 32G15.

Partially supported by the Swiss National Science Foundation, Contract 20-68181.02.

real one, in particular in [6] the hyperbolic description was guided by the natural choice of a pants decomposition and Fenchel–Nielsen coordinates given by the real structures. Here, we have to work with marked Riemann surfaces in the Teichmüller space for the hyperbolic description of our group action while the algebraic one is made with unmarked Riemann surfaces in the moduli space. In particular, the two groups are different. The first one, G_{Q}, is a group of transformations of a special type of hyperbolic quadrilateral and can be identified with the Teichmüller modular group of the sphere with 5 points removed. The second is the symmetric group S_5, giving the 5 points on the quotient a symmetric role, and naturally appears as a quotient of G_{Q}. These two actions correspond and we give here this correspondence in terms of equations and of generators for the Fuchsian groups (see Theorem 4.6 and Table 3).

The fact that the two groups are different means that G_{Q} intersects the Teichmüller modular group of genus two surfaces and thus allows to interpret this difference in terms of Dehn twists.

But more interestingly and surprisingly, the action of the whole group G_{Q} can be interpreted in terms of half-twists (see Theorem 5.1). Thus, by merging Theorems 4.6 and 5.1, we obtain the main result of this paper which can be expressed as the following.

Theorem. *Let S be a genus 2 Riemann surface having a non-trivial involution φ.*

Then, on the one hand, the Fuchsian group has a set of generators of the form

$$(e_1e_3)^2, e_3e_2e_1, e_1e_3e_2, e_3e_4e_1, e_1e_3e_4,$$

where $e_i, \in \mathrm{PSL}_2(\mathbf{R})$, $i = 1, \ldots, 4$, with $\mathrm{tr}(e_i) = 0$, and $\mathrm{tr}\big(\prod_{i=1}^4 e_i\big) = 0$, and on the other hand the underlying algebraic curve has a normalized equation of the form

$$y^2 = (x^2 - 1)(x^2 - a)(x^2 - b).$$

For such a surface S, the half-twists along certain geodesics lead to surfaces having a non-trivial involution. Sets of generators for the Fuchsian groups and normalized equations for these surfaces can be explicitly deduced from those of S.

We have, in particular, the correspondence between half-twists and changes of equation given in Table 1.

The action of $G_{\mathrm{Q}} = \langle \eta_1, \ldots, \eta_4 \rangle$ for η_i, $i = 1, \ldots, 4$ as in Table 1, *induces an action of S_5 on the underlying algebraic curves.*

	word representing the (oriented) geodesic under which the half-twist is made	ordered e_i's for the Fuchsian group	parameters for equations
	none	(e_1, e_2, e_3, e_4)	(a, b)
η_1	$(e_3e_2)^2$	$(e_1, e_3, e_3e_2e_3, e_4)$	$(1-a, b(1-a)/(b-a))$
η_2	$(e_4e_3)^2$	$(e_1, e_2, e_4, e_4e_3e_4)$	$(a(1-b)/(a-b), (1-b))$
η_3	$(e_1e_4)^2$	$(e_1e_4e_1, e_2, e_3, e_1)$	$((b-a)/(b-1), b/(b-1))$
η_4	$(e_2e_3e_4)^2$	$(e_2e_3e_4e_1, e_2, e_3, e_4)$	$(1-a, 1-b)$

Table 1.

2. Notation and preliminaries

We recall briefly some classical definitions and notation.

Definition 2.1. Let S be a Riemann surface of genus 2, and τ be the hyperelliptic involution on S.

An automorphism $\varphi \in \mathrm{Aut}(S)$, with $\varphi \neq \tau$ is said to be *non-trivial.*

For any hyperelliptic Riemann surface S, the hyperelliptic involution τ is in the center of $\mathrm{Aut}(S)$. The *reduced automorphism group of* S is then

$$\mathrm{Aut}^r(S) = \mathrm{Aut}(S)/\tau.$$

The classification of Riemann surfaces of genus two in terms of their reduced automorphism group is due to Bolza ([4]). It is summarized Table 2 as well as the inclusions between families.

Except for F_5, every Riemann surface of genus two having a non-trivial automorphism has at least one non-trivial involution, and then belongs to F_2.

Let S be a Riemann surface, τ the hyperelliptic involution and φ a non-trivial involution on S. The involutions φ and $\varphi\tau$ have two fixed points, say p_1 and p_2 for φ and q_1 and q_2 for $\varphi\tau$, that satisfy

$$\varphi(q_1) = q_2, \qquad \tau(p_1) = p_2.$$

Lemma 2.2. *Let S, τ, φ as before. The covering $p_\varphi\colon S \longrightarrow S/\langle\varphi,\tau\rangle \simeq \mathbf{P}^1(\mathbf{C})$ is ramified over 5 points among which 3 are the images of the Weierstrass points and the last two are $p = p_\varphi(p_1) = p_\varphi(p_2)$ and $q = p_\varphi(q_1) = p_\varphi(q_2)$. Those five points and the triple of them that lift to the set of Weierstrass points determine S completely.*

Proof. This follows directly from the fact that the surfaces of genus 2, are, as all the hyperelliptic algebraic curves, determined by their Weierstrass points. ◻

Family	Aut^r	Classical form for the equation
F_2	$\mathbf{Z}/2\mathbf{Z}$	$y^2 = (x^2-1)(x^2-a)(x^2-b)$
F_4	D_2	$y^2 = (x^2-1)(x^2-a)(x^2-1/a)$
F_6	D_3	$y^2 = x^6 - 2\,a\,x^3 + 1$
F_{12}	D_6	$y^2 = x^6 + 1$
F_{24}	S_4	$y^2 = x(x^4-1)$
F_5	$\mathbf{Z}/5\mathbf{Z}$	$y^2 = x^5 - 1$

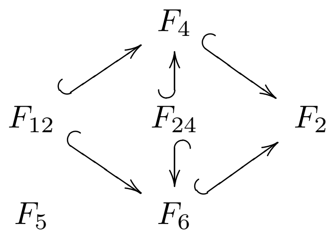


Table 2.

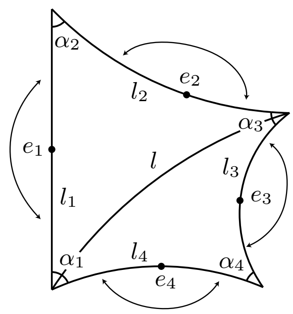
Corollary 2.3. *Let $r_1,\ldots,r_5$ be five distinct points on $\mathbf{P}^1(\mathbf{C})$. There exist at most* 10 *different surfaces S_j, τ_j, φ_j, $j=1,\ldots,10$, in the family F_2 such that the coverings p_{φ_j} are ramified over the r_i's.*

Proof. Each S_j correspond to the choice of a triple $\{r_k,r_l,r_m\}$ of points among the r_i's that lift to the Weierstrass points of S_j. □

3. Marked quadrilaterals

Let S, τ, φ as before. The hyperbolic structure on S induces, via p_φ, a structure of hyperbolic sphere with five cone points of angle π on the quotient $S/\langle\tau,\varphi\rangle$.

Such a surface can always be obtained by pasting the sides of a hyperbolic quadrilateral with interior angles adding up to π, as on Figure 1.



$$\alpha_1+\alpha_2+\alpha_3+\alpha_4=\pi$$

Figure 1.

This observation induces a particular presentation for the Fuchsian group of such a surface and motivates the following definitions.

Definition 3.1. An ordered system $Q = (e_1, e_2, e_3, e_4)$, $e_i \in \mathrm{PSL}_2(\mathbf{R})$, is a *marked quadrilateral* if it satisfies the following conditions:

(i) $\mathrm{tr}(e_i) = 0$, $i = 1, \dots, 4$.
(ii) $\mathrm{tr}\bigl(\prod_{i=1}^{4} e_i\bigr) = 0$.
(iii) The e_i's are positioned clockwise around the quadrilateral of vertices are the fixed points of $e_1e_2e_3e_4$, $e_2e_3e_4e_1$, $e_3e_4e_1e_2$ and $e_4e_1e_2e_3$ (see Figure 2).

Remark 3.2. Using trace relations, one can easily show that the quadrilateral of (iii) above is a convex domain, delimited by the axes of the hyperbolic transformations $e_1e_2e_3$, $e_2e_3e_4$, $e_3e_4e_1$, $e_4e_1e_2$. As it is uniquely determined we will also denote it by Q.

Definition 3.3. We will denote by Q be the set of all marked quadrilaterals modulo the relation

$$(e_1, e_2, e_3, e_4) \sim (e_1', e_2', e_3', e_4') \iff \exists\, \gamma \in \mathrm{PSL}_2(\mathbf{R}),\ \mathrm{e_i'} = \gamma \mathrm{e_i} \gamma^{-1},\ \mathrm{i} = 1, \dots 4.$$

Definition 3.4. Given a surface S_0 of signature $(0; 2, 2, 2, 2, 2)$, a *quadrilateral fundamental domain for* S_0 is a marked quadrilateral $Q = (e_1, e_2, e_3, e_4)$ such that $\Gamma_0(Q) = \langle e_1, e_2, e_3, e_4 \rangle$ is a Fuchsian group for S_0.

We will denote by Q_{S_0} the set of all quadrilateral fundamental domains for S_0 under the relation $\sim$.

Conversely, given $Q \in \mathrm{Q}$, will denote by $S_0(Q)$ the surface $\mathbf{H}/\Gamma_0(Q) = \mathbf{H}/\langle e_1, e_2, e_3, e_4 \rangle$.

Remarks 3.5. 1. As the e_i's are elliptic transformations of order 2, they completely determine their fixed points. So we may also denote by e_i the fixed point of e_i (notably on figures).

2. As a marked quadrilateral is defined up to direct isometry and separates into two triangles, it is also characterized by the following set of five lengths:

– the lengths l_i of the i-th sides given by:

$$\cosh\bigl(\tfrac{1}{2} l_i\bigr) = \tfrac{1}{2} |\,\mathrm{tr}(e_{i+1}e_{i+2}e_{i+3})|,$$

– the length l of the first diagonal given by

$$\cosh\bigl(\tfrac{1}{2} l\bigr) = \tfrac{1}{2} |\,\mathrm{tr}(e_1e_2e_3e_4e_3e_4e_1e_2)| = \tfrac{1}{2} |(\mathrm{tr}(e_1e_2))^2 + (\mathrm{tr}(e_3e_4))^2 - 2|.$$

3.1. Transformations of a marked quadrilateral

Definition 3.6. (1) We define the following transformations on Q:
(i) the circular permutation:

$$\sigma_0\colon (e_1, e_2, e_3, e_4) \longrightarrow (e_2, e_3, e_4, e_1),$$

(ii) σ_1 (see Figure 2):

$$\sigma_1(e_1, e_2, e_3, e_4) \longrightarrow (e_3, e_2, e_3e_4e_1e_2, e_3e_4e_3).$$

(2) We denote by G_{Q}, the group

$$G_{\mathrm{Q}} = \langle \sigma_0, \sigma_1 \rangle$$

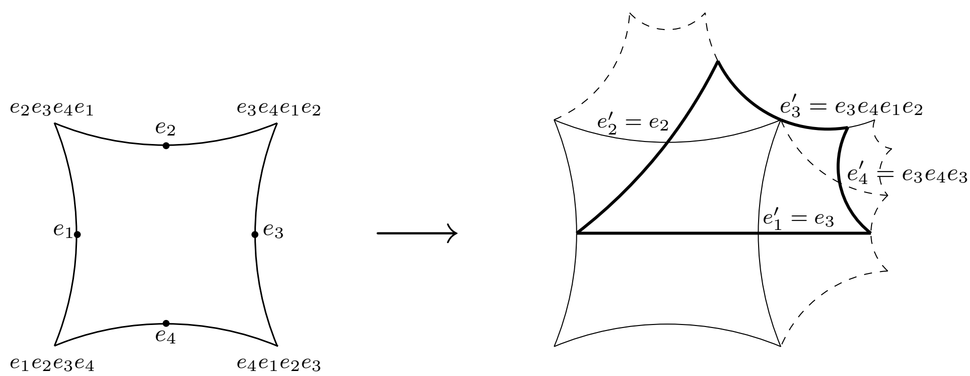


Figure 2. The transformation σ_1.

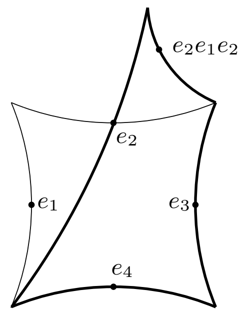
Remarks 3.7. Let S_0 be a hyperbolic surface of genus 0 with five cone points of angle π. Then σ_0 and σ_1 preserve Q_{S_0}.

The transformations σ_0 and σ_1 are of different nature. While σ_0 only operates on the marking of Q but leaves the unmarked quadrilateral unchanged, σ_1 is mainly devoted to changing the choice of the point among the five cone points on the sphere S_0 which correspond to the vertices of Q.

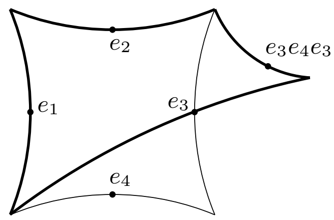
For further use, we introduce the following transformations in G_{Q}:

Note that σ_2 and σ_3 are of infinite order and that they do not have fixed points in G_{Q}.

Proposition 3.8. *Let S_0 be a hyperbolic surface of signature $(0; 2, 2, 2, 2, 2)$. G_{Q} acts transitively on Q_{S_0}.*



$$\sigma_2 = \sigma_0^2(\sigma_1^2\sigma_0^3)^3\sigma_0^2$$
$$\sigma_2(e_1, e_2, e_3, e_4) = (e_2, e_2e_1e_2, e_3, e_4)$$



$$\sigma_3 = (\sigma_0\sigma_1)^3$$
$$\sigma_3(e_1, e_2, e_3, e_4) = (e_1, e_2, e_3e_4e_3, e_3)$$

Proof. Let Q and Q' be two quadrilaterals of Q_{S_0}. We build a sequence of quadrilaterals of Q_{S_0} using elements of G_{Q}.

Let $a_1, \ldots, a_4$ be the four geodesic arcs in S_0 corresponding to the sides of Q. The a_i's are oriented such that they all have the same source.

Start, if necessary, with a transformation of the form $\sigma_1\sigma_0^k$ leading to a quadrilateral Q_1, such that the vertices of Q and Q_1 correspond to the same point of S_0.

For each a_i, consider the number k_{i,Q_1} of connected components of $a_i \cap \overset{\circ}{Q}_1$. Let $a_{1,1}, \ldots, a_1, k_{1,Q_1}$ be the corresponding connected component.

We cut S_0 along the sides of Q_1.

We treat the a_i's in the order given by the marking.

For a_1:

If $k_{1,Q_1} = 0$, then a_1 corresponds to one of the sides of Q_1, and we go to a_2.

If $k_{1,Q_1} \neq 0$, we give the arcs $a_{1,1}, \ldots, a_{1,k_{1,Q_1}}$ the a_1's orientation.

Using a transformation of the form σ_0^k, we get a quadrilateral Q_2 such that the source of $a_{1,1}$ in Q_2 is at the intersection of the first and the fourth sides. Its end point is then necessarily on the second or the third side of Q_2. Using σ_2 in the first case and σ_3 in the second one, we build a quadrilateral Q_3 such that the number k_{1,Q_3} of connected components of $a_1 \cap \overset{\circ}{Q}_3$ is strictly smaller than $k_{1,Q_2} = k_{1,Q_1}$. Assume that one of the $a_{1,k}$'s, say a_{1,k_0} penetrates into the triangle of sides a, b, c, where a is a part of $a_{1,1}$, b is a part of the side of Q_3 which is not a side of Q_3, and c is a part of the side of Q_2 which is not a side of Q_3. Then, the arc a_{1,k_0} must leave this triangle through b or c, the $a_{1,k}$'s being disjoint. Thus, the arc is at the same time cut and pasted once. Therefore the number of connected components does not increase. As this situation is possible only in the case $k_0 > 1$, we have $k_{1,Q_3} < k_{1,Q_2}$.

We treat similarly the path a_2, a_3, a_4, each construction respecting the preceding ones. □

Lemma 3.9. *Given a generic surface* S_0 *of signature* $(0;2,2,2,2,2)$, G_{Q} *operates without fixed points on* Q_{S_0}.

Proof. Let $Q \in \mathrm{Q}_{S_0}$ and $\sigma \in G_{\mathrm{Q}}$ such that $\sigma \cdot Q = Q$. Then σ induces an isometry on S_0. □

3.2. Identification of G_{Q} with the modular group $\Gamma_{0,5}$. The group G_{Q} acts on sets of four generators of Fuchsian groups of a sphere with five punctures. It is then naturally linked with the modular group of a sphere with five points removed, $\Gamma_{0,5}$. More precisely, given a transformation $\sigma \in G_{\mathrm{Q}}$, one can associate the isotopy class of h_σ to σ, where h_σ is a homeomorphism of the sphere preserving the set of five points and corresponding to the deformation from any quadrilateral Q to $\sigma(Q)$ (the quadrilateral being simply connected) mapping the interior onto the interior and the i-th side onto the i-th side.

Define the following transformations of G_{Q}:

$$\begin{aligned}
\eta_1\colon (e_1,e_2,e_3,e_4) &\longrightarrow (e_1,e_3,e_3e_2e_3,e_4),\\
\eta_2\colon (e_1,e_2,e_3,e_4) &\longrightarrow (e_1,e_2,e_4,e_4e_3e_4),\\
\eta_3\colon (e_1,e_2,e_3,e_4) &\longrightarrow (e_1e_4e_1,e_2,e_3,e_1),\\
\eta_4\colon (e_1,e_2,e_3,e_4) &\longrightarrow (e_2e_3e_4e_1,e_2,e_3,e_4).
\end{aligned}$$

Then, with the topological interpretation below, if $r_1,\dots,r_4$ are the points of $S_0(Q)$ corresponding to the middle of the sides and r_5 is the point corresponding to the vertices, each η_i corresponds to an homeomorphism φ_i where φ_i is the identity outside a disk D_i enclosing r_{i+1} and r_{i+2} (subscript modulo 5) and φ_i exchanges r_{i+1} and r_{i+2}. According to J.S. Birman (see [3, Theorem 4.5, p. 164 and Remark, p. 165]), this means that the set of geometric transformations $\{\eta_1,\eta_2,\eta_3,\eta_4\}$ is a set of generators for $\Gamma_{0,5}$ with the following full list of relations:

$$\begin{aligned}
\eta_i\eta_j &= \eta_j\eta_i, \qquad |i-j| \geqslant 2,\\
\eta_i\eta_{i+1}\eta_i &= \eta_{i+1}\eta_i\eta_{i+1},\\
\eta_1\eta_2\eta_3\eta_4^2\eta_3\eta_2\eta_1 &= 1,\\
(\eta_1\eta_2\eta_3\eta_4)^5 &= 1.
\end{aligned}$$

We remark without expanding the computations that the following correspondence between the σ_i's and the η_i's allows one to find a full list of relations for

$\Gamma_{0,5}$ with the minimal set of generators $\{\sigma_0, \sigma_1\}$:

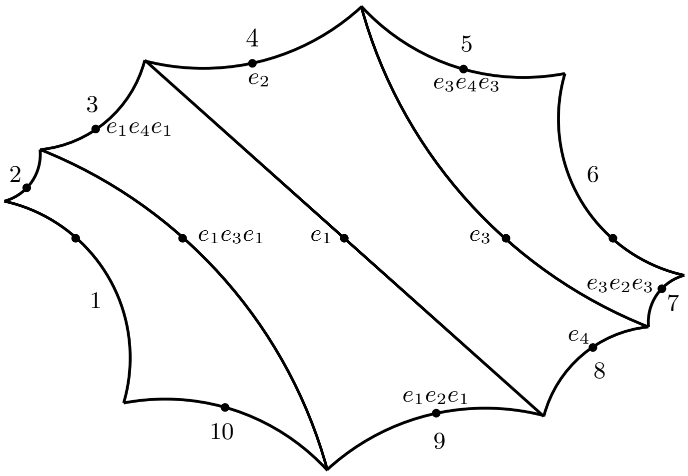
$$\begin{aligned}
\eta_1 &= \sigma_0(\sigma_1^2\sigma_0^3)^3\sigma_0^3, \\
\eta_2 &= (\sigma_1^2\sigma_0^3)^3, & \sigma_0 &= \eta_4^2\eta_3\eta_2\eta_1, \\
\eta_3 &= \sigma_0^3(\sigma_1^2\sigma_0^3)^3\sigma_0, & \sigma_1 &= \eta_2^{-1}\eta_3^{-1}\eta_4^{-2}\eta_2^{-1}\eta_4, \\
\eta_4 &= \sigma_0^2\sigma_3\sigma_2\sigma_0^3\sigma_1.
\end{aligned}$$

4. The genus two coverings

4.1. The surface S_Q. Given $Q \in \mathrm{Q}$, we construct a genus 2 cover S_Q of the sphere $S_0(Q)$ as shown in Figure 3. In other words, we construct a set of generators for a Fuchsian group Γ_Q of signature $(2,0)$, namely $\Gamma_Q = \langle (e_1e_3)^2, e_3e_2e_1, e_1e_3e_2, e_3e_4e_1, e_1e_3e_4 \rangle$, from the generators (e_1,e_2,e_3,e_4) of the Fuchsian group $\Gamma_0(Q)$ of $S_0(Q)$. Note that the Weierstrass points of the surface S_Q correspond to the conjugacy classes in Γ_Q of the centers of the elliptic transformations

$$e_2,\ e_1e_4e_1,\ e_4,\ e_1e_2e_1,\ e_1e_2e_3e_4,\ e_2e_3e_4e_1$$

(and to the middles of the sides labelled 2, 3, 4, 5, 7, 8, 9, 10 and the vertices of the polygon on Figure 3).



1—6, 2—4, 3—5, 7—9, 8—10

Figure 3.

Note also that S_Q has two non-trivial involutions, the fixed points of the first being the conjugacy classes of e_1, and $e_3e_1e_3$, those of the second being e_3 and $e_1e_3e_1$.

By analogy with Lemma 2.2, we have:

Proposition 4.1. *Let S be a surface of F_2 with a non-trivial involution φ and $S_0 = S/\langle\varphi,\tau\rangle$. Choose $Q \in \mathrm{Q}_{S_0}$ such that the images p and q of the fixed points of φ and $\varphi\tau$ via p_φ are on the first and the third side of Q.*

Then S is isometric to S_Q.

Proposition 4.2. *The map $\mathrm{Q} \ni Q \longmapsto S_Q$ is an injection of Q into the Teichmüller space of Riemann surfaces of genus 2, $\mathscr{T}_2$.*

Proof. We choose $Q_0 = (e_1^0, e_2^0, e_3^0, e_4^0) \in \mathrm{Q}$ as a quadrilateral of reference. As a quadrilateral is simply connected, for any $Q \in \mathrm{Q}$, there exists, up to isotopy, a unique homeomorphism $\eta_Q \colon Q_0 \longrightarrow Q$ such that $\eta_Q \circ e_i^0 = e_i$.

This condition ensures that η_Q is extendable to a homeomorphism $\hat{\eta}_Q$ such that

$$\begin{array}{ccc} Q_0 & \xrightarrow{\eta_Q} & Q_0 \\ \big\downarrow & & \big\downarrow \\ S_{Q_0} & \xrightarrow{\hat{\eta}_Q} & S_Q \end{array}$$

is commutative.

The couple $(S_Q, \hat{\eta}_Q)$ is then a marked Riemann surface.

The injectivity follows from the construction of $\hat{\eta}_Q$. □

We will denote

$$F_2(\mathrm{Q}) = \{S_Q, Q \in \mathrm{Q}\} \subset \mathscr{T}_2.$$

Corollary 4.3. *The induced action of G_Q on $F_2(\mathrm{Q})$ is generically fixed point free.*

Remark 4.4. While F_2 is a subspace of the space of isometry classes of Riemann surfaces of genus 2, $\mathscr{M}_2$, $F_2(\mathrm{Q})$ is a subspace of the Teichmüller space of genus 2, $\mathscr{T}_2$. It is well known that the moduli space $\mathscr{M}_2$ is a quotient of $\mathscr{T}_2$ by the modular group, generated by Dehn twists. We will be concerned by this point of view in Section 5.

4.2. Equations for surfaces in $F_2(\mathrm{Q})$—induced action on F_2. The classical normalization of the equations for the surfaces of F_2 under the form $y^2 = P(x)$ is the one given in Table 2. More precisely, given a surface $S \in F_2$, we choose the involutions φ and $\varphi\tau$ so that they lift $x \longmapsto -x$. We also impose that one of the Weierstrass points has coordinates $(1,0)$.

This choice is equivalent to the choice of a (global) coordinate x on the quotient $S/\langle\varphi,\tau\rangle$ such that

(1) the images of the fixed points of φ and $\varphi\tau$ via p_φ are mapped onto 0 and ∞,

(2) the image of a pair of the Weierstrass points exchanged by φ and $\varphi\tau$ is mapped via p_φ onto 1.

Given such an x, the remaining two points of $S/\langle\varphi,\tau\rangle$ are mapped upon a and b and

$$y^2 = (x^2 - a)(x^2 - 1)(x^2 - b) \tag{4.4}$$

is an equation of S.

Note that conditions (1) and (2) do not determine precisely the choice of the coordinate x, since x/a, x/b, $1/x$, a/x, and b/x would also fufill them.

As we want to describe the action of G_{Q} on $F_2(\mathrm{Q})$ in terms of equations, we will, given a surface S_Q, make the choice of x precise by taking into account the geometry of Q as follows.

Let $Q \in \mathrm{Q}$ and let $S_0 = S_0(Q)$ the genus 0 surface obtained by gluing the sides of Q. We then choose a coordinate x_Q depending on the position of the cone points of S_0 on Q. We denote by $r_{1,Q}, \ldots, r_{5,Q}$ these points in the order given by the marking of Q (in other words, if $Q = (e_1, e_2, e_3, e_4)$, then $r_{i,Q}$, $1 \leqslant i \leqslant 4$ is the conjugacy class of e_i in $\Gamma_0(Q) = \langle e_1, e_2, e_3, e_4\rangle$, and $r_{5,Q}$ is the conjugacy class of $e_1e_2e_3e_4$).

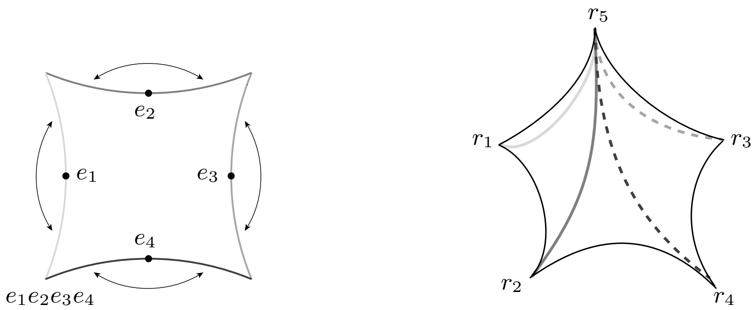


Figure 4.

More precisely, we choose x_Q such that:

$$x_Q(r_{1,Q}) = 0, \qquad x_Q(r_{3,Q}) = \infty, \qquad x_Q(r_{5,Q}) = 1. \tag{4.5}$$

We then call the couple

$$(a,b) = \big(x_Q(r_{2,Q}), (x_Q(r_{4,Q})\big)$$

normalized equation parameters for S_Q.

Let now $Q \in \mathrm{Q}$, and $\sigma \in G_{\mathrm{Q}}$. As $S_0(Q)$ and $S_0\big(\sigma(Q)\big)$ are isometric, we have

$$\{r_{1,Q}, \ldots, r_{5,Q}\} = \{r_{1,\sigma(Q)}, \ldots, r_{5,\sigma(Q)}\}.$$

This means that σ acts as a permutation on the set of the cone points of S_0. More precisely, we associate to σ the permutation $\bar\sigma \in S_5$ defined for all $Q \in \mathrm{Q}$ and $i \in \{1,\dots,5\}$ by

$$r_{i,\sigma(Q)} = r_{\bar\sigma^{-1}(i),Q}.$$

The map $\sigma \longmapsto \bar\sigma$ is a group homeomorphism as we have for all $Q \in \mathrm{Q}$ and $i \in \{1,\dots,5\}$

$$r_{i,\sigma'\sigma(Q)} = r_{\bar\sigma'^{-1}(i),\sigma(Q)} = r_{\bar\sigma^{-1}\bar\sigma'^{-1}(i),\sigma(Q)} = r_{\overline{(\sigma'\sigma)}^{-1}(i),\sigma(Q)}.$$

The image is the subgroup of S_5 generated by the images of the generators σ_0 and σ_1, for which we have

$$\sigma_0.(r_{1,Q}, r_{2,Q}, r_{3,Q}, r_{4,Q}, r_{5,Q}) = (r_{2,Q}, r_{3,Q}, r_{4,Q}, r_{1,Q}, r_{5,Q});\ \text{thus}\ \bar\sigma_0 = (4,3,2,1),$$
$$\sigma_1.(r_{1,Q}, r_{2,Q}, r_{3,Q}, r_{4,Q}, r_{5,Q}) = (r_{3,Q}, r_{2,Q}, r_{5,Q}, r_{4,Q}, r_{1,Q});\ \text{thus}\ \bar\sigma_1 = (1,5,3).$$

The map is surjective, since $(4,3,2,1)$ and $(1,5,3)$ together generate S_5. Its kernel is the subgroup H_{Q} of those transformations such that for all $Q \in \mathrm{Q}$ and $i \in \{1,\dots,5\}$

$$r_{i,Q} = r_{i,\sigma(Q)}.$$

The normalized equation parameters for S_Q depend only on the position of the cone points on Q. Then, for $\sigma \in G_{\mathrm{Q}}$, it is $\bar\sigma \in S_5$ rather than σ that acts on them. This action is as follows: the coordinates x_Q and $x_{\sigma(Q)}$ on $S_0(Q) = S_0(\sigma(Q))$ are exchanged by the unique transformation $A_{\bar\sigma,Q}$ of $\mathbf{P}^1$ mapping $x_Q(r_{1,\sigma(Q)})$ onto 0, $x_Q\big(r_{3,\sigma(Q)}\big)$ onto ∞ and $x_Q\big(r_{1,\sigma(Q)}\big)$ onto 1, i.e., $A_{\bar\sigma,Q}$ is defined by:

$$\begin{aligned} A_{\bar\sigma,Q}(z) &= \left(\frac{z - x_Q(r_{1,\sigma(Q)})}{z - x_Q(r_{3,\sigma(Q)})}\right)\left(\frac{x_Q(r_{5,\sigma(Q)}) - x_Q(r_{3,\sigma(Q)})}{x_Q(r_{5,\sigma(Q)}) - x_Q(r_{1,\sigma(Q)})}\right) \\ &= \left(\frac{z - x_Q(r_{\bar\sigma^{-1}(1),Q})}{z - x_Q(r_{\bar\sigma^{-1}(3),Q})}\right)\left(\frac{x_Q(r_{\bar\sigma^{-1}(5),Q}) - x_Q(r_{\bar\sigma^{-1}(3),Q})}{x_Q(r_{\bar\sigma^{-1}(5),Q}) - x_Q(r_{\bar\sigma^{-1}(1),Q})}\right) \end{aligned}$$

and we have

$$\begin{aligned} \bar\sigma.\big(x_Q(r_2,Q), x_Q(r_4,Q)\big) &= \big(x_\sigma(Q)\big(r_2,\sigma(Q)\big), x_\sigma(Q)\big(r_4,\sigma(Q)\big)\big) \\ &= \big(A_{\bar\sigma,Q}\big(x_Q(r_{2,\sigma(Q)})\big), A_{\bar\sigma,Q}\big(x_Q(r_{4,\sigma(Q)})\big)\big) \\ &= \big(A_{\bar\sigma,Q}\big(x_Q(r_{\bar\sigma^{-1}(2),Q})\big), A_{\bar\sigma,Q}\big(x_Q(r_{\bar\sigma^{-1}(4),Q})\big)\big). \end{aligned}$$

For the generators σ_0 and σ_1 of G_{Q}, and a couple (a,b) of normalized equation parameters, we have

$$\begin{aligned} \bar\sigma_0 &= (4,3,2,1), & \bar\sigma_1 &= (1,5,3), \\ A_{\bar\sigma_0}(z) &= \frac{z-a}{z-b}\frac{1-b}{1-a}, & A_{\bar\sigma_1}(z) &= \frac{1}{1-z}, \\ \bar\sigma_0.(a,b) &= \left(\frac{1-b}{1-a}, \frac{a\,(1-b)}{b\,(1-a)}\right), & \bar\sigma_1.(a,b) &= \left(\frac{1}{1-a}, \frac{1}{1-b}\right). \end{aligned}$$

We have thus proved:

Theorem 4.6. *The action of G_{Q} on $F_2(\mathrm{Q})$ $(\subset \mathscr{T}_2)$ induces an action of the symmetric group S_5 on F_2 given in terms of parameters of equations by the generators*

$$\bar{\sigma}_0\colon (a,b) \longmapsto \left(\frac{1-b}{1-a}, \frac{a\,(1-b)}{b\,(1-a)}\right), \qquad \bar{\sigma}_1 : (a,b) \longmapsto \left(\frac{1}{1-a}, \frac{1}{1-b}\right).$$

According to Corollary 2.3, given a generic surface of F_2, its quotient under its automorphism group has ten different genus two covers in F_2. A representative of each of these isomorphy classes is given in Table 3.

σ	$\bar{\sigma}$	$\bar{\sigma}.(a,b)$
Id	Id	(a,b)
σ_0	$(4,3,2,1)$	$((1-b)/(1-a), a(1-b)/b(1-a))$
σ_1	$(1,5,3)$	$(1/(1-a), 1/(1-b))$
σ_1^2	$(1,3,5)$	$((-1+a)/a, (-1+b)/b))$
$\sigma_1\sigma_0$	$(1,4)(2,5,3)$	$((-1+a)/(-b+a), (b(-1+a)/(-b+a))$
$(\sigma_1\sigma_0)^2$	$(2,3,5)$	$(1/(1-a), -b/(-b+a))$
$(\sigma_1\sigma_0)^3$	$(1,4)$	$((b-a)/(b-1), b/(b-1))$
$\sigma_1^2\sigma_0$	$(2,3)(1,4,5)$	$((b-a)/(b-1), (a-b)/(a(b-1))$
$\sigma_2 = \sigma_0^2(\sigma_1^2\sigma_0^3)^3\sigma_0^2$	$(1,2)$	$(a/(a-1), (a-b)/(a-1))$
$\sigma_3 = (\sigma_0\sigma_1)^3$	$(3,4)$	$(a(1-b)/(a-b), 1-b)$

Table 3.

Remarks 4.7. (1) While the action of G_{Q} on $F_2(\mathrm{Q})$ is generically fixed point free, the induced action of S_5 on F_2 is such that there exists a subgroup of S_5, namely $S_{\{1,3\}} \times S_{\{2,4,5\}}$, that preserves isometry classes.

(2) Call K_{Q} the group of transformations σ such that $\bar{\sigma} \in S_{\{1,3\}} \times S_{\{2,4,5\}}$. It is neither normal in G_{Q} nor contains a normal subgroup of G_{Q}. Then S_5 is the smallest group whose action on the set of genus two Riemann surfaces ramified over 5 points of $\mathbf{P}^1$ is transitive.

Table 4 gives a representative in K_{Q} for each element of $S_{1,3} \times S_{2,4,5}$.

σ	$\bar{\sigma}$	$\bar{\sigma}.(a,b)$	σ	$\bar{\sigma}$	$\bar{\sigma}.(a,b)$
Id	Id	(a,b)	$\sigma_0\sigma_1\sigma_0$	$(1,3)(2,5)$	$(a,a/b)$
$\sigma_2\sigma_0\sigma_2\sigma_0^{-1}\sigma_2$	$(2,4)$	(b,a)	$\sigma_0\sigma_1^2\sigma_0$	$(1,3)(4,5)$	$(b/a,b)$
σ_0^2	$(1,3)(2,4)$	$(1/b,1/a)$	$\sigma_2\sigma_0\sigma_2\sigma_0^{-1}\sigma_2\sigma_0^{-1}\sigma_1\sigma_0$	$(2,5)$	$(a/b,1/b)$
$\sigma_0^2\sigma_2\sigma_0\sigma_2\sigma_0^{-1}\sigma_2$	$(1,3)$	$(1/a,1/b)$	$\sigma_2\sigma_0\sigma_2\sigma_0^{-1}\sigma_2\sigma_0^{-1}\sigma_1^2\sigma_0$	$(4,5)$	$(1/a,b/a)$
$\sigma_0^{-1}\sigma_1\sigma_0$	$(2,5,4)$	$(1/b,a/b)$	$\sigma_2\sigma_0\sigma_2\sigma_0^{-1}\sigma_2\sigma_0\sigma_1\sigma_0$	$(1,3)(2,4,5)$	$(a/b,a)$
$\sigma_0^{-1}\sigma_1^2\sigma_0$	$(2,4,5)$	$(b/a,1/a)$	$\sigma_2\sigma_0\sigma_2\sigma_0^{-1}\sigma_2\sigma_0\sigma_1^2\sigma_0$	$(1,3)(2,5,4)$	$(b,b/a)$

Table 4.

Remark 4.8. In [6] the authors identify the space $\mathscr{P}_2$ of pairs of pants having the lengths of two boundary geodesics equal, with the space $\mathscr{M}_{\mathbf{R}}^{(2,3,0)}(\mathbf{Z}/2\,\mathbf{Z}\times\mathbf{Z}/2\,\mathbf{Z})$ of the real genus 2 curves with 3 real components and whose real automorphisms group contains $\mathbf{Z}/2\,\mathbf{Z}\times\mathbf{Z}/2\,\mathbf{Z}$. More precisely, given a pair of pants P of $\mathscr{P}_2$, they consider on the one hand the surface S_P obtained by gluing 2 copies of P with twist 0 on each component, and on the other hand the real algebraic curve whose real components correspond to the boundary components of P. The surface S_P clearly belong to F_2 as the isometry exchanging the two boundary components of same lengths on P can be extended to an isometry φ of S_P. The involution φ can be normalized in $(x,y)\longmapsto(-x,y)$ and the algebraic curve has equation $y^2=(x^2-a)(x^2-1)(x^2-b)$, $0<a<1<b<1$.

The authors then define (see [6, 5.17]) a D_5-action on $\mathscr{P}_2$ both in terms of the lengths (l_1,l_2) (where a pair of pants has one boundary component of length $2l_1$, and two of lengths $2l_2$) and of the parameters (a,b) of real equation above. We will show that our Theorem 4.6 is in fact a generalization of this result.

Let P be the pair of pants given by length (l_1,l_2). We denote, as in [6], by $\hat{l}_1$ the length of the common perpendicular to the two boundary components of length $2\,l_2$, and by $\hat{l}_2$ that of the common perpendicular arcs to the boundary of length $2\,l_1$ and each of those of length $2\,l_2$. Those arcs cut P into two copies of a right-angled hyperbolic hexagon given by the lengths $(l_1,\hat{l}_2,l_2,\hat{l}_1,l_2,\hat{l}_2)$ in cyclic order. Each copy of this hexagon can be cut into two isometric mirror pentagons along the common perpendicular to the sides of length l_1 and $\hat{l}_1$ (see Figure 5).

The hyperbolic surface S_P is isometric to the surface S_{Q_P} where Q_P is the quadrilateral obtained from P as on Figure 5.

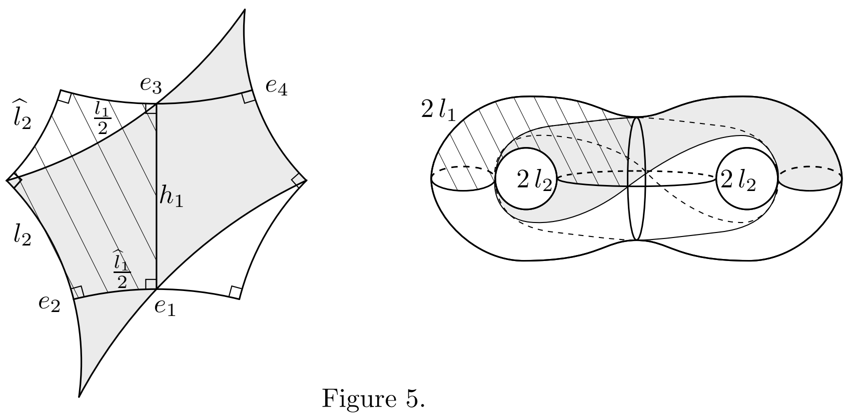


Figure 5.

It is easily shown, using, for example, trigonometric formulas in triangles and trace relations (given in [5]), that Q_P is given by the following trace relations:

$$
(4.8)\qquad
\begin{aligned}
2\,|\operatorname{tr}(e_2e_3)| &= |\operatorname{tr}(e_1e_3)|\,|\operatorname{tr}(e_1e_2)| = 4\,\cosh(h_1)\cosh\big(\tfrac{1}{2}\hat{l}_1\big),\\
2\,|\operatorname{tr}(e_2e_3)| &= |\operatorname{tr}(e_1e_3)|\,|\operatorname{tr}(e_1e_2)| = 4\,\cosh(h_1)\cosh\big(\tfrac{1}{2}\hat{l}_1\big),\\
2\,|\operatorname{tr}(e_2e_3e_4)| &= |\operatorname{tr}(e_1e_2)|\,|\operatorname{tr}(e_3e_4e_1)| = 4\,\cosh\big(\tfrac{1}{2}\hat{l}_1\big)\cosh(l_2),\\
2\,|\operatorname{tr}(e_4e_1e_2e_1)| &= |\operatorname{tr}(e_3e_4e_1)|\,|\operatorname{tr}(e_3e_1e_2)| = 4\,\cosh(l_2)\cosh(\hat{l}_2),\\
2\,|\operatorname{tr}(e_4e_1e_2)\,| &= |\operatorname{tr}(e_3e_4)|\,|\operatorname{tr}(e_3e_1e_2)| = 4\,\cosh(\hat{l}_2)\cosh\big(\tfrac{1}{2}l_1\big),\\
2\,|\operatorname{tr}(e_1e_4)| &= |\operatorname{tr}(e_3e_4)|\,|\operatorname{tr}(e_1e_3)| = 4\,\cosh\big(\tfrac{1}{2}l_1\big)\cosh(h_1).
\end{aligned}
$$

Conversely if a marked quadrilateral $Q = (e_1, e_2, e_2, e_4)$ verifies relations (4.8), one can build from Q the right-angled hexagon as in Figure 5 and thus the pair of pants P. This pair of pants P is then given by the lengths

$$(l_1, l_2) = \big(\operatorname{arccosh}\big(\tfrac{1}{2}|\operatorname{tr}(e_3e_4)|^2 - 1\big), \operatorname{arccosh}\big(|\operatorname{tr}(e_3e_4e_1)|\big)\big).$$

From the algebraic point of view, the fact that the boundary components of P correspond to the real components of the algebraic curve

$$y^2 = (x^2-1)(x^2-a)(x^2-b), \qquad 0 < a < 1 < b$$

ensures that the coordinate induced on the quotient $S_P/\langle\varphi,\tau\rangle$ via $(x,y)\longmapsto x^2$ is in fact the coordinate x_{Q_P} defined by (4.5). In particular (a,b) are the normalized equation parameters for S_{Q_P}.

Let us now consider the transformations of G_{Q}:

$$
\begin{aligned}
\varrho_1 &: (e_1,e_2,e_3,e_4) \longmapsto (e_2e_3e_4e_1, e_2, e_1, e_1e_3e_1),\\
\varrho_2 = \sigma_0^2 &: (e_1,e_2,e_3,e_4) \longmapsto (e_3, e_4, e_1, e_2).
\end{aligned}
$$

Straightforward computations show that ϱ_1 and ϱ_2 preserve relations (4.8), that ϱ_1 is of order 5, and that the subgroup $\langle\varrho_1,\varrho_2\rangle$ of G_{Q} is isomorphic to the dihedral group D_5.

Using the above correspondence between this description and that given in [6], and the expression of $\bar{\varrho}_1$ and $\bar{\varrho}_2$ in terms of normalized equation parameters, we get

$$
\begin{aligned}
\varrho_1.(l_1,l_2) &= (2\,h_1, \hat{l}_2), &\qquad \bar{\varrho}_1.(a,b) &= \left(\frac{b\,(1-a)}{b-a}, \frac{b}{b-a}\right),\\
\varrho_2.(l_1,l_2) &= (\hat{l}_1, \hat{l}_2), &\qquad \bar{\varrho}_1.(a,b) &= \left(\frac{1}{b}, \frac{1}{a}\right).
\end{aligned}
$$

These are exactly the generators for D_5 as given in [6].

Finally, we note that in [6], each orbit under D_5 consisted of five a priori different isometry classes (i.e., complex isomorphy classes) while the actions of G_{Q} and S_5 involve 10 different surfaces. The remaining five surfaces were in fact obtained by transporting the D_5-action from $\mathscr{P}_2$ onto a space of Riemann surfaces with a half-twist. The next section is devoted to showing that the action of G_{Q} can in fact be completely interpreted in term of half-twists.

Examples 4.9. We give here some exact examples. The first two surfaces are isometric, and thus complex isomorphic as algebraic curves, to surfaces that can be found in [6]. However, for both examples the reduced automorphism groups contain at least two different involutions. For each of them, the involutions considered here and in [6] are such that the genus 0 quotient are not isometric. This in particular implies that the genus two curves described here are not real isomorphic to those in [6] and that the transformed surfaces are not isometric to those under the D_5-action in [6].

I have never found Example 3 in the literature.

1. Let Q_0 be the totally regular quadrilateral, i.e., such that $\sigma_0(Q_0) = Q_0$, defined in terms of length of its sides and first diagonal by:

$$\cosh(l_i) = 3 + 2\sqrt{2}, \qquad \cosh(l) = 4\sqrt{2} + 5.$$

Then the pair (a, b) of normalized equation parameters for S_{Q_0} must satisfy

$$\overline{\sigma_0}.(a, b) = (a, b), \qquad \text{i.e.,} \qquad a = -i \text{ and } b = i \text{ or } a = i \text{ and } b = -i,$$

and thus $y^2 = (x^2 - 1)(x^4 + 1)$ is an equation for S_{Q_0}. As an algebraic curve, S_{Q_0} is complex (but not real) isomorphic to that of equation

$$y^2 = (x^2 - 1)\big(x^2 - \big(3 + 2\sqrt{2}\,\big)\big)\big(x^2 - \big(3 - 2\sqrt{2}\,\big)\big).$$

2. Let Q_1 be the quadrilateral such that $Q_1 = \sigma_1(Q_1)$, then the surface S_{Q_1} must satisfy

$$\overline{\sigma_1}.(a, b) = (a, b) \quad \text{i.e.,} \quad \{a, b\} = \big\{\tfrac{1}{2}(1 + i\sqrt{3}\,), \tfrac{1}{2}\big(1 - i\sqrt{3}\,\big)\big\}.$$

3. Let Q_2 be the quadrilateral given by (in terms of hyperbolic cosine of length of the sides and of the first diagonal)

$$\big(3 + 2\sqrt{3}\,, 3 + 2\sqrt{3}\,, 3 + 2\sqrt{3}\,, 3 + 2\sqrt{3}\,, 3 + 2\sqrt{3}\,\big).$$

Note that Q_2 is obtained by gluing two copies of a triangle with interior angles $\big(\frac{1}{6}\pi, \frac{1}{6}\pi, \frac{1}{6}\pi\big)$ we obtain Q_2. This implies that $D_4 \subset \mathrm{Aut}(S_{Q_2})$ (and thus belongs to the family F_4 of Table 2) and that the surface S_{Q_2}/D_4 has an automorphism of order 3. We then have

$$b = \frac{1}{a}, \qquad \frac{a + \dfrac{1}{a}}{2} = \pm i\sqrt{3}\,.$$

From easy but technical considerations (see [1] for more details) on real structures and on the position of the unit circle on the quotient, one can also deduce that in fact $\frac{1}{2}(a+1/a) = i\sqrt{3}$, and that the normalized equation parameters are:

$$(a,b) = \left(a, \frac{1}{a}\right) = \big(i(\sqrt{3}-2), i(\sqrt{3}+2)\big).$$

As the surface belongs to F_4, one can find only six a priori non-isometric transformed surfaces under the actions of G_{Q} and S_5. We give them in terms of length of the quadrilaterals and in normalized equation parameters in Table 5.

Note that the two last surfaces have the real structure induced by $x \longmapsto 1/\bar{x}$. Note also the two first ones have the real structures induced by $x \longrightarrow i\,(\bar{x}-i)/(\bar{x}+i)$ (I wish to thank R. Silhol for this last remark). They are thus isomorphic as they are conjugated.

One can easily verify that the third and the fourth have no real structures. Exact examples without real structure are very rare in the literature.

σ	$\cosh(l_1)$	$\cosh(l_2)$	$\cosh(l_3)$	$\cosh(l_4)$	$\cosh(l)$	a	b
Id	$3+2\sqrt{3}$	$3+2\sqrt{3}$	$3+2\sqrt{3}$	$3+2\sqrt{3}$	$3+2\sqrt{3}$	$i(\sqrt{3}-2)$	$i(\sqrt{3}+2)$
σ_0	$3+2\sqrt{3}$	$3+2\sqrt{3}$	$3+2\sqrt{3}$	$3+2\sqrt{3}$	$11+6\sqrt{3}$	$-i(\sqrt{3}+2)$	$-i(\sqrt{3}-2)$
σ_1	$7+4\sqrt{3}$	$1+\sqrt{3}$	$3+2\sqrt{3}$	$5+3\sqrt{3}$	$9+5\sqrt{3}$	$1-i(\sqrt{3}+2)$	$1-i(\sqrt{3}-2)$
$\sigma_1^2\sigma_0$	$3+2\sqrt{3}$	$1+\sqrt{3}$	$7+4\sqrt{3}$	$5+3\sqrt{3}$	$5+3\sqrt{3}$	$1+i(\sqrt{3}-2)$	$1+i(\sqrt{3}+2)$
σ_2	$11+6\sqrt{3}$	$3+2\sqrt{3}$	$3+2\sqrt{3}$	$3+2\sqrt{3}$	$3+2\sqrt{3}$	$1-i(\sqrt{3}+2)$	$(1+i(\sqrt{3}+2))^{-1}$
$(\sigma_1\sigma_0)^3$	$51+30\sqrt{3}$	$3+2\sqrt{3}$	$3+2\sqrt{3}$	$3+2\sqrt{3}$	$21+12\sqrt{3}$	$1+i(\sqrt{3}-2)$	$(1-i(\sqrt{3}-2))^{-1}$

Table 5.

5. Twist and half-twists

Recall that K_{Q} is the subgroup of transformations σ in G_{Q} such that $\bar{\sigma}$ as in Section 4.2 belongs to $S_{1,3} \times S_{2,4,5}$. As for any quadrilateral Q the surfaces S_Q and $S_{\sigma(Q)}$ are isometric but correspond to a priori different points in Teichmüller space $\mathscr{T}_2$, they are linked by an element of the modular group.

On the other hand, $K_{\mathrm{Q}} \subsetneq G_{\mathrm{Q}}$, and the main result of this section is that G_{Q} is in fact generated by half-twists in the following sense . Let $Q = (e_1, e_2, e_3, e_4) \in \mathrm{Q}$. An oriented geodesic of the surface S_Q is represented by a word $m(e_1, e_2, e_3, e_4)$ in the letters e_1, e_2, e_3, and e_4 up to conjugacy in Γ_Q. The word m then represents a free homotopy class for surfaces in $\{S_Q,\ Q \in \mathrm{Q}\}$.

Let $\sigma \in G_{\mathrm{Q}}$. We will say that σ corresponds to a half-twist (or a Dehn twist) along a geodesic m if given any $\mathrm{Q} = (e_1, e_2, e_3, e_4)$, one can go from S_Q to $S_{\sigma(Q)}$ by a half-twist (or a Dehn twist) along the geodesic $m(e_1, e_2, e_3, e_4)$ of S_Q. Of course, as on the one hand the elements of G_{Q} preserve $F_2(\mathrm{Q})$ and on the other hand only the twists along the geodesics stable under $\mathrm{aut}(S)$ preserve $F_2(\mathrm{Q})$, $m(e_i)$ has to be conjugated in the Fuchsian group Γ_Q to $e_1 m(e_i) e_1$ or $e_1 m(e_i)^{-1} e_1$. We will also consider simultaneous Dehn twists along $m(e_i)$ and $e_1 m(e_i) e_1$, in the case where the two corresponding geodesics are disjoint.

5.1. Half-twists. We show that the generators for G_{Q} given in Section 3.2 act on $F(\mathrm{Q})$ as half-twists.

Theorem 5.1. *The transformation* $\eta_1\colon (e_1, e_2, e_3, e_4) \longrightarrow (e_1, e_3, e_3e_2e_3, e_4)$ *is a half-twist along the geodesic represented by the word* $(e_3e_2)^2 = w^1(e_1, e_2, e_3, e_4)$.

The transformation $\eta_2\colon (e_1, e_2, e_3, e_4) \longrightarrow (e_1, e_2, e_4, e_4e_3e_4)$ *is a half-twist along the geodesic represented by the word* $(e_4e_3)^2 = w^2(e_1, e_2, e_3, e_4)$.

The transformation $\eta_3\colon (e_1, e_2, e_3, e_4) \longrightarrow (e_1e_4e_1, e_2, e_3, e_1)$ *is a half-twist along the geodesic represented by the word* $(e_1e_4)^2 = w^3(e_1, e_2, e_3, e_4)$.

The transformation $\eta_4\colon (e_1, e_2, e_3, e_4) \longrightarrow (e_2e_3e_4e_1, e_2, e_3, e_4)$ *is a half-twist along the geodesic represented by the word* $(e_2e_3e_4)^2 = w^4(e_1, e_2, e_3, e_4)$.

The group $G_{\mathrm{Q}} = \langle \eta_1, \eta_2, \eta_3, \eta_4 \rangle$ *is thus generated by half-twists.*

Proof. We treat the four cases with the same argument. Namely, for any Q, we exhibit pants decompositions $\mathscr{D}_i$ and $\mathscr{D}'_i$ on S_Q and $S_{\eta_i(Q)}$, respectively, such that:

(1) The four pairs of pants obtained by cutting S_Q along $\mathscr{D}_i$ and $S_{\eta_i(Q)}$ along $\mathscr{D}'_i$ are isometric.

(2) The three geodesics of $\mathscr{D}_i$ on S_Q and that of $\mathscr{D}'_i$ on $S_{\eta_i(Q)}$ correspond to the same decomposition according to the marking: if $Q = (e_1, e_2, e_3, e_4)$ and $\eta_i(Q) = (e^i_1, e^i_2, e^i_3, e^i_4)$, there exist three words in four letters $w^i = w^i_1, w^i_2$ and w^i_3 such that

$$\mathscr{D}_i = \big(w^i_1(e_j), w^i_2(e_j), w^i_3(e_j)\big), \qquad \mathscr{D}'_i = \big(w^i_1(e^i_j), w^i_2(e^i_j), w^i_3(e^i_j)\big).$$

Conditions (1) imply in particular that the length parts of the Fenchel–Nielsen coordinates of S_Q and $S_{\eta_i(Q)}$ associated to the pants decompositions $\mathscr{D}_i$ and $\mathscr{D}'_i$, respectively, are the same. With condition (2), this means that the transformation η_i corresponds to a change of these Fenchel–Nielsen parameters that affects only their twist part. In other words, η_i corresponds to a product of twist deformations (not necessarily Dehn twists) along the geodesics of the decomposition $\mathscr{D}_i$ of the marked Riemann surface S_Q.

It remains to show that the values of the twist parameters are 0 for w_2^i and w_3^i and $\frac{1}{2}$ for w_1^i.

The first part is achieved by considering geodesics crossing w_2^i and w_3^i and observing that their word expression is unchanged by η_i.

To show that the twist is $\frac{1}{2}$ on w_1^i, we use the underlying algebraic curves observing that η_i^2 belongs to K_{Q} while η_i does not.

We take:

for η_1: $w_1^1(e_i) = (e_3e_2)^2$, $w_2^1(e_i) = e_1e_2e_3$, $w_3^1(e_i) = e_2e_3e_1$,
for η_2: $w_1^2(e_i) = (e_4e_3)^2$, $w_2^2(e_i) = e_3e_4e_1$, $w_3^2(e_i) = e_1e_3e_4$,
for η_3: $w_1^3(e_i) = (e_1e_4)^2$, $w_2^2(e_i) = e_3e_1e_4$, $w_3^2(e_i) = e_1e_4e_3$,
for η_4: $w_1^4(e_i) = (e_2e_3e_4)^2$, $w_2^4(e_i) = e_2e_3e_4e_3$, $w_3^4(e_i) = e_3e_2e_3e_4$.

Now consider the geodesic represented by the word $(e_2e_3e_4)^2$. It crosses the geodesics represented by $w_2^1(e_i)$ and $w_3^1(e_i)$ (or $w_2^2(e_i)$ and $w_3^2(e_i)$) but not the one represented by $w_1^1(e_i)$ (or $w_1^2(e_i)$) and its word expression is invariant by occurrences of η_1 (or η_2). In particular, its length is unchanged. According to A. Douady in [7, Exposé 7], this means that the twist parameter along the geodesic represented by $w_2^1(e_i)$ and $w_3^1(e_i)$ when applying η_1 (or $w_2^2(e_i)$ and $w_3^2(e_i)$ when applying η_2) is zero.

The same arguments for the geodesics represented by $(e_2e_1e_4)^2$ for η_3 and $(e_2e_2)^2$ for η_4 shows the nullity of the twist parameter on $w_2^3(e_i)$ and $w_3^3(e_i)$ when applying η_3 and on $w_2^4(e_i)$ and $w_3^4(e_i)$ when applying η_4. ▫

Remark 5.2. Theorems 4.6 and 5.1 together allow to see some half-twists and Dehn twists on the equation of the associated algebraic curve.

Remark 5.3. As working on groups with representation by generators and relations is not an easy thing, we do not have a precise idea of what is possible in terms of twists with elements of G_{Q}.

Note for example that simultaneous half-twists along disjoint geodesics exchanged by the non-trivial involutions of surfaces S_Q do not correspond to elements of G_{Q} in general. We give an example that illustrates this fact. Let $P \in \mathscr{P}_2$, and S_P the genus two surface constructed from P as in (4.8). According to [6], for a generic P, the transforms of S_P under G_{Q} either have a real structure with 2 components or a real structure with 3 components. Now consider the surface $\widetilde{S}_P$ obtained from S_P by making simultaneous half-twists along the two real components exchanged by the involution of S_P. Then, one can show that for a generic P, S_P has no real structure with more than one component, and thus does not belong to the transforms of S_P. See [1] for more details.

5.2. Dehn twists along the sides of the quadrilateral. There are several motivations to not consider only Dehn twists as double half-twists in our situation. First, as mentionned in Remark 5.3, simultaneous half-twists are not allowed in general while we will see below an example of a correspondence between an element of G_Q and simultaneous Dehn twists.

Another motivation to deal with Dehn twists separately is their difference in nature with half-twists: the Dehn twists are defined on the topological surface T_2 of genus two, while half-twists only make sense on the (marked) Riemann surfaces. In particular, the composition of different transformations corresponding to half-twists can only be considered as successive operations on successively different Riemann surfaces. On the other hand, it makes sense to compose Dehn twists even along geodesics which are not disjoint on the topological surface T_2.

We will show that in a way the group law in the mapping class group and in the subgroup K_{Q} and G_{Q} are reversed.

Let $Q=(e_1,e_2,e_3,e_4)\in \mathrm{Q}$, we choose S_Q as a model for T_2. As S_Q is a hyperbolic surface, each homotopy class of a closed path c on T_2 is represented by a unique geodesic of S_Q, i.e., the conjugacy class of a word w in the letters e_1,e_2,e_3 and e_4. We will denote by $\tau_{w(e_i)}$ the change of the marking corresponding to the twist along c on T_2.

We have then

Proposition 5.4. *Let $\tau=\tau_{w_1(e_i)}\circ\cdots\circ\tau_{w_{k_1}(e_i)}$ and $\tau'=\tau_{m'_1(e_i)}\circ\cdots\circ\tau_{m'_{k_2}(e_i)}$ be two products of Dehn twists along geodesics of S_Q, for any $Q=(e_1,e_2,e_3,e_4)\in \mathrm{Q}$.*

Assume that there exist transformations σ and σ' in K_{Q} such that for any $Q=(e_1,e_2,e_3,e_4)\in\mathrm{Q}$,

$$S_{\sigma(Q)}=\tau(S_Q)\quad\text{and}\quad S_{\sigma'(Q)}=\tau'(S_Q).$$

Then

(i) $\tau\circ\tau'(S_Q)=S_{\sigma'\circ\sigma(Q)}$.

(ii) *For $\tilde\sigma\in K_{\mathrm{Q}}$, and $\tau^{\tilde\sigma}=\tau_{w_1(\tilde\sigma(e_i))}\circ\cdots\circ\tau_{w_{k_1}(\tilde\sigma(e_i))}$, we have*

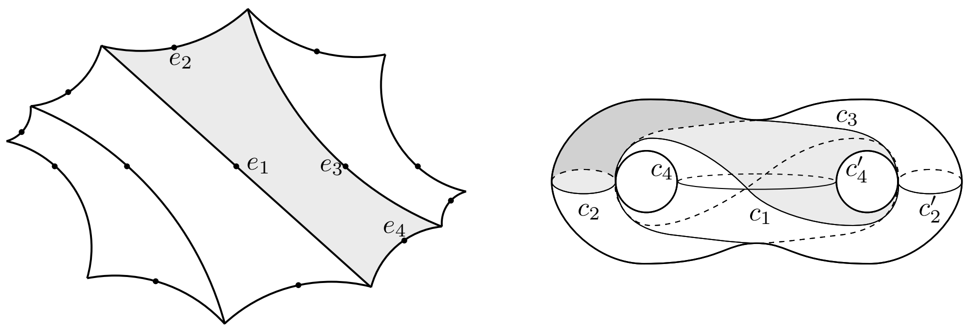
$$\tau_{\tilde\sigma}(S_Q)=S_{\tilde\sigma^{-1}\sigma\tilde\sigma(Q)}.$$

Proof. It is well known that if α is an homeomorphism of a surface T_2, and c is a homotopy class of a simple closed path on T_2, then $\tau_{\alpha(c)}=\alpha\tau_c\alpha^{-1}$ (see for example [3]).

Point (i) is a direct consequence of this fact. We have

$$\begin{aligned}\tau\circ\tau'(S_Q)&=\tau\circ\big(\tau_{w_1(e_i)}\circ\cdots\circ\tau_{w_{k'}(e_i)}\big)(S_Q)\\&=\tau\circ\Big(\big(\tau^{-1}\,\tau_{\tau(m'_1(e_i))}\,\tau\big)\circ\cdots\circ\big(\tau^{-1}\,\tau_{\tau(m'_{k'}(e_i))}\,\tau\big)\Big)\,(S_Q)\\&=\big(\tau_{\tau(m'_1(e_i))}\circ\cdots\circ\tau_{\tau(m'_{k'}(e_i))}\big)\circ\tau\,(S_Q)\\&=\big(\tau_{\tau(m'_1(e_i))}\circ\cdots\circ\tau_{\tau(m'_{k'}(e_i))}\big)\,(S_{\sigma(Q)})\\&=\big(\tau_{m'_1(\sigma(e_i))}\circ\cdots\circ\tau_{m'_{k'}(\sigma(e_i))}\big)\,(S_{\sigma(Q)})=\tau'{}_{\sigma}(S_{\sigma(Q)})=S_{\sigma'\circ\sigma(Q)}.\end{aligned}$$

(ii) As $\tilde{\sigma}$ belongs to K_{Q}, there exists $\tilde{\tau} \in \Gamma_2$ such that $\tilde{\tau}(S_Q) = S_{\tilde{\sigma}(Q)}$ and $\tilde{\tau}(w_k(e_i)) = \tilde{\tau}(w_k(\tilde{\sigma}(e_i)))$, and the result follows from (i). □



$c_1(e_i) = (e_2e_3e_4)^2 \quad c_2(e_i) = (e_3e_4e_1) \quad c_2'(e_i) = e_1e_3e_4$
$c_3(e_i) = (e_4e_1e_2)^2 \quad c_4(e_i) = (e_1e_2e_3) \quad c_4'(e_i) = (e_2e_3e_1)$

Figure 6.

We end by showing in an example how, given a Dehn twist or a product of Dehn twists along geodesics exchanged by the involutions of S_Q, $Q = (e_1, e_2, e_3, e_4)$, one can recover "by hand" the corresponding transformation of G_{Q}.

Consider the topological model on the right-hand side of Figure 6 for S_Q.

Let $\tau_2 = \tau_{(e_3e_4e_1)} \circ \tau_{(e_1e_3e_4)} = \tau_{(e_1e_3e_4)} \circ \tau_{(e_3e_4e_1)}$. We obtain the correspondence between τ_2 and the transformation $\sigma = \sigma_0\sigma_2\sigma_3\sigma_0^3\sigma_1\sigma_0$ using the techniques developed in Section 3 as follows. We first determine the homotopy classes of the images under τ_2 of the geodesics of S_Q corresponding to the sides of the different copies of Q. The corresponding geodesics are mapped, in the quotient $S_0(Q) = S_0\big(\sigma(Q)\big)$, onto the sides of the quadrilateral fundamental domain $\sigma(Q)$.

Then, if one cuts the quotient along the sides of Q, the sides of $\sigma(Q)$ appear as geodesic arcs on Q (see Figure 7).

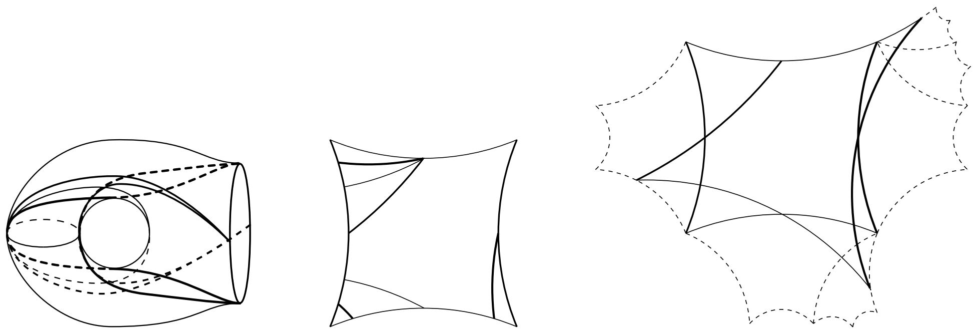


Figure 7.

The transformation σ is then built using the technique indicated in the proof of Proposition 3.8.

Using Theorem 5.1, the twist τ_1 along $(e_2e_3e_4)^2$ corresponds to η_4^2 and Lemma 5.4(ii) we get the following correspondences for the Dehn twists along geodesics corresponding to the sides of the different copies of Q in S_Q:

$\tau_1 = \tau_{(e_2e_3e_4)^2}$ corresponds to
$\eta_4^2\colon (e_1,e_2,e_3,e_4) \longmapsto ((e_2e_3e_4)^2, e_2, e_3, e_4)$,

$\tau_2 = \tau_{(e_3e_4e_1)} \circ \tau_{(e_1e_3e_4)} = \tau_{(e_1e_3e_4)} \circ \tau_{(e_3e_4e_1)}$
corresponds to $\sigma\colon (e_1,e_2,e_3,e_4) \mapsto (e_1, e_3e_4e_1e_2, e_3, e_4)$,

$\tau_3 = \tau_{(e_4e_1e_2)^2}\tau_1^{\sigma_0^2(e_i)}$ corresponds to
$\sigma_0^2\eta_4^2\sigma_0^2\colon (e_1,e_2,e_3,e_4) \longmapsto (e_1, e_2, (e_4e_1e_2)^2e_3, e_4)$,

$\tau_4 = \tau_{(e_1e_2e_3)} \circ \tau_{(e_3e_1e_2)} = \tau_{(e_3e_1e_2)} \circ \tau_{(e_1e_2e_3)} = \tau_2^{\sigma_0^2(e_i)}$ corresponds to
$\sigma_0^2\sigma\sigma_0^2\colon (e_1,e_2,e_3,e_4) \longmapsto (e_1, e_2, e_3, e_1e_2e_3e_4)$.

Acknowledgements. I would like to express my gratitude to Peter Buser and Robert Silhol for many helpful suggestions.

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Received August 28, 2003