Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 29, 2004, 329–344

CYCLIC EXTENSIONS OF SCHOTTKY UNIFORMIZATIONS

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Abstract. A conformal automorphism $\phi\colon S \to S$ of a closed Riemann surface S of genus $p \geq 2$ is said to be of Schottky type if there is a Schottky uniformization of S for which ϕ lifts. In the case that ϕ is of Schottky type, we have associated a geometrically finite Kleinian group K, generated by the uniformizing Schottky group G and any of the liftings of ϕ. We have that K contains G as a normal subgroup and K/G is cyclic. In this note we describe, up to topological equivalence, all possible groups K obtained in this way. Equivalently, if we are given a handlebody M^3 of genus $p \geq 2$ and an orientation preserving homeomorphism of finite order ϕ, then we proceed to describe, up to topological equivalence, the hyperbolic structures of the orbifold M^3/ϕ having bounded by below injectivity radius.

1. Introduction

Notions as Kleinian groups, region of discontinuity, limit set, function groups, geometrically finite Kleinian groups, Riemann surfaces, Riemann orbifolds, signatures, etc., can be found in the excellent book of B. Maskit [18] and the references in there.

A *Schottky group of genus zero* is just the trivial group. A Schottky group of positive genus is defined as follows. Assume we have a collection of $2p$ $(p > 0)$ pairwise disjoint simple loops, say C_1, $C_1', \dots, C_p$ and C_p', in the Riemann sphere bounding a common region $\mathscr{D}$ of connectivity $2p$, and that there are loxodromic transformations $A_1, \dots, A_p$ so that $A_j(C_j) = C_j'$ and $A_j(\mathscr{D}) \cap \mathscr{D} = \emptyset$, for each $j = 1, 2, \dots, p$. The group G, generated by $A_1, \dots, A_p$, is called a *Schottky group of genus* $p > 0$. The collection of loops C_1, $C_1', \dots, C_p$ and C_p', is called a *fundamental system of loops* of G with respect to the generators $A_1, \dots, A_p$. Generalities on Schottky groups can be found, for instance, in [3] and 16]. Each Schottky group of genus $p > 0$ defines a geometrically finite complete hyperbolic structure on a handlebody of genus p with injectivity radii bounded below. Reciprocally, every geometrically finite complete hyperbolic structure on a handlebody of genus p with injectivity radii bounded below is given by a Schottky group of genus $p > 0$. If we denote by Ω the region of discontinuity of a Schottky group G of genus p, then the quotient $S = \Omega/G$ turns out to be a closed Riemann surface of genus p.

2000 Mathematics Subject Classification: Primary 30F10, 30F40, 30F50.

Partially supported by projects Fondecyt 1030252, 1030373 and UTFSM 12.03.21.

The reciprocal is valid by the retrosection theorem [14] (see [2] for a modern proof using quasiconformal deformation theory). A triple $(\Omega, G, P\colon \Omega \to S)$ is called a *Schottky uniformization* of a closed Riemann surface S if G is a Schottky group with Ω as its region of discontinuity and $P\colon \Omega \to S$ is a holomorphic regular covering with G as covering group.

Let us assume we have a closed Riemann surface S and a group H of automorphisms of it. We say that H is of *Schottky type* if there is a Schottky uniformization of S for which every element of H lifts.

As already observed, each Schottky group of genus $p > 0$ corresponds to a geometrically finite complete hyperbolic structure on a handlebody of genus p. In this way, we may interpret the notion of a Schottky type automorphism as follows. Assume we have a pair (S, ϕ), where S is a closed orientable surface of genus p and $\phi\colon S \to S$ is an orientation preserving homeomorphism of S, homotopically of finite order. As consequence of Nielsen's realization problem [13], we may just consider S as a closed Riemann surface of genus p and ϕ a finite-order conformal automorphism of S. We have that S is the boundary of infinitely many handlebodies of genus p. We may ask for the existence of one of these handlebodies for which the homeomorphism ϕ extends continuously. The existence of such a handlebody is equivalent to the existence of a Schottky uniformization of S for which ϕ lifts. In [20], [19] and [22] can be found some results in this direction using three-dimensional techniques.

The problem of deciding when H is of Schottky type has been discussed in [5], [7], [8], [9], [10], [11] and [12] for different class of conformal groups and certain cyclic anticonformal groups. In fact, we have the following results.

Theorem 1.1 ([7]). *Let S be a closed Riemann surface of genus $p \geq 1$ and H be a group of conformal automorphism of S. If H is of Schottky type, then it must satisfy condition* (A).

Theorem 1.2 ([7]). *Let S be a closed Riemann surface of genus $p \geq 1$ and $\phi\colon S \to S$ be a conformal automorphism. The cyclic group $H = \langle \phi \rangle$ is of Schottky type if and only if H satisfies condition* (A).

Condition (A). Let S be a closed Riemann surface and $\phi\colon S \to S$ be a conformal automorphism. Let $H = \langle \phi \rangle$. If $a \in S$ is a fixed point of some $h \in H - \{I\}$, then we denote by $R(h, a) \in (-\pi, \pi]$ the rotation number of h about a, and we denote by $H(a)$ the stabilizer subgroup of a in H. We say that H satisfies the *condition* (A) if all fixed points of the non-trivial elements of H can be put into pairs satisfying the following properties.

(A1) If $\{a, b\}$ is such a pair, then $a \neq b$, $H(a) = H(b)$ and $R(h, a) = -R(h, b)$, for each $h \in H(a) = H(b)$ of order greater than two.

(A2) If $\{a, b\}$ and $\{r, t\}$ are two such pairs, then either $\{a, b\} \cap \{r, t\} = \emptyset$ or $\{a, b\} = \{r, t\}$.

(A3) If $\{a,b\}$ is a pair, then there is no element $t \in H$ so that $t(a) = b$.

(A4) If a is fixed point of some non-trivial element of H, then there is another fixed point b so that $\{a,b\}$ is one of the above pairs.

A pairing of the non-trivial elements of H satisfying (A1)–(A4) will called a *Schottky pairing.* In Section 2 we recall a short argumentation of the necessary part of Theorem 1.2.

Corollary 1.1. *Let S be a closed Riemann surface and $\phi\colon S \to S$ be a conformal automorphism. If either* (i) *no non-trivial power ϕ^k has fixed points*; *or* (ii) *ϕ has order 2, then $H = \langle \phi \rangle$ is of Schottky type.*

Let us consider a closed Riemann surface S of genus p and a group H of conformal automorphism of S of Schottky type. In this case there is a Schottky uniformization $(\Omega, G, P\colon \Omega \to S)$ satisfying the property that for every $h \in H$ there is some conformal automorphism $\hat{h}\colon \Omega \to \Omega$ for which $h \circ P = P \circ \hat{h}$. As the region of discontinuity Ω of a Schottky group is known to be a domain of type O_{AD} [1], we have that the conformal automorphism $\hat{h}$ is the restriction of a Möbius transformation. Let K be the group generated by the liftings $\hat{h}$, $h \in H$, and the Schottky group G. We have that K contains G as a normal subgroup and that K/G is isomorphic to H. It follows that Ω is also the region of discontinuity of K and, if H is finite, then G will be of finite index in K, in particular, K will be a geometrically finite function group. In the particular case H is a cyclic group of Schottky type, say generated by a conformal automorphism $\phi\colon S \to S$, we have that K/G is a cyclic group. In the case that ϕ has finite order, the case when the genus of S is at least 2, we have that G has finite index in K and, in particular, K is geometrically finite. In any case, we have $S/\phi = \Omega/K$, obtaining in this way a uniformization $(\Omega, K, Q\colon \Omega \to S/\phi)$ so that $Q = L \circ P$, where $L\colon S \to S/\phi$ is the branched covering induced by the action of ϕ on S. We may have many different Schottky uniformizations of S for which ϕ lifts. In particular, we may get many different geometrically finite groups K and respective uniformizations. The purpose of this note is to describe all such possible groups K we obtain by the above process (see Section 3) up to topological conjugacy.

As noted before, the above can be also described in terms of handlebodies. If we are given a handlebody M^3 and a finite order orientation preserving homeomorphism $\phi\colon M^3 \to M^3$, then we have the orbifold $\mathscr{O} = M^3/\phi$. Each hyperbolic structure on M^3, of bounded by below injectivity radius (i.e. a Schottky group G), for which ϕ acts as a hyperbolic isometry (i.e. of Schottky type for such a Schottky uniformization), we get a hyperbolic orbifold structure on $\mathscr{O}$, of bounded by below injectivity radius (i.e. a geometrically finite Kleinian group K containing G as a finite index normal subgroup). In this way, to give a description, up to topological equivalence, of the geometrically finite groups K as above is the same as to describe, up to topological equivalence, the hyperbolic orbifold structure on $\mathscr{O}$, of bounded by below injectivity radius.

2. Automorphisms of Schottky type

2.1. Low genus.

2.1.1. *Genus zero.* If $p=0$, then by the classical uniformization theorem, we have that S is conformally equivalent to the Riemann sphere. The only Schottky uniformization in this case is given by the trivial group. In particular, all automorphisms are trivially of Schottky type.

2.1.2. *Genus one.* If $p=1$, then we have that the cases of (i) automorphism of order 6 with fixed points and (ii) the automorphism of order 4 with fixed points, do not satisfy the condition (A). Moreover, all other types of automorphisms keep homotopically invariant some non-dividing simple closed curve on the respective tori. It follows that these automorphisms are of Schottky type. In this case the groups K we obtain are of one of the following types: (i) a Schottky group of genus 1, say generated by A, where G is generated by A^n, some positive integer n; or (ii) a group generated by a loxodromic transformation A and an elliptic transformation L, of finite order, so that $L\circ A = A\circ L$; or (iii) a group freely generated by two elliptic transformations of order 2, say E_1 and E_2.

2.2. Hyperbolic situation. Let us assume $p\geq 2$, which is the case we will consider from now on in this note. In this case we have that ϕ has finite order.

We proceed first to give a short argumentation of the necessity part of Theorem 1.2. Let S be a closed Riemann surface of genus $p\geq 2$ and a conformal automorphism $\phi\colon S\to S$, of order n, which is of Schottky type. Let $(\Omega, G, P\colon \Omega\to S)$ be a Schottky uniformization of S for which ϕ lifts. As already observed, we have a geometrically finite function group K, generated by G and any lifting of ϕ. The group K contains G as a normal subgroup of index n and K/G is a cyclic group of order n. As K is a finite extension of G and G contains no parabolic transformations, then neither K does. This observation together the results in [6] asserts that: if $\hat{h}\in K$ is an elliptic transformation, then both fixed points of $\hat{h}$ belong to Ω or there is a loxodromic transformation in G commuting with $\hat{h}$. For each elliptic transformation in K, with both fixed points in Ω, we consider the pair obtained as the projections of its two fixed points. In this way, we produce a collection of pairs, on the surface S, on which every fixed point of non-trivial powers of ϕ appears. The non-existence of parabolic transformations in K asserts that these pairs satisfy the conditions (A1), (A2) and (A4). We proceed to check condition (A3). Assume we have $\hat{h}\in K$, an elliptic transformation with $\mathrm{Fix}(\hat{h})=\{a,b\}\subset\Omega$. If there is some transformation $\hat{t}\in K$ so that $\hat{t}(a)=b$, then the non-existence of parabolics in K ensures that $\hat{t}(b)=a$. It follows that $\hat{t}^2=I$ and, in particular, $\hat{t}$ defines an automorphism t of order 2 in $H=\langle\phi\rangle$ which permutes the projected points $a'=P(a)$ and $b'=P(b)$. The points $a',b'\in S$ are fixed points of the automorphism in H induced by $\hat{h}$, say $h=\phi^k$. If h has order two, then we have that the cyclic group H contains two different elements of order 2, a contradiction. If the order of h is at least 3, then we have that $R(h,a') = -R(h,b') \in (-\pi,\pi)$. But as t is conformal and commutes with h, we also have that

$$R(h,a') = R(t \circ h \circ t^{-1}, a') = R(h,b'),$$

a contradiction. It follows that our pairs also satisfy condition (A3). In this way we have obtained a Schottky pairing for ϕ as desired.

Remark 2.1. As before, let S be a closed Riemann surface of genus $p \geq 2$ and $\phi\colon S \to S$ be a conformal automorphism, say of finite order n. Set $H = \langle \phi \rangle$.

(1) It is not difficult to see that given a Schottky pairing for a group H, there is a new Schottky pairing with the extra property that: *if $\{p,q\}$ is a pair, then for every $h \in H$ we have that $\{h(p), h(q)\}$ is also one of our pairs.*

(2) If ϕ is a conformal automorphism of Schottky type, then ϕ^k is also of Schottky type, for all $k = 1,\dots,n-1$.

(3) If H satisfies condition (A), then it is clear that S/H cannot have signature of type $(0,3;m,n,t)$. For example, let us consider Klein's surface S of genus $p=3$, the only, up to biholomorphisms, closed Riemann surface of genus three admitting a group of conformal automorphisms of maximal order $84(p-1) = 168$. We have that S admits a conformal automorphism ϕ of order 7. The quotient S/ϕ has signature $(0,3;7,7,7)$, in particular, ϕ is not of Schottky type.

Let S be a closed Riemann surface of genus $p \geq 2$ and $\phi\colon S \to S$ a conformal automorphism of order n. We know from [21] that $n \leq 2(2p+1)$. The observation (3), in the above remark, and Riemann–Hurwitz's formula permits us to obtain the following fact with respect to the order of a Schottky-type conformal automorphism.

Proposition 2.1. *Let S be a closed Riemann surface of genus $p \geq 2$ and $\phi\colon S \to S$ a Schottky-type conformal automorphism of order n. If $q \geq 0$ denotes the genus of the quotient S/ϕ, then:*

(i) *If $q \geq 2$, then $n \leq (p-1)$.*

(ii) *If $q = 1$, then $n \leq 2(p-1)$.*

(iii) *If $q = 0$, then $n \leq 6(p-1)$. Moreover, if the quotient has at least 6 branch values, then $n \leq 2(p-1)$.*

Proof. Set $R = S/\phi$, the quotient Riemann orbifold, and $q \geq 0$ its genus. We denote by $\pi\colon S \to R$ the regular branched covering induced by the action of ϕ.

As we are assuming ϕ of Schottky type, we have from Theorem 1.2 the existence of a Schottky pairing for $H = \langle \phi \rangle$, which we may write as:

$$\bigl\{\{a_{1,1}, b_{1,1}\},\dots,\{a_{1,s_1}, b_{1,s_1}\},\{a_{2,1}, b_{2,1}\},\dots,\{a_{2,s_2}, b_{2,s_2}\},\dots,$$
$$\{a_{r,1}, b_{r,1}\},\dots,\{a_{r,s_r}, b_{r,s_r}\}\bigr\},$$

so that $\pi(a_{j,i}) = x_j$ and $\pi(b_{j,i}) = y_j$, for $j = 1, \dots, r$ and $i = 1, \dots, s_j$. In this way, the branch locus of $\pi\colon S \to R$ is given by

$$B = \{x_1, y_1, \dots, x_r, y_r\}.$$

Let $H_j = H(a_{j,1}) = H(b_{j,1}) = \langle \phi^{k_j} \rangle$ be the stabilizer in H of $a_{j,1}$, for $j = 1, \dots, r$. If n_j denotes the order of H_j, then we also set $l_j = n/n_j \in \left\{1, \dots, \frac{1}{2}n\right\}$. With this notation, we have that the signature of R is:

$$(q, 2r; n_1, n_1, n_2, n_2, \dots, n_r, n_r).$$

Riemann–Hurwitz's formula [4] reads, in this case, as:

$$(*) \qquad p = n(q + r - 1) + 1 - \sum_{j=1}^{r} l_j.$$

If $q \geq 2$, as $l_j \leq \frac{1}{2}n$, then $(*)$ asserts that

$$n \leq \frac{2}{2+r}(p-1) \leq (p-1).$$

If $q = 1$, then $(*)$ asserts that $r > 0$ and

$$n \leq \frac{2}{r}(p-1) \leq 2(p-1).$$

If $q = 0$, then $(*)$ asserts that $r \geq 2$. In the case that $r \geq 3$ we have that

$$n \leq \frac{2}{r-2}(p-1) \leq 2(p-1).$$

In the case $r = 2$, we have

$$(**) \qquad p - 1 + l_1 + l_2 = n.$$

As $p \geq 2$, we cannot have $l_1 = l_2 = \frac{1}{2}n$. We may assume that $l_1 \leq l_2$. If we have that $l_2 = \frac{1}{2}n$, then we must have $l_1 \leq \frac{1}{3}n$. In this case, $(**)$ asserts that $n \leq p - 1 + \frac{1}{3}n + \frac{1}{2}n$, from which we obtain

$$n \leq 6(p-1). \ \square$$

Example 2.1. Let us consider $K = \langle A, B : A^3 = B^2 = I \rangle$, where $A(z) = e^{2\pi i/3}z$ and B is an elliptic element of order two that keeps invariant a simple closed curve $L \subset \mathbf{C}$, contained in the interior of the sector $\mathrm{Arg}(z) \in \left(-\frac{1}{3}\pi, +\frac{1}{3}\pi\right)$, and permutes both topological discs bounded by it. The subgroup

$$G = \langle U = B \circ A \circ B \circ A^{-1}, V = B \circ A^{-1} \circ B \circ A \rangle$$

is a normal subgroup of K, in fact, the commutator subgroup of K. We have that G is a Schottky group of genus $p = 2$; a fundamental system of loops is given by $W(L)$, $B\bigl(W(L)\bigr)$, $W^{-1}(L)$ and $B\bigl(W^{-1}(L)\bigr)$, with respect to the Schottky generators U and V. Let us denote by Ω the region of discontinuity of K, which is the same as for G. The closed Riemann surface $S = \Omega/G$ has genus $p = 2$ and admits a Schottky type conformal automorphism $\phi\colon S \to S$, induced by the transformation $A \circ B \in K$, of order $6(p-1) = 6$. These surfaces are given by the algebraic curves

$$y^2 = (x^3 - 1)(x^3 - \lambda), \qquad \lambda > 1,$$

and the automorphism ϕ is represented by

$$\phi = \begin{cases} x \mapsto e^{2\pi i/3}x, \\ y \mapsto -y. \end{cases}$$

3. Topological classification

Consider a closed Riemann surface S, of genus $p \geq 2$, and a Schottky type conformal automorphism $\phi\colon S \to S$, say of order $n \geq 2$. Set $H = \langle \phi \rangle \cong \mathbf{Z}/n\mathbf{Z}$. As seen before, each Schottky uniformization $(\Omega, G, P\colon \Omega \to S)$ for which ϕ lifts produces a geometrically finite function group K, containing the Schottky group G as a normal subgroup of index n. In this way we get a uniformization $(\Omega, K, Q\colon \Omega \to R)$ of the quotient orbifold $R = S/\phi$. In this section we proceed to describe, up to topological equivalence, all such uniformizations of R we obtain in this way.

In the first part we proceed to see that K belongs to a certain family of groups constructed as a free product of certain basic groups in terms of Klein–Maskit's combination theorems. Next, we give a simple condition to decide which of the groups on such a family are the possible ones.

Notation. We keep the following notation. The quotient Riemann orbifold $R = S/\phi$ has genus $q \geq 0$ and we denote by $\pi\colon S \to R$ the branched regular covering induced by the action of ϕ. As we are assuming ϕ of Schottky type, we have from Theorem 1.2 the existence of a Schottky pairing for $H = \langle \phi \rangle$:

$$\bigl\{\{a_{1,1}, b_{1,1}\}, \dots, \{a_{1,s_1}, b_{1,s_1}\}, \dots, \{a_{r,1}, b_{r,1}\}, \dots, \{a_{r,s_r}, b_{r,s_r}\}\bigr\},$$

so that $\pi(a_{j,i}) = x_j$ and $\pi(b_{j,i}) = y_j$, for $j = 1, \dots, r$ and $i = 1, \dots, s_j$. The branch locus of $\pi\colon S \to R$ is then given by

$$B = \{x_1, y_1, \dots, x_r, y_r\}.$$

Let $H_j = H(a_{j,1}) = H(b_{j,1}) = \langle \phi^{k_j} \rangle$ be the stabilizer in H of $a_{j,1}$, for $j = 1, \ldots, r$. If n_j denotes the order of H_j, then we set $l_j = n/n_j \in \{1, \ldots, \frac{1}{2}n\}$. In this way, the signature of R is

$$(q, 2r; n_1, n_1, n_2, n_2, \ldots, n_r, n_r).$$

We may assume that ϕ^{k_j} is a geometric generator, that is, the rotation number of ϕ^{k_j} at $a_{j,1}$ is given as

$$R(\phi^{k_j}, a_{j,1}) = 2\pi/n_j.$$

3.1. A family of geometrically finite groups. Assume we have a Schottky uniformization $(\Omega, G, P\colon \Omega \to S)$ of the surface S for which ϕ lifts. Let K be the group generated by G and any lifting of ϕ. We have noted that K is a geometrically finite Kleinian group, containing G as a finite-index normal subgroup, with Ω as region of discontinuity and $\Omega/K = R$. Let us denote by $Q\colon \Omega \to R$ the branched regular covering induced by K so that $Q = \pi \circ P$.

Let $\alpha \subset S$ be a simple closed geodesic of minimal hyperbolic length that lifts to a loop on Ω by P. We have that either:

(i) $\phi(\alpha) = \alpha$; or
(ii) $\phi(\alpha) \cap \alpha = \emptyset$.

The reason of this is that if $\phi(\alpha) \cap \alpha \neq \emptyset$ and $\phi(\alpha) \neq \alpha$, then as both α and $\phi(\alpha)$ lift to loops on Ω, we may find a simple closed loop (homotopically non-trivial) that also lifts to a loop on Ω of smaller hyperbolic length than of α, a contradiction.

It follows that the translates of α under H define a collection of pairwise disjoint simple closed geodesics. Let S' be the surface obtained from S after cutting along such a collection of simple closed geodesics. By the construction, we have that H permutes the connected components of S'.

If S' still has a connected component of positive genus, we proceed to find a simple closed geodesic β of minimal length inside S' that lifts to a loop on Ω via P. This is possible since the Schottky uniformization is a lowest regular planar covering of S. We have that β is disjoint from any of the translates of α under H and that either:

(i) $\phi(\beta) = \beta$; or
(ii) $\phi(\beta) \cap \beta = \emptyset$.

Let S'' be the surface obtained from S' after cutting along such a collection of simple closed geodesics. By the construction, we have that H permutes the connected components of S''. If some of the connected componets of S'' still have positive genus, then we proceed in a similar fashion. As S has finite genus, the above procedure ends after a finite number of steps.

At the end, we end with a collection

$$\alpha_1, \ldots, \alpha_m \subset S$$

of pairwise disjoint simple closed geodesics so that:

(i) $\phi(\alpha_j) = \alpha_j$ or $\phi(\alpha_j) \cap \alpha_j = \emptyset$;
(ii) $\phi^l(\alpha_j) \cap \phi^t(\alpha_k) = \emptyset$, for all l, t and $j \neq k$;
(iii) each geodesic loop α_j lifts to a loop on Ω by $P\colon \Omega \to S$;
(iv) $S - \bigcup_{j=1}^{m} \alpha_j$ has no component of positive genus; and
(v) the components of $S - \bigcup_{j=1}^{m} \alpha_j$ are permuted by the cyclic group H.

Let us observe that we may replace each of the above geodesic loops by homotopic simple loops (not longer geodesic ones) with the property that none of them contains a fixed point of a non-trivial power of ϕ. From now on we assume this assumption and we forget about geodesic status.

Let us denote by $\beta_1, \dots, \beta_s \subset R$ the projection of the simple loops $\alpha_1, \dots, \alpha_m$ under $\pi\colon S \to R$. Clearly, $s \leq m$. The above asserts that each β_j is a simple loop and that

(vi) $\beta_1, \dots, \alpha_s$ are pairwise disjoint;
(vii) each of the loops β_j is disjoint from the branch locus of $\pi\colon S \to R$; and
(viii) $R - \bigcup_{j=1}^{s} \beta_j$ has no component of positive genus.

It follows in particular that $q \leq s$ and that we may find a subcollection of loops (reordering indices if necessary) $\beta_1, \dots, \beta_q \subset R$ so that $R - \bigcup_{j=1}^{q} \beta_j$ is connected and planar. Let us consider the corresponding simple closed loops $\alpha_1, \dots, \alpha_q \subset S$ (and their translates under H) so that $\phi(\alpha_j) = \beta_j$ (up to a permutation of indices of the original collection if necessary).

Each loop β_j , for $j = 1, \dots, q$, lifts to a loop on Ω under $Q\colon \Omega \to R$. If $\hat{\beta}_j \subset \Omega$ is any one of such liftings of β_j, then $\hat{\beta}_j$ is also a lifting of α_j. We have two possibilities for $\hat{\beta}_j$:

(a) $k(\hat{\beta}_j) \cap \hat{\beta}_j = \emptyset$, for all $k \in K - \{I\}$; or
(b) there is a geometrical elliptic transformation $\theta_j \in K$ so that:
  (b.1) $k(\hat{\beta}_j) = \hat{\beta}_j$ for $k \in \langle \theta_j \rangle$; and
  (b.2) $k(\hat{\beta}_j) \cap \hat{\beta}_j = \emptyset$ for all $k \in K - \langle \theta_j \rangle$.

Choose a base point $x \in \beta_j$ and a base point $y \in \hat{\beta}_j$ so that $Q(y) = x$. In case (a) we have that the lifting of β starting at y defines a closed loop, and in case (b) we have that the lifting of β starting at y defines a simple arc ending at $\theta_j(y)$. As θ_j descends by $P\colon \Omega \to S$ to a power of ϕ, we have that the order m_j of θ_j is a divisor of n. Also, as the covering $Q\colon \Omega \to R$ is regular, the above is independent of the choice of the lifting loop. We associate to the loop β_j the weight m_j.

Next, we proceed to construct a collection of pairwise disjoint simple arcs

$$\widetilde{\delta}_1, \dots, \widetilde{\delta}_r \subset R - \bigcup_{j=1}^{q} \beta_j,$$

so that $\widetilde{\delta_j}$ connects x_j with y_j and is disjoint from the other branch values.

The arc $\widetilde{\delta_j}$ lifts to S to exactly l_j pairwise disjoint packages $\mathscr{F}_{j,1},\dots,\mathscr{F}_{j,l_j}$, of simple arcs. There are permutations $\sigma_j \in \mathscr{S}_{l_j}$ (the symmetric group on l_j letters) so that each package $\mathscr{F}_{j,i}$ consists of exactly n_j simple arcs, each one connecting the points $a_{j,i}$ with $b_{j,\sigma_j(i)}$, which are cyclically permuted by ϕ^{k_j}. The packages are cyclically permuted by ϕ.

We may give an orientation of $\widetilde{\delta_j}$ in order that the starting point of an arc in the family $\mathscr{F}_{j,i}$ starts at $a_{j,i}$ and ends at $b_{j,\sigma_j(i)}$.

We may reorder the points $b_{j,i}$ so that the lifted arc corresponding to $\widetilde{\beta_j}$ that starts at $a_{j,i}$ ends at $b_{j,i}$, that is, σ_j is the trivial permutation. Such a change gives us another Schottky pairing for the automorphism ϕ.

For each $j \in \{1,\dots,r\}$, we consider two of the arcs in $\mathscr{F}_{j,1}$, say $\delta_{j,1}$ and $\delta_{j,2}$, so that $\phi^{k_j}(\delta_{j,1}) = \delta_{j,2}$. Recall that we have assumed ϕ^{k_j} to be a geometrical generator of H_j.

The simple loop $\delta_j = \delta_{j,1} \cup \delta_{j,2}$ has two possible types of liftings under the Schottky uniformization $P\colon \Omega \to S$.

(P1) δ_j lifts to a loop; or

(P2) δ_j lifts to an arc, connecting two limit points of G.

If $n_j > 2$, then (P2) is not possible. This is just a consequence of the fact that no simple arc is invariant under an elliptic transformation of order bigger than 2. It follows that situation (P2) only may happen if $n_j = 2$. In such a case, we must have the existence of a pair of elliptic transformations of order 2, say $\gamma_1,\gamma_2 \in K$, so that $\gamma_1 \circ \gamma_2 \in G$. Each of the elements of order 2 above must have both of its fixed points in Ω. In fact, γ_j has one of its fixed points in Ω since it projects to the point $a_{j,1}$ or $b_{j,1}$. As consequence of the results in [6], we have that the other fixed point also belongs to Ω.

In this way, we may change our Schottky pairing by a new one so that both fixed points of each elliptic element of order 2 in K project to a pair.

Now, in situation (P1) we have the existence of an elliptic transformation $\varepsilon \in K$ of order n_j with both fixed points projecting to the pair $\{a_{j,1}, b_{j,1}\}$.

All the above arguments assert that we may assume that our Schottky pairing is such that each pair comes from the projection of both fixed points of an elliptic transformation in K (of course assuming they belong to Ω [6]).

Since K is a geometrically finite function group, we have as a consequence of the classification of function groups in [17], that the above information describes K up to topological equivalence. More precisely, the group K can be geometrically constructed as follows. Fix integers

$$(*) \quad \begin{cases} n \in \{2, 3, \ldots\}, \\ a, b, c, d \in \{0, 1, 2, \ldots\}, \\ n_1, \ldots, n_d \in \{3, \ldots, n\}, \\ m_1, \ldots, m_b \in \{2, \ldots, n\}, \\ n_j \text{ and } m_j \text{ divisors of } n, \end{cases}$$

so that:

(i) $q = a + b$;
(ii) $c + d = r$;
(iii) $2c$ is the number of branch values on R of order 2;
(iv) $2d$ is the number of branch values on R of orders greater than 2.

Let us consider loxodromic transformations

$$\tau_1, \ldots, \tau_a, \eta_1, \ldots, \eta_b \in K,$$

and elliptic transformations

$$\theta_1, \ldots, \theta_b, \gamma_1, \ldots, \gamma_c, \varepsilon_1, \ldots, \varepsilon_d \in K,$$

so that

(v) the order of θ_j is m_j;
(vi) the order of γ_j is 2 (they only appear in the case n is even);
(vii) the order of ε_j is n_j; and
(viii) $\eta_j \circ \theta_j = \theta_j \circ \eta_j$ commute.

We assume the existence of a collection simple loops as shown in Figure 1.

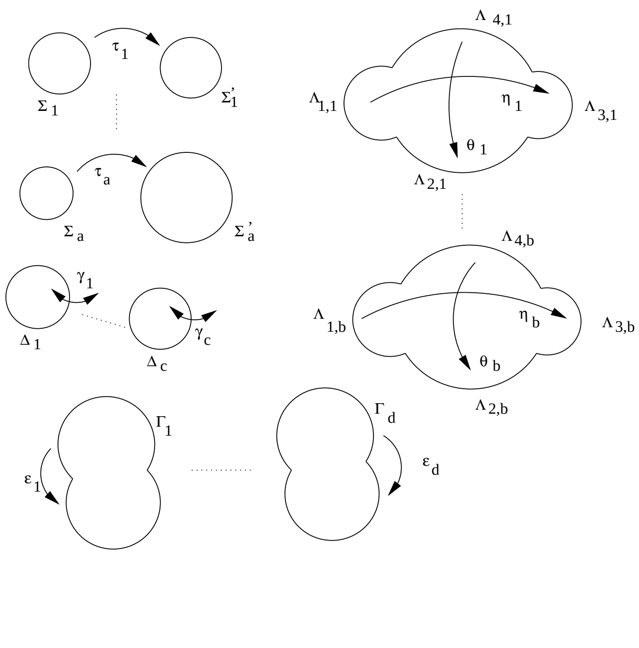

Figure 1.

In this way,

$$K \cong \underbrace{\mathbf{Z} * \cdots * \mathbf{Z}}_{a} * \underbrace{(\mathbf{Z} \oplus \mathbf{Z}/m_1\mathbf{Z}) * \cdots * (\mathbf{Z} \oplus \mathbf{Z}/m_b\mathbf{Z})}_{b} * \underbrace{\mathbf{Z}/2\mathbf{Z} * \cdots * \mathbf{Z}/2\mathbf{Z}}_{c} * \underbrace{\mathbf{Z}/n_1\mathbf{Z} * \cdots * \mathbf{Z}/n_d\mathbf{Z}}_{d}.$$

The Schottky group G is a normal subgroup of K, of index n, so that $S/\phi = \Omega/K$, where the cyclic group generated by ϕ is given by K/G. It follows from Riemann–Hurwitz's formula that the following equality should be true:

$$(**) \qquad p = n\big(a + b + \tfrac{1}{2}c + d - 1\big) + 1 - n\sum_{j=1}^{d} 1/n_j.$$

The above tells us how to construct the geometrically finite Kleinian group K in terms of certain parameters given by integers.

3.1.1. *The family $\mathscr{F}$.* Let us denote by $\mathscr{F}$ the family of groups constructed as above, from Maskit–Klein's combination theorems, using integers as in $(*)$ restricted to the equality $(**)$.

As consequences of the arguments above, we have that the groups we are looking for are members of the family $\mathscr{F}$, but it may happen that some of the groups in $\mathscr{F}$ are not. We will say that a group $K \in \mathscr{F}$ is *admissible* if K has a Schottky subgroup G as normal subgroup so that K/G is a finite cyclic group.

Before going into the problem of deciding which groups in $\mathscr{F}$ are admissible, we must observe that Maskit–Klein's combination theorems together with the above arguments permit us to see the following.

Theorem 3.1. (1) *The algebraic structure of any group in $\mathscr{F}$ is uniquely determined by the data*:

$$\begin{cases} n \in \{2,3,\ldots\}, \\ a,b,c,d \in \{0,1,\ldots,\}, \\ 3 \le n_1,\ldots,n_d \le n \text{ divisors of } n \text{ (up to permutation)}, \\ 2 \le m_1,\ldots,m_b \le n \text{ divisors of } n \text{ (up to permutation)}. \end{cases}$$

(2) *All groups in $\mathscr{F}$ with the same algebraic structure are quasiconformally equivalent.*

(3) *If $K^* \in \mathscr{F}$ has data as in* (1), *then the signature of Ω/K^* is*

$$\Big(q = a + b, 2(c+d); \underbrace{2,\ldots,2}_{2c}, n_1, n_1, n_2, n_2, \ldots, n_d, n_d\Big).$$

3.2. Admissible groups in $\mathscr{F}$. Now we proceed to decide which groups in the family $\mathscr{F}$ are admissible. Assume we have given integers $p, n \geq 2$, $a, b, c, d \in \{0, 1, 2, \ldots\}$, and positive integer divisors of n, say $n_1, \ldots, n_d$, $m_1, \ldots, m_b$, all of them satisfying the conditions $(*)$ and equality $(**)$. By Klein–Maskit's combination theorem we construct a geometrically finite function group $K \in \mathscr{F}$ as above. The signature of Ω/K (Ω is the region of discontinuity of K) is given by

$$\Big(q = a + b, 2(c + d); \underbrace{2, \ldots, 2}_{2c}, n_1, n_1, n_2, n_2, \ldots, n_d, n_d\Big).$$

Lemma 3.1. *Let $K \in \mathscr{F}$ be constructed as above from Klein–Maskit's combination theorems. If we have a surjective homomorphism*

$$\Phi\colon K \to \mathbf{Z}/n\mathbf{Z} = \langle U : U^n = I \rangle,$$

with torsion-free kernel G, then G is necessarily a Schottky group of genus

$$p = n\big(a + b + \tfrac{1}{2}c + d - 1\big) + 1 - n \sum_{j=1}^{d} 1/n_j.$$

In particular, the closed Riemann surface $S = \Omega/G$ admits a Schottky type conformal automorphism $\phi\colon S \to S$, of order n, induced by K/G. In other words, the group K is admissible.

Proof. By Klein–Maskit's combination theorems, all elements different from the identity are either: (i) loxodromic; or (ii) conjugates of non-trivial powers of some θ_j, or conjugates of some γ_j, or (iii) conjugates of non-trivial powers of some ε_j.

If the kernel G of Φ is torsion free, then we have that G is a purely loxodromic, geometrically finite function group with its limit set a Cantor set. By the classification of function groups [17], we must have that G is in fact a Schottky group. Riemann–Hurwitz's formula asserts that the genus of G is in fact p. □

As a consequence of Lemma 3.1, in order for our group K to be admissible, we need to have the existence of a surjective homomorphism

$$\Phi\colon K \to \mathbf{Z}/n\mathbf{Z} = \langle U : U^n = I \rangle,$$

with a torsion-free kernel.

3.2.1. *First case.* If we have $a > 0$ or $b > 0$, then the existence of Φ is clear. In fact, we only need to define Φ as follows:

(1) if n is even and $c \neq 0$, then $\Phi(\gamma_j) = U^{n/2}$;
(2) if $b \neq 0$, then set $\Phi(\theta_j) = U^{n/m_j}$;
(3) if $d \neq 0$, then set $\Phi(\varepsilon_j) = U^{n/n_j}$;
(4) if $a > 0$, then set $\Phi(\tau_1) = U$ and send the rest of generators to the identity element;
(5) if $a = 0$, then set $\Phi(\eta_1) = U$ and we send the rest of generators to the identity element.

3.2.2. *Second case.* If we have $a = b = 0$, then the existence of Φ is not in general true. In this case we need to have that:

(1) If $c > 0$, then n is even and $\langle U : U^n = I\rangle = \langle U^{n/2}, U^{n/n_1}, \ldots, U^{n/n_d}\rangle$, which is equivalent to having $\mathrm{GCD}\,(n/2, n/n_1, \ldots, n/n_d) = 1$.
(2) If $c = 0$, then $d > 0$ and $\langle U : U^n = I\rangle = \langle U^{n/n_1}, \ldots, U^{n/n_d}\rangle$, which is equivalent to having $\mathrm{GCD}\,(n/n_1, \ldots, n/n_d) = 1$.

As a consequence of all the above, we have the following conditions to determine those groups in $\mathscr{F}$ we are looking for.

Theorem 3.2. *Assume we have given integers $p, n \geq 2$, $a, b, c, d \in \{0, 1, 2, \ldots\}$, and positive integers divisors of n, say $n_1, \ldots, n_d$, $m_1, \ldots, m_b$, all of them satisfying the conditions $(*)$ and equality $(**)$. Let K be the geometrically finite group constructed from Klein–Maskit's combination theorem as above. If either* (i) *$a + b > 0$; or* (ii) *$a = b = 0$, $c > 0$ and $GCD(n/2, n/n_1, \ldots, n/n_d) = 1$; or* (iii) *$a = b = c = 0$ and $GCD(n/n_1, \ldots, n/n_d) = 1$, then K is admissible.*

Now, Theorems 3.1 and 3.2 give us the classification of all admissible geometrically finite Kleinian groups we are looking for.

Example 3.1. We assume we have a closed Riemann surface of genus $p = 2$ and a conformal automorphism $\phi\colon S \to S$ of order n of Schottky type. As consequence of Proposition 2.1, we have that $n \in \{2, 3, 4, 5, 6\}$. On the other hand, a conformal automorphism of order $n = 5$ of a closed Riemann surface has exactly 3 fixed points, this situation is not possible by condition (A). In particular, $n \in \{2, 3, 4, 6\}$.

(1) $n = 2$. In this case we have $d = 0$ and $m_j = 2$. Equation $(**)$ then gives us the following possible solutions:
   (1.1) $a = b = 0$, $c = 3$; in which case the group K is generated by the involutions γ_1, γ_2 and γ_3. The Schottky group G is generated by $\gamma_1 \circ \gamma_2$ and $\gamma_1 \circ \gamma_3$.
   (1.2) $a = 0$, $b = c = 1$; in which case the group K is generated by the transformations η_1, θ_1 and γ_1. The Schottky group is generated by η_1 and $\eta_1 \circ \gamma_1$.
   (1.3) $a = 1$, $b = 0$, $c = 1$; in which case the group K is generated by the transformations τ_1 and γ_1. The Schottky group is generated by τ_1 and $\gamma_1 \circ \tau_1 \circ \gamma_1$.

(2) $n = 3$. In this case we have $c = 0$ and $n_j = m_i = 3$. Equation $(**)$ then gives us as unique solution $a = b = 0$, $d = 2$. In this case the group K is generated by the transformations of order three ε_1 and ε_2. The Schottky group G is generated by $\varepsilon_1 \circ \varepsilon_2$ and $\varepsilon_1^{-1} \circ \varepsilon_2^{-1}$.

(3) $n = 4$. In this case we have $b = b_1 + b_2$, where $n_1 = \cdots = n_d = 4$, $m_1 = \cdots = m_{b_1} = 2$ and $m_{b_1+1} = \cdots = m_b = 4$. Equation $(**)$ then gives us as unique solution $a = b = 0$, $c = d = 1$. In this case the group K is generated by the transformation of order four ε_1 and the involution γ_1. The Schottky group G is generated by $\varepsilon_1^2 \circ \gamma_1$ and $\varepsilon_1^{-1} \circ \gamma_1 \circ \varepsilon_1^{-1}$.

(4) $n = 6$. In this case we have $d = d_1 + d_2$, $b = b_1 + b_2 + b_3$, so that $n_1 = \cdots = n_{d_1} = 3$, $n_{d_1+1} = \cdots = n_d = 6$, $m_1 = \cdots = m_{b_1} = 2$, $m_{b_1+1} = \cdots = m_{b_2} = 3$, and $m_{b_2+1} = \cdots = m_b = 6$. Equation $(**)$ then gives us

$$7 = 6\big(a + b + \tfrac{1}{2}c\big) + 5d - d_1.$$

We cannot have $a = b = c = 0$, since (as $d_1 \leq d$) in that case $7 = 5d - d_1 \geq 4d$, from which we may obtain a contradiction. It follows that

$$a + b + \tfrac{1}{2}c \geq \tfrac{1}{2}.$$

Again using the fact that $5d - d_1 \geq 4d$, we obtain that the unique solutions are $a = b = 0$, $c = d = d_1 = 1$. In this case the group K is generated by the transformation of order three ε_1 and the involution γ_1, and the Schottky group G is generated by $\gamma_1 \circ \varepsilon_1 \circ \gamma_1 \circ \varepsilon_1^{-1}$ and $\gamma_1 \circ \varepsilon_1^{-1} \circ \gamma_1 \circ \varepsilon_1$ (see Example 2.1).

Remark 3.1. The arguments done above permit us to describe all topological classes of geometrically finite groups K. Assume we have fixed one of these admissible groups K. A homeomorphism of the Riemann sphere that conjugates K into itself is called a *geometrical normalizer* of K. We say that two Schottky subgroups G_1 and G_2 of K are *equivalent* if there is a geometrical normalizer of K, say f, so that $f \circ G_1 \circ f^{-1} = G_2$. A difficult problem, at least for the author, is to give a classification of the equivalence classes of Schottky subgroups of K.

References

[1] Ahlfors, L. V., and L. Sario: Riemann Surfaces. - Princeton Univ. Press., Princeton, NJ, 1960.

[2] Bers, L.: Automorphic forms for Schottky groups. - Adv. in Math. 16, 1975, 332–361.

[3] Chuckrow, V.: On Schottky groups with applications to Kleinian groups. - Ann. of Math. 88, 1968, 47–61.

[4] Farkas, H. M., and I. Kra: Riemann Surfaces. - Grad. Texts in Math. 71, Springer-Verlag, 1991 (second edition).

[5] Hidalgo, R. A.: On Γ-hyperelliptic Schottky groups. - Notas Soc. Mat. Chile 8, 1989, 27–36.

[6] HIDALGO, R. A.: The mixed elliptically fixed point property for Kleinian groups. - Ann. Acad. Sci. Fenn. Ser. A I Math. 19, 1994, 247–258.

[7] HIDALGO, R. A.: On Schottky groups with automorphisms. - Ann. Acad. Sci. Fenn. Ser. A I Math. 19, 1994, 259–289.

[8] HIDALGO, R. A.: Schottky uniformizations of closed Riemann surfaces with abelian groups of conformal automorphisms. - Glasgow Math. J. 36, 1994, 17–32.

[9] HIDALGO, R. A.: Γ-hyperelliptic Riemann surfaces. - Proyecciones 17, 1998, 77–117.

[10] HIDALGO, R. A.: Dihedral groups are of Schottky type. - Proyecciones 18, 1999, 23–48.

[11] HIDALGO, R. A.: $\mathscr{A}_4$, $\mathscr{A}_5$, $\mathscr{S}_4$ and $\mathscr{S}_5$ of Schottky type. - Rev. Mat. Univ. Compl. Madrid 15, 2002.

[12] HIDALGO, R. A., and A. F. COSTA: Anticonformal automorphisms and Schottky coverings. - Ann. Acad. Sci. Fenn. Math. 26, 2001, 489–508.

[13] KERCKHOFF, S. P.: The Nielsen realization problem. - Ann. of Math. (2) 117, 1983, 235–265.

[14] KOEBE, P.: Über die Uniformisierung der Algebraischen Kurven II. - Math. Ann. 69, 1910, 1–81.

[15] MASKIT, B.: Characterization of Schottky groups. - J. Anal. Math. 19, 1967, 227–230.

[16] MASKIT, B.: Self-maps on Kleinian groups. - Amer. J. Math. XCIII:3, 1973, 840–856.

[17] MASKIT, B.: On the classification of Kleinian groups II-signatures. - Acta Math. 138, 1977, 17–42.

[18] MASKIT, B.: Kleinian Groups. - Grundlehren Math. Wiss., Springer-Verlag, 1987.

[19] MCCULLOUGH, D., A. MILLER, and B. ZIMMERMANN: Group actions on handlebodies. - Proc. London Math. Soc. 59, 1989, 373–416.

[20] MILLER, A., and B. ZIMMERMANN: Large groups of symmetries of handlebodies. - Proc. Amer. Math. Soc. 106, 1989, 829–838.

[21] WIMAN, A.: Ueber die hyperelliptischen Curven und diejenigen vom Geschlechte $p = 3$, welche eindeutigen Transformationen in sich zulassen. - Bihang Kungl. Svenska Vetenskaps-Akademiens Handlingar, 21, 1, no. 1, 1985, 1–23.

[22] ZIMMERMANN, B.: Über Homöomorphismen n-dimensionaler Henkelkörper und endliche Erweiterungen von Schottky-Gruppen. - Comment. Math. Helv. 56, 1981, 474–486.

Received 30 September 2003