Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 29, 2004, 459–469

INTEGRAL MEANS OF ANALYTIC FUNCTIONS

Daniel Girela and José Ángel Peláez

Universidad de Málaga, Facultad de Ciencias, Departamento de Análisis Matemático
Campus de Teatinos, ES-29071 Málaga, Spain; girela@uma.es, pelaez@anamat.cie.uma.es

Abstract. If $0<p<\infty$ and f is an analytic function in the unit disc $\Delta=\{z\in\mathbf{C}:|z|<1\}$, we set, as usual,

$$M_p(r,f)=\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\right)^{1/p},\qquad 0<r<1.$$

Given $p\in(0,\infty)$, we let $\mathscr{F}_p$ denote the space of those functions f which are analytic in Δ and satisfy $M_p(r,f')=\mathrm{O}(1/(1-r))$, as $r\to1$. In this paper we obtain sharp estimates on the growth of the integral means $M_p(r,f)$, $f\in\mathscr{F}_p$.

1. Introduction and main results

Let Δ denote the unit disc $\{z\in\mathbf{C}:|z|<1\}$. If $0<r<1$ and g is an analytic function in Δ, we set

$$I_p(r,g)=\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(re^{i\theta})|^p d\theta,\qquad 0<p<\infty,$$
$$M_p(r,g)=I_p(r,g)^{1/p},\qquad 0<p<\infty,$$
$$M_\infty(r,g)=\max_{|z|=r}|g(z)|.$$

For $0<p\le\infty$ the Hardy space H^p consists of those functions g, analytic in Δ, for which

$$\|g\|_{H^p}=\sup_{0<r<1}M_p(r,g)<\infty.$$

We refer to [4] for the theory of Hardy spaces.

It is well known that there is a close relation between the integral means of an analytic function and those of its derivative. A classical result of Hardy and Littlewood [8], [9] (see Theorem 5.5 of [4]) asserts that if $0<p\le\infty$, $\alpha>1$ and f is an analytic function in Δ, then

$$M_p(r,f')=\mathrm{O}\left(\frac{1}{(1-r)^\alpha}\right),\qquad \text{as } r\to1, \tag{1}$$

2000 Mathematics Subject Classification: Primary 30D55.

The authors are partially supported by a grant from “La Junta de Andalucía” (FQM-210) and by an MCyT grant, BFM2001-1736, Spain.

if and only

$$M_p(r,f) = \mathrm{O}\left(\frac{1}{(1-r)^{\alpha-1}}\right), \qquad \text{as } r \to 1. \tag{2}$$

If $0 < \alpha < 1$ and f satisfies (1) then it follows that $f \in H^p$ (see Theorem 5.1 and Theorem 5.4 of [4] for the case $1 \le p \le \infty$ and Remark 1 below for the case $0 < p < 1$).

Now it remains to consider the case $\alpha = 1$. Studying this case is the main object of this paper.

Applying the continuous form of Minkowski's inequality, in the case $1 \le p < \infty$, and simply integration of the derivative, in the case $p = \infty$, yield:

If $1 \le p \le \infty$ and f is an analytic function in Δ which satisfies

$$M_p(r,f') = \mathrm{O}\left(\frac{1}{1-r}\right), \qquad \text{as } r \to 1, \tag{3}$$

then

$$M_p(r,f) = \mathrm{O}\left(\log\frac{1}{1-r}\right), \qquad \text{as } r \to 1. \tag{4}$$

This result is certainly sharp for $p = \infty$ and for $p = 1$. Indeed:

(i) $f(z) = \log\bigl(1/(1-z)\bigr)$, $z \in \Delta$, satisfies

$$M_\infty(r,f') \sim \frac{1}{1-r} \qquad \text{and} \qquad M_\infty(r,f) \sim \log\frac{1}{1-r}, \qquad \text{as } r \to 1.$$

(ii) The function $f(z) = 1/(1-z)$, $z \in \Delta$, satisfies

$$M_1(r,f') \sim \frac{1}{1-r} \qquad \text{and} \qquad M_1(r,f) \sim \log\frac{1}{1-r}, \qquad \text{as } r \to 1.$$

However, in Theorem 1 and Theorem 2 we obtain better estimates for $1 < p < \infty$ and we also obtain sharp estimates in the case $0 < p < 1$.

Theorem 1. (a) *If $2 < p < \infty$ and f is an analytic function in Δ such that*

$$M_p(r,f') = \mathrm{O}\left(\frac{1}{1-r}\right), \qquad \text{as } r \to 1, \tag{5}$$

then

$$M_p(r,f) = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{\beta}\right), \qquad \text{as } r \to 1, \text{ for all } \beta > \frac{1}{2}. \tag{6}$$

(b) *Furthermore, this result is sharp in the sense that there exists a function* f*, analytic in* Δ*, which satisfies* (5) *for every* $p \in (2,\infty)$ *and such that*

$$M_p(r,f) \asymp \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{1/2}\right) \qquad \text{as } r \to 1, \text{ for every } p \in (2,\infty).$$

Theorem 2. (a) *If* $0 < p \le 2$ *and* f *is an analytic function in* Δ *which satisfies* (5) *then*

$$M_p(r,f) = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{1/p}\right), \qquad \text{as } r \to 1. \tag{7}$$

(b) *Furthermore, this result is sharp*: *For every* $p \in (0,2]$ *the function* $f(z) = 1/(1-z)^{1/p}$, $z \in \Delta$, *satisfies* (5) *and*

$$M_p(r,f) \asymp \left(\left(\log\frac{1}{1-r}\right)^{1/p}\right), \qquad \text{as } r \to 1. \tag{8}$$

2. Proof of the main results

For simplicity, for $0 < p \le \infty$ we let $\mathscr{F}_p$ be the space of those functions f which are analytic in Δ and satisfy (3). Notice that $\mathscr{F}_\infty$ coincides with the space $\mathscr{B}$ of Bloch functions [1]. Also, since $M_p(r,g)$ increases with p, we have

$$\mathscr{B} = \mathscr{F}_\infty \subset \mathscr{F}_q \subset \mathscr{F}_p, \qquad 0 < p < q < \infty. \tag{9}$$

Let us recall that Clunie and MacGregor [3] and Makarov [11] proved the following result.

Theorem A. *If* $f \in \mathscr{B}$ *then*

$$M_p(r,f) = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{1/2}\right), \qquad \text{as } r \to 1,$$

for all $p \in (0,\infty)$.

Theorem A is sharp. Indeed, if

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}, \qquad z \in \Delta,$$

then $f \in \mathscr{B}$ (see e.g. Lemma 2.1 of [1]) and

$$M_p(r,f) \asymp \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{1/2}\right), \qquad \text{as } r \to 1,$$

for all $p \in (0,\infty)$. Notice that, bearing in mind (9), this proves part (b) of Theorem 1.

Proof of Theorem 1(a). Take $p \in (2,\infty)$ and $f \in \mathscr{F}_p$. As noted above, we have

$$M_p(r,f) = \mathrm{O}\left(\log\frac{1}{1-r}\right), \qquad \text{as } r \to 1. \tag{10}$$

Using a result of Hardy [7] (see p. 126 of [12]), we see that

$$\frac{d}{dr}\left[rI_p'(r,f)\right] = \frac{p^2 r}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|^{p-2}|f'(re^{it})|^2\,dt,$$

which, since $I_p'(r,f) \ge 0$, implies

$$I_p''(r,f) \le \frac{p^2}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|^{p-2}|f'(re^{it})|^2\,dt. \tag{11}$$

For $0<r<1$, we set

$$E_{1,1}(r) = \left\{ t \in [-\pi,\pi] : |f'(re^{it})| \le \frac{|f(re^{it})|}{(1-r)\log\dfrac{1}{1-r}} \right\}, \tag{12}$$

and

$$E_{2,1}(r) = [-\pi,\pi] \setminus E_{1,1}(r). \tag{13}$$

Using (11), (12), (13) and bearing in mind that $p>2$, we deduce that

$$\begin{aligned} I_p''(r,f) &\le \frac{p^2}{2\pi}\int_{E_{1,1}(r)} |f(re^{it})|^{p-2}|f'(re^{it})|^2\,dt \\ &\qquad + \frac{p^2}{2\pi}\int_{E_{2,1}(r)} |f(re^{it})|^{p-2}|f'(re^{it})|^2\,dt \\ &\le p^2\left[\frac{I_p(r,f)}{(1-r)^2\left(\log\dfrac{1}{1-r}\right)^2} + \left((1-r)\log\frac{1}{1-r}\right)^{p-2} I_p(r,f')\right]. \end{aligned} \tag{14}$$

Using this, (5) and (10), we obtain

$$I_p''(r,f) = \mathrm{O}\left(\frac{1}{(1-r)^2}\left(\log\frac{1}{1-r}\right)^{p-2}\right), \qquad \text{as } r \to 1,$$

which, integrating twice, gives

$$I_p(r,f) = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{p-1}\right), \qquad \text{as } r \to 1, \tag{15}$$

which is better than (10).

This process can be iterated. We define a sequence $\{\alpha_k\}_{k=1}^{\infty}$ inductively as follows:

$$\begin{cases} \alpha_1 = 1, \\ \alpha_{k+1} = \dfrac{1}{p}\left(p - 2\displaystyle\sum_{j=1}^{k}\left(\alpha_j - \frac{1}{2}\right)\right), \qquad k = 1, 2, \ldots . \end{cases} \tag{16}$$

We have

$$p\alpha_{k+1} = p\alpha_k - 2\left(\alpha_k - \tfrac{1}{2}\right) = (p-2)\alpha_k + 1, \qquad k \geq 1. \tag{17}$$

Arguing by induction we can easily see that

$$\alpha_k > \tfrac{1}{2}, \qquad k = 1, 2, \ldots, \tag{18}$$

which, together with (17), implies that the sequence $\{\alpha_k\}_{k=1}^{\infty}$ is decreasing and, hence, convergent. Using again (17), we deduce that

$$\lim_{k\to\infty} \alpha_k = \tfrac{1}{2}. \tag{19}$$

Let us remark that (15) can be written as

$$I_p(r,f) = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{p\alpha_2}\right), \qquad \text{as } r \to 1.$$

We are going to prove inductively that

$$I_p(r,f) = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{p\alpha_j}\right), \qquad \text{as } r \to 1, \tag{20}$$

for all j. Since $\alpha_j \downarrow \frac{1}{2}$, this will finish the proof.

We already know that (20) is true for $j = 1$ and $j = 2$. Suppose that it is true for $j = k$, that is, suppose that

$$I_p(r,f) = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{p\alpha_k}\right), \qquad \text{as } r \to 1. \tag{21}$$

For $0 < r < 1$, we set

$$E_{1,k}(r) = \left\{ t \in [-\pi, \pi] : |f'(re^{it})| \le \frac{|f(re^{it})|}{(1-r)\left(\log \frac{1}{1-r}\right)^{\alpha_k}} \right\}, \tag{22}$$

and

$$E_{2,k}(r) = [-\pi, \pi] \setminus E_{1,k}(r). \tag{23}$$

Using (11), (22) and (23), we obtain

$$\begin{aligned} I_p''(r,f) &\le \frac{p^2}{2\pi} \int_{E_{1,k}(r)} |f(re^{it})|^{p-2} |f'(re^{it})|^2 \, dt \\ &\quad + \frac{p^2}{2\pi} \int_{E_{2,k}(r)} |f(re^{it})|^{p-2} |f'(re^{it})|^2 \, dt \\ &\le p^2 \left[\frac{I_p(r,f)}{(1-r)^2 \left(\log \frac{1}{1-r}\right)^{2\alpha_k}} \right. \\ &\quad \left. + (1-r)^{p-2} \left(\log \frac{1}{1-r}\right)^{(p-2)\alpha_k} I_p(r,f') \right]. \end{aligned} \tag{24}$$

Using (5), (21) and (24), we deduce that

$$I_p''(r,f) = \mathrm{O}\left(\frac{1}{(1-r)^2} \left(\log \frac{1}{1-r}\right)^{(p-2)\alpha_k} \right), \qquad \text{as } r \to 1,$$

and then, integrating twice and using (17), it follows that

$$\begin{aligned} I_p(r,f) &= \mathrm{O}\left(\left(\log \frac{1}{1-r}\right)^{(p-2)\alpha_k + 1} \right) \\ &= \mathrm{O}\left(\left(\log \frac{1}{1-r}\right)^{p\alpha_{k+1}} \right), \qquad \text{as } r \to 1. \end{aligned} \tag{25}$$

This is (20) for $j = k+1$. □

We do not know whether or not (6) can be substituted by

$$M_p(r,f) = \mathrm{O}\left(\left(\log \frac{1}{1-r}\right)^{1/2} \right), \qquad \text{as } r \to 1, \tag{26}$$

in Theorem 1(a). We know that any Bloch function satisfies (26) for any $p \in (2, \infty)$ but the question of determining whether or not the condition $f \in \mathscr{F}_p$, $2 < p < \infty$, is enough to conclude (26) remains open. It is worth noticing that no counterexample (if any) can be given by a power series with Hadamard gaps because of the following result.

Proposition 1. *Let f be an analytic function in Δ given by a power series with Hadamard gaps,*

$$f(z) = \sum_{j=1}^{\infty} a_j z^{n_j}, \qquad \text{with } \frac{n_{j+1}}{n_j} \geq \lambda > 1 \text{ for all } j. \tag{27}$$

Then the following conditions are equivalent.

(i) $f \in \mathscr{B}$.
(ii) $f \in \mathscr{F}_p$ *for some* $p \in (0,\infty)$.
(iii) $f \in \mathscr{F}_p$ *for all* $p \in (0,\infty)$.
(iv) $\sup_{j\geq 1} |a_j| < \infty$.

Proof. The implication (i) $\Rightarrow$ (ii) follows from (9). Bearing in mind that f' is also given by a power series with Hadamard gaps, (ii) $\Rightarrow$ (iii) follows using Theorem 8.20 in Chapter V of [14], Vol. I.

We turn now to prove that (iii) $\Rightarrow$ (iv). Hence, suppose that f is given by (27) and satisfies (iii). Then there exists a positive constant C such that

$$M_1(r, f') \leq C\frac{1}{1-r}, \qquad 0 < r < 1. \tag{28}$$

Now,

$$zf'(z) = \sum_{j=1}^{\infty} n_j a_j z^{n_j},$$

and then, using Cauchy's formula and (28), we obtain

$$n_j|a_j| = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f'(z)}{z^{n_j}}\, dz \right| \leq \frac{M_1(r,f')}{r^{n_j-1}} \leq \frac{C}{r^{n_j-1}(1-r)}, \qquad 0 < r < 1.$$

Taking $r = 1 - 1/n_j$ we obtain (iv).

Finally, the implication (iv) $\Rightarrow$ (i) follows from Lemma 2.1 of [1]. □

Before embarking into the case $0 < p \leq 2$, let us introduce a new family of spaces. For $0 < p < \infty$, the space of Dirichlet type $\mathscr{D}^p_{p-1}$ consists of all functions f which are analytic in Δ and satisfy

$$\int_\Delta (1-|z|^2)^{p-1} |f'(z)|^p \, dA(z) < \infty.$$

Here, $dA(z) = (1/\pi)\, dx\, dy$ denotes the normalized Lebesgue area measure in Δ. The spaces $\mathscr{D}^p_{p-1}$ are closely related to Hardy spaces. Indeed, a classical result of Littlewood and Paley [10] asserts that

$$H^p \subset \mathscr{D}^p_{p-1}, \qquad 2 \leq p < \infty. \tag{29}$$

On the other hand, we have

$$\mathscr{D}^p_{p-1} \subset H^p, \qquad 0 < p \le 2, \tag{30}$$

(see [10] and [13]). Notice that $\mathscr{D}^2_1 = H^2$. However, we remark that if $p \neq 2$ then $H^p \neq \mathscr{D}^p_{p-1}$. A number of results about the spaces $\mathscr{D}^p_{p-1}$ have been recently proved in [2] and [6]. In particular, estimates on the growth of the integral means of $\mathscr{D}^p_{p-1}$-functions have been obtained in [6]. We remark that

$$\mathscr{D}^p_{p-1} \subset \mathscr{F}_p, \qquad 0 < p < \infty. \tag{31}$$

Indeed, the condition $f \in \mathscr{D}^p_{p-1}$ is equivalent to saying that

$$\int_0^1 (1-r)^{p-1} I_p(r, f')\, dr < \infty, \tag{32}$$

and, taking into account that $I_p(r,f')$ is an increasing function of r, we easily see that (32) implies that $M_p(r,f') = \mathrm{o}\big(1/(1-r)\big)$, as $r \to 1$.

In view of (31), the results contained in this paper complement those of [6].

Our proof of Theorem 2 will be based on (30). Actually, we shall use the following result which follows from (30) by the closed graph theorem.

Theorem B. *If $0 < p \le 2$ then there exists a positive constant C_p, which only depends on p, such that*

$$\|f\|^p_{H^p} \le C_p\left(|f(0)|^p + \int_\Delta (1-|z|^2)^{p-1} |f'(z)|^p\, dA(z)\right), \tag{33}$$

for every $f \in \mathscr{D}^p_{p-1}$.

Proof of Theorem 2. Take $p \in (0,2]$ and $f \in \mathscr{F}_p$. Assume, without loss of generality, that $f(0) = 0$. For $0 < r < 1$, set $f_r(z) = f(rz)$, $z \in \Delta$. In what follows we shall be using the convention that C will denote a positive constant which may depend on p and f but not on r or ϱ, and which is not necessarily the same at different occurrences. Applying (33) to f_r, $0 < r < 1$, and using that $f \in \mathscr{F}_p$, yield

$$\begin{aligned} I_p(r,f) &\le C \int_0^1 (1-\varrho)^{p-1} I_p(r\varrho, f')\, d\varrho \le C \int_0^1 \frac{(1-\varrho)^{p-1}}{(1-r\varrho)^p}\, d\varrho \\ &= C\left(\int_0^r \frac{(1-\varrho)^{p-1}}{(1-r\varrho)^p}\, d\varrho + \int_r^1 \frac{(1-\varrho)^{p-1}}{(1-r\varrho)^p}\, d\varrho\right), \qquad 0 < r < 1. \end{aligned} \tag{34}$$

Since $r\varrho < \varrho$ and $r\varrho < r$, $0 < r, \varrho < 1$, (34) implies

$$\begin{aligned} I_p(r,f) &\le C\left(\int_0^r \frac{1}{1-\varrho}\, d\varrho + \frac{1}{(1-r)^p} \int_r^1 (1-\varrho)^{p-1}\, d\varrho\right) \\ &= \mathrm{O}\left(\log\frac{1}{1-r}\right), \qquad \text{as } r \to 1. \end{aligned} \tag{35}$$

This finishes the proof of part (a) of Theorem 2. Part (b) is clear. □

Remark 1. If $0 < p < 1$, $0 < \alpha < 1$, and f is an analytic function in Δ which satisfies (1) with $f(0) = 0$, then, arguing as in the proof of Theorem 2, we obtain

$$I_p(r, f) \le C \int_0^1 \frac{(1-\varrho)^{p-1}}{(1-r\varrho)^{\alpha p}}\, d\varrho \le C \int_0^1 (1-\varrho)^{p(1-\alpha)-1}\, d\varrho, \qquad 0 < r < 1,$$

and hence it follows that $f \in H^p$ as we asserted in Section 1.

3. Univalent functions in the classes $\mathscr{F}_p$

A complex-valued function defined in Δ is said to be univalent if it is analytic and one-to-one there. We refer to [5] and [12] for the theory of these functions. Throughout the paper, $\mathscr{U}$ will stand for the class of all univalent functions in Δ. The aim of this section is studying the growth of the integral means $M_p(r, f)$ of functions $f \in \mathscr{U} \cap \mathscr{F}_p$.

It is well known that $\mathscr{U} \subset H^p$, if $0 < p < \frac{1}{2}$ (see e.g. Theorem 3.16 of [4]). Hence, if $0 < p < \frac{1}{2}$ and $f \in \mathscr{U}$, then $M_p(r, f) = \mathrm{O}(1)$, as $r \to 1$. We can prove the following result for $p \ge \frac{1}{2}$.

Theorem 3. *If $\frac{1}{2} \le p < \infty$ and $f \in \mathscr{U} \cap \mathscr{F}_p$, then*

$$M_p(r, f) = \mathrm{O}\left(\left(\log \frac{1}{1-r}\right)^{1/p}\right), \qquad \text{as } r \to 1.$$

The following proposition will be used in the proof of Theorem 3.

Proposition 2. *If $\frac{1}{2} \le p < \infty$ and $f \in \mathscr{F}_p$, then*

$$M_\infty(r, f) = \mathrm{O}\left(\left(\frac{1}{1-r}\right)^{1/p}\right), \qquad \text{as } r \to 1. \tag{36}$$

Proof. Arguing as in the proof of Theorem 5.9 of [4] and using Minkowski's inequality in continuous form (notice that $2p \ge 1$), we see that there exists a constant C which only depends on p such that

$$\begin{aligned} M_\infty(r, f) &\le C \frac{M_{2p}\left(\frac{1}{2}(1+r), f\right)}{(1-r)^{1/2p}} \\ &\le C\left[\frac{|f(0)| + \int_0^{(1+r)/2} M_{2p}(s, f')\, ds}{(1-r)^{1/2p}}\right], \qquad 0 < r < 1. \end{aligned} \tag{37}$$

Reasoning again as in the proof of Theorem 5.9 of [4] and using that $f \in \mathscr{F}_p$, we obtain that

$$M_{2p}(s, f') \le C \frac{M_p\left(\frac{1}{2}(1+s), f'\right)}{(1-s)^{1/2p}} \le C \frac{1}{(1-s)^{1+1/2p}}, \qquad 0 < s < 1. \tag{38}$$

Putting together (37) and (38) we deduce that

$$M_\infty(r,f) = \mathrm{O}\left(\frac{\int_0^{(1+r)/2}(1-s)^{-1-1/2p}\,ds}{(1-r)^{1/2p}}\right) = \mathrm{O}\left(\left(\frac{1}{1-r}\right)^{1/p}\right), \qquad \text{as } r \to 1.$$

This finishes the proof. □

Proof of Theorem 3. Let p and f be as in Theorem 3. Assume, without loss of generality that $f \in S$, that is $f(0) = 0$ and $f'(0) = 1$. It follows from (11) that

$$rI_p'(r,f) \le C_p M_\infty^p(r,f),$$

(see p. 127 of [12]) and then, using (36), we deduce that

$$I_p(r,f) \le I_p\left(\frac{1}{2}, f\right) + 2C_p \int_{1/2}^r M_\infty^p(s,f)\,ds \le I_p\left(\frac{1}{2}, f\right) + 2C_p \int_{1/2}^r \frac{1}{1-s}\,ds$$
$$= \mathrm{O}\left(\log\frac{1}{1-r}\right), \qquad \text{as } r \to 1. \ \square$$

Remark 2. Notice that if $\frac{1}{2} \le p < \infty$ then the function

$$f(z) = 1/(1-z)^{1/p}, \qquad z \in \Delta,$$

belongs to $\mathscr{U} \cap \mathscr{F}_p$ and satisfies

$$M_p(r,f) \asymp \left(\left(\log\frac{1}{1-r}\right)^{1/p}\right), \qquad \text{as } r \to 1.$$

Hence, Theorem 3 is sharp.

References

[1] Anderson, J. M., J. Clunie, and Ch. Pommerenke: On Bloch functions and normal functions. - J. Reine Angew. Math. 270, 1974, 12–37.

[2] Baernstein, A., II, D. Girela, and J. A. Peláez: Univalent functions, Hardy spaces and spaces of Dirichlet type. - Illinois J. Math. (to appear).

[3] Clunie, J. G., and T. H. MacGregor: Radial growth of the derivative of univalent functions. - Comment. Math. Helv. 59, 1984, 362–365.

[4] Duren, P. L.: Theory of H^p Spaces, Second edition. - Dover, Mineola, New York, 2000.

[5] Duren, P. L.: Univalent Functions. - Springer-Verlag, New York–Berlin–Heidelberg–Tokyo, 1983.

[6] Girela, D., and J. A. Peláez: Growth properties and sequences of zeros of analytic functions in spaces of Dirichlet type. - Preprint, 2004.

[7] Hardy, G. H.: The mean value of the modulus of an analytic function. - Proc. London Math. Soc. (2) 14, 1915, 269–277.

[8] Hardy, G. H., and J. E. Littlewood: Some properties of conjugate functions. - J. Reine Angew. Math. 167, 1931, 405–423.

[9] Hardy, G. H., and J. E. Littlewood: Some properties of fractional integrals, II. - Math. Z. 34, 1932, 403–439.

[10] Littlewood, J. E., and R. E. A. C. Paley: Theorems on Fourier series and power series. II. - Proc. London Math. Soc. 42, 1936, 52–89.

[11] Makarov, N. G.: On the distortion of boundary sets under conformal mappings. - Proc. London Math. Soc. (3) 51, 1985, 369–384.

[12] Pommerenke, Ch.: Univalent Functions. - Vandenhoeck und Ruprecht, Göttingen, 1975.

[13] Vinogradov, S. A.: Multiplication and division in the space of the analytic functions with area integrable derivative, and in some related spaces. - Zap. Nauchn. Sem. S.-Petersburg. Otdel. Mat. Inst. Steklov. (POMI) 222, 1995, 45–77, 308 (Russian). English translation in: J. Math. Sci. 87, 1997, 3806–3827.

[14] Zygmund, A.: Trigonometric Series. Vol. I and Vol. II, Second edition. - Cambridge Univ. Press, Cambridge, 1959.

Received 10 March 2004