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ON Q-HOMEOMORPHISMS

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Abstract. Space BMO-quasiconformal mappings satisfy a special modulus inequality that is used to define the class of Q-homeomorphisms. In this class we study distortion theorems, boundary behavior, removability and mapping problems. Our proofs are based on extremal length methods and properties of BMO functions.

1. Introduction

Let D be a domain in $\mathbf{R}^n$, $n \geq 2$, and let $Q\colon D \to [1,\infty]$ be a measurable function.

Definition 1.1. We say that a homeomorphism $f\colon D \to \overline{\mathbf{R}^n}$ is a Q-*homeomorphism* if

$$M(f\Gamma) \leq \int_D Q(x)\varrho^n(x)\,dm(x) \tag{1.2}$$

for every family Γ of paths in D and every admissible function ϱ for Γ.

Here we use only open paths $\gamma\colon (a,b) \to \overline{\mathbf{R}^n}$. We say that γ joins sets E and F in a domain D if $\gamma\bigl(\bigl((a,b)\bigr) \subset D$ and γ is a restriction of a closed path $\overline{\gamma}\colon [a,b] \to \mathbf{R}^n$ such that $\overline{\gamma}(a) \in E$ and $\overline{\gamma}(b) \in F$. The family of all paths which join E and F in D will be denoted by $\Gamma(E,F;D)$. Recall that, given a family of paths Γ in a domain D, a Borel function $\varrho\colon \mathbf{R}^n \to [0,\infty]$ is called *admissible* for Γ, abbreviated $\varrho \in \operatorname{adm}\Gamma$, if

$$\int_\gamma \varrho(x)\,|dx| \geq 1 \tag{1.3}$$

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for each $\gamma \in \Gamma$. The (*conformal*) *modulus* $M(\Gamma)$ of Γ is defined as

$$M(\Gamma) = \inf_{\varrho \in \operatorname{adm} \Gamma} \int_D \varrho^n(x)\, dm(x). \tag{1.4}$$

An example of $Q(x)$-homeomorphisms is provided by a class of homeomorphisms $f \in \mathrm{W}^{1,n}_{\mathrm{loc}}(D)$ whose dilatation majorant Q is in $\mathrm{L}^{n-1}_{\mathrm{loc}}(D)$, see Theorem 2.19 below.

For $f\colon D \to \mathbf{R}^n$ with partial derivatives a.e. and $x \in D$, we let $f'(x)$ denote the Jacobian matrix of f at x or the differential operator of f at x, if it exists, by $J(x) = J(x,f) = \det f'(x)$ the Jacobian of f at x, and by $|f'(x)|$ the operator norm of $f'(x)$, i.e., $|f'(x)| = \max\bigl\{|f'(x)h| : h \in \mathbf{R}^n,\ |h| = 1\bigr\}$. We also let $l\bigl(f'(x)\bigr) = \min\bigl\{|f'(x)h| : h \in \mathbf{R}^n,\ |h| = 1\bigr\}$. The *outer dilatation* of f at x is defined by

$$K_O(x) = K_O(x,f) = \begin{cases} \dfrac{|f'(x)|^n}{|J(x,f)|}, & \text{if } J(x,f) \neq 0, \\ 1, & \text{if } f'(x) = 0, \\ \infty, & \text{otherwise,} \end{cases} \tag{1.5}$$

the *inner dilatation* of f at x by

$$K_I(x) = K_I(x,f) = \begin{cases} \dfrac{|J(x,f)|}{l\bigl(f'(x)\bigr)^n}, & \text{if } J(x,f) \neq 0, \\ 1, & \text{if } f'(x) = 0, \\ \infty, & \text{otherwise,} \end{cases} \tag{1.6}$$

and the *maximal dilatation*, or in short the *dilatation*, of f at x by

$$K(x) = K(x,f) = \max\bigl(K_O(x), K_I(x)\bigr), \tag{1.7}$$

cf. [MRV] and [Re_1]. Note, that $K_I(x) \le K_O(x)^{n-1}$ and $K_O(x) \le K_I(x)^{n-1}$, see e.g. 1.2.1 in [Re_1], and, in particular, $K_O(x)$, $K_I(x)$ and $K(x)$ are simultaneously finite or infinite. $K(x,f) < \infty$ a.e. is equivalent to the condition that a.e. either $\det f'(x) \ge 0$ or $f'(x) = 0$, cf. [GI] and [IŠ].

Definition 1.8. Given a function $Q\colon D \to [1,\infty]$, we say that a sense-preserving homeomorphism $f\colon D \to \mathbf{R}^n$ is $Q(x)$-*quasiconformal*, abbr. $Q(x)$-*qc*, if $f \in \mathrm{W}^{1,n}_{\mathrm{loc}}(D)$ and

$$K(x,f) \le Q(x) \quad \text{a.e.} \tag{1.9}$$

Definition 1.10. We say that $f\colon D \to \mathbf{R}^n$ is *BMO-quasiconformal*, abbr. *BMO-qc*, if f is $Q(x)$-qc for some BMO function $Q\colon D \to [1,\infty]$.

Here BMO stands for the function space introduced by John and Nirenberg [JN], see also [RR]. Recall that a real-valued function $\varphi \in \mathrm{L}^1_{\mathrm{loc}}(D)$ is said to be of *bounded mean oscillation* in D, abbr. $\varphi \in \mathrm{BMO}(D)$, if

$$\|\varphi\|_* = \sup_{B\subset D} \frac{1}{|B|}\int_B |\varphi(x) - \varphi_B|\, dx < \infty, \tag{1.11}$$

where the supremum is taken over all balls B in D and

$$\varphi_B = \frac{1}{|B|}\int_B \varphi(x)\, dx \tag{1.12}$$

is the mean value of the function φ over B. It is well known that $\mathrm{L}^\infty(D) \subset \mathrm{BMO}(D) \subset \mathrm{L}^p_{\mathrm{loc}}(D)$ for all $1 \le p < \infty$.

Since $\mathrm{L}^\infty(D) \subset \mathrm{BMO}$, the class of BMO-qc mappings obviously contains all qc mappings. We show that many properties of qc mappings hold for BMO-qc mappings. Note that Q-homeomorphisms, $Q(x)$-qc and BMO-qc mappings are Möbius invariants and hence the concepts extend to mappings $f\colon D \to \overline{\mathbf{R}^n} = \mathbf{R}^n \cup \{\infty\}$ as in the qc theory.

The subject of Q-homeomorphisms is interesting on its own right and has applications to much wider classes of mappings which we plan to investigate elsewhere. In this paper we study various properties as distortion, removability, boundary behavior and mapping properties of Q-homeomorphisms under various conditions on Q. Then the corresponding properties of $Q(x)$-qc mappings $f\colon D \to \mathbf{R}^n$, $n \ge 2$, with $Q \in \mathrm{L}^{n-1}_{\mathrm{loc}}$ are obtained as simple consequences of Theorem 2.19 below. A special attention is paid to BMO-qc mappings in $\mathbf{R}^n$, $n \ge 3$.

The study of related maps for $n = 2$ started by David [Da] and Tukia [Tu]. Recently Astala, Iwaniec, Koskela and Martin considered mappings with dilatation controlled by BMO functions for $n \ge 3$, see e.g. [IKM] and [AIKM]. It is necessary to note the activity of the related investigations of mappings of finite distortion, see e.g. [KKM$_1$], [KKM$_2$], [IKO], [IKMS], [KR], [KO], [MV$_1$] and [MV$_2$]. The present paper is a continuation of our study of BMO-qc mappings in the plane [RSY$_{1-3}$], cf. [IM], see also [Sa], and a similar geometric approach is used throughout.

For $a, b \in \overline{\mathbf{R}^n}$ and $E, F \subset \overline{\mathbf{R}^n}$ we let $q(a,b)$, $q(E)$ and $q(E,F)$ denote the spherical (chordal) distance between the points a and b, the spherical diameter of E and the spherical distance between E and F, respectively. We denote by $B^n(a,r)$ the Euclidean ball $|x-a| < r$ in $\mathbf{R}^n$ with center a and radius r, $S^n(a,r) = \partial B^n(a,r)$. We also let $B^n(r) = B^n(0,r)$ and $\mathbf{B}^n = B^n(1)$, $S^n = \partial \mathbf{B}^n$.

2. Preliminaries

2.1. Proposition. *Let $f\colon D \to \mathbf{R}^n$ be a $Q(x)$-qc mapping. Then*
(i) *f is differentiable a.e.,*
(ii) *f satisfies Lusin's property (N),*
(iii) *$J_f(x) \geq 0$ a.e.*
If, in addition, $Q \in \mathrm{BMO}(D)$, or if more generally $Q \in \mathrm{L}^{n-1}_{\mathrm{loc}}$, then $f^{-1} \in \mathrm{W}^{1,n}_{\mathrm{loc}}\bigl(f(D)\bigr)$, and
(iv) *f^{-1} is differentiable a.e.,*
(v) *f^{-1} has the property (N),*
(vi) *$J_f(x) > 0$ a.e.*

Proof. (i) and (ii) follow from the corresponding results for $\mathrm{W}^{1,n}_{\mathrm{loc}}$ homeomorphisms, see [Re$_2$] and [Re$_3$]. In view of (i) and the fact that f is sense-preserving, (iii) follows by Rado–Reichelderfer [RR*, p. 333].

Now, if $Q \in \mathrm{BMO}$, then Q and hence $K(x,f)$ belongs to $\mathrm{L}^p_{\mathrm{loc}}$ for all $p < \infty$ and, in particular, to $\mathrm{L}^{n-1}_{\mathrm{loc}}$. Hence, by Theorem 6.1 in [HK], $f^{-1} \in \mathrm{W}^{1,n}_{\mathrm{loc}}\bigl(f(D)\bigr)$ and thus (iv)–(vi) follow.

2.2. Lemma. *Let Q be a positive BMO function in $\mathbf{B}^n, n \geq 3$, and let $A(t) = \{x \in \mathbf{R}^n : t < |x| < e^{-1}\}$. Then for all $t \in (0, e^{-2})$,*

$$\int_{A(t)} \frac{Q(x)\,dm(x)}{(|x| \log 1/|x|)^n} \leq c \tag{2.3}$$

where $c = c_1\|Q\|_* + c_2 Q_1$, and c_1 and c_2 are positive constants which depend only on n. Here $\|Q\|_*$ is the BMO norm of Q and Q_1 is the average of Q over $B^n(1/e)$.

Proof. Fix $t \in (0, e^{-2})$, and set

$$\eta(t) = \int_{A(t)} \frac{Q(x)\,dm(x)}{(|x| \log 1/|x|)^n}. \tag{2.4}$$

For $k = 1, 2, \ldots$, write $t_k = e^{-k}$, $A_k = \{x \in \mathbf{R}^n : t_{k+1} < |x| < t_k\}$, $B_k = B^n(t_k)$ and let Q_k be the mean value of $Q(x)$ in B_k. Choose an integer N, such that $t_{N+1} \leq t < t_N$. Then $A(t) \subset A(t_{N+1}) = \bigcup_{k=1}^{N+1} A_k$, and

$$\eta(t) \leq \int_{A(t_{N+1})} \frac{Q(x)}{|x|^n \log^n 1/|x|}\,dx = S_1 + S_2 \tag{2.5}$$

where

$$S_1 = \sum_{k=1}^{N+1} \int_{A_k} \frac{Q(x) - Q_k}{|x|^n \log^n 1/|x|}\,dx \tag{2.6}$$

and

$$S_2 = \sum_{k=1}^{N+1} Q_k \int_{A_k} \frac{dx}{|x|^n \log^n 1/|x|}. \tag{2.7}$$

Since $A_k \subset B_k$ and for $x \in A_k$, $|x|^{-n} \le \Omega_n e^n/|B_k|$, where $\Omega_n = |\mathbf{B}^n|$ and since $\log 1/|x| > k$, it follows that

$$|S_1| \le \Omega_n \sum_{k=1}^{N+1} \frac{1}{k^n} \frac{e^n}{|B_k|} \int_{B_k} |Q(x) - Q_k|\, dx \le \Omega_n e^n \|Q\|_* \sum_{k=1}^{N+1} \frac{1}{k^n}$$

and, thus,

$$|S_1| \le 2\Omega_n e^n \|Q\|_* \tag{2.8}$$

because, for $p \ge 2$,

$$\sum_{k=1}^{\infty} \frac{1}{k^p} < 2. \tag{2.9}$$

To estimate S_2, we first obtain from the triangle inequality that

$$Q_k = |Q_k| \le \sum_{l=2}^{k} |Q_l - Q_{l-1}| + Q_1. \tag{2.10}$$

Next we show that, for $l \ge 2$,

$$|Q_l - Q_{l-1}| \le e^n \|Q\|_*. \tag{2.11}$$

Indeed,

$$\begin{aligned} |Q_l - Q_{l-1}| &= \frac{1}{|B_l|} \left| \int_{B_l} (Q(x) - Q_{l-1})\, dx \right| \\ &\le \frac{e^n}{|B_{l-1}|} \int_{B_{l-1}} |Q(x) - Q_{l-1}|\, dx \le e^n \|Q\|_*. \end{aligned}$$

Thus, by (2.10) and (2.11),

$$Q_k \le Q_1 + \sum_{l=2}^{k} e^n \|Q\|_* \le Q_1 + k e^n \|Q\|_*, \tag{2.12}$$

and, since

$$\int_{A_k} \frac{dx}{|x|^n \log^n 1/|x|} \le \frac{1}{k^n} \int_{A_k} \frac{dx}{|x|^n} = \omega_{n-1} \frac{1}{k^n}, \tag{2.13}$$

where ω_{n-1} is the $(n-1)$-measure of S^{n-1}, it follows that

$$S_2 \le \omega_{n-1} \sum_{k=1}^{N+1} \frac{Q_k}{k^n} \le \omega_{n-1} Q_1 \sum_{k=1}^{N+1} \frac{1}{k^n} + \omega_{n-1} e^n \|Q\|_* \sum_{k=1}^{N+1} \frac{1}{k^{(n-1)}}.$$

Thus, for $n \ge 3$, we have by (2.9) that

$$S_2 \le 2\omega_{n-1} Q_1 + 2\omega_{n-1} e^n \|Q\|_*. \tag{2.14}$$

Finally, from (2.8) and (2.14) we obtain (2.3), where $c = c_1 Q_1 + c_2 \|Q\|_*$, and c_1 and c_2 are constants which depend only on n.

2.15. Remark. For $n \ge 2$, $0 < t < e^{-2}$, and $A(t)$ as in Lemma 2.2, let Γ_t denote the family of all paths joining the spheres $|x| = t$ and $|x| = e^{-1}$ in $A(t)$. Then the function ϱ given by

$$\varrho(x) = \frac{1}{(\log\log 1/t)|x| \log 1/|x|} \tag{2.16}$$

for $x \in A(t)$ and $\varrho(x) = 0$ otherwise, belongs to $\operatorname{adm}\Gamma_t$.

The following lemma provides the standard lower bound for the modulus of all paths joining two continua in $\mathbf{R}^n$, see [Ge$_1$], [Vu, 7.37]. The lemma involves the constant λ_n which depends only on n and which appears in the asymptotic estimates of the modulus of the Teichmüller ring $R_n(t) = \mathbf{R}^n \setminus \big([-\infty, 0] \cup [t, \infty]\big)$.

2.17. Lemma. *Let E and F be two continua in $\overline{\mathbf{R}^n}$, $n \ge 2$, with $q(E) \ge \delta_1 > 0$ and $q(F) \ge \delta_2 > 0$, and let Γ be the family of paths joining E and F. Then*

$$M(\Gamma) \ge \frac{\omega_{n-1}}{\left(\log \dfrac{2\lambda_n}{\delta_1 \delta_2}\right)^{n-1}} \tag{2.18}$$

where ω_{n-1} is the $(n-1)$-measure of S^{n-1}.

2.19. Theorem. *Let $f\colon D \to \mathbf{R}^n$ be a $Q(x)$-qc mapping with $Q \in \mathrm{L}^{n-1}_{\mathrm{loc}}(D)$. Then, for every family Γ of paths in D and every $\varrho \in \operatorname{adm}\Gamma$,*

$$M(f\Gamma) \le \int_D Q(x) \varrho^n(x)\, dm(x), \tag{2.20}$$

i.e., f is a Q-homeomorphism.

Proof. Since $Q \in \mathrm{L}^{n-1}_{\mathrm{loc}}$, we may apply Proposition 2.1. Thus $f^{-1} \in \mathrm{W}^{1,n}_{\mathrm{loc}}\big(f(D)\big)$ and hence $f^{-1} \in \mathrm{ACL}^{n}_{\mathrm{loc}}\big(f(D)\big)$, see e.g. [Maz, p. 8]. Therefore, by Fuglede's theorem, see [Fu] and [Vä$_1$, p. 95], if $\widetilde{\Gamma}$ is the family of all paths $\gamma \in f\Gamma$ for which f^{-1} is absolutely continuous on all closed subpaths of γ, then $M(f\Gamma) = M(\widetilde{\Gamma})$. Also, by Proposition 2.1, f^{-1} is differentiable a.e. Hence, as in the qc case, see [Vä, p. 110], given a function $\varrho \in \operatorname{adm}\Gamma$, we let $\tilde{\varrho}(y) = \varrho\big(f^{-1}(y)\big)|(f^{-1})'(y)|$ for $y \in f(D)$ and $\tilde{\varrho}(y) = 0$ otherwise. Then we obtain that for $\tilde{\gamma} \in \widetilde{\Gamma}$

$$\int_{\tilde{\gamma}} \tilde{\varrho}\, ds \geq \int_{f^{-1}\circ\tilde{\gamma}} \varrho\, ds \geq 1,$$

and consequently $\tilde{\varrho} \in \operatorname{adm}\widetilde{\Gamma}$.

By Proposition 2.1, both f and f^{-1} are differentiable a.e. and have (N)-property and $J(x,f) > 0$ a.e., and since f^{-1} is a homeomorphism in $\mathrm{W}^{1,n}_{\mathrm{loc}}(D)$, we can use the integral transformation formula and obtain

$$\begin{aligned} M(f\Gamma) = M(\widetilde{\Gamma}) &\leq \int_{f(D)} \tilde{\varrho}^n\, dm(y) \\ &= \int_{f(D)} \varrho\big(f^{-1}(y)\big)^n |(f^{-1})'(y)|^n\, dm(y) \\ &= \int_{f(D)} \frac{\varrho\big(f^{-1}(y)\big)^n}{l\big(f'(f^{-1}(y))\big)^n}\, dm(y) \\ &= \int_{f(D)} \varrho\big(f^{-1}(y)\big)^n K_I\big(f^{-1}(y), f\big) J(y, f^{-1})\, dm(y) \\ &\leq \int_{f(D)} \varrho\big(f^{-1}(y)\big)^n Q\big(f^{-1}(y)\big) J(y, f^{-1})\, dm(y) \\ &= \int_D Q(x)\varrho(x)^n\, dm(x). \end{aligned}$$

The proof follows.

2.21. Corollary. *Every BMO-qc mapping is a Q-homeomorphism with some $Q \in \mathrm{BMO}$.*

3. Distortion theorems

3.1. Theorem. *Let $f\colon \mathbf{B}^n \to \overline{\mathbf{R}^n}$ be a Q-homeomorphism with $Q \in \mathrm{BMO}(\mathbf{B}^n)$. If $q\big(\overline{\mathbf{R}^n} \setminus f\big(B^n(1/e)\big)\big) \geq \delta > 0$, then for all $|x| < e^{-2}$*

$$q\big(f(x), f(0)\big) \leq \frac{C}{(\log 1/|x|)^{\alpha}} \tag{3.2}$$

where C and α are positive constants which depend only on n, δ, the BMO norm $\|Q\|_*$ of Q and the average Q_1 of Q over the ball $|x| < 1/e$.

Proof. Fix $t \in (0, e^{-2})$. Let $A(t)$, Γ_t and ϱ be as in Remark 2.15 and let $\delta_t = q\big(f\big(B^n(t)\big)\big)$. Then, by Remark 2.15, $\varrho \in \operatorname{adm}\Gamma_t$, and

$$M(f\Gamma_t) \le \int_{\mathbf{R}^n} Q\varrho^n \, dm. \tag{3.3}$$

In view of (2.3),

$$\int_{\mathbf{R}^n} Q\varrho^n \, dm = \int_{A(t)} Q\varrho^n \, dm \le \frac{c}{(\log\log 1/t)^{n-1}} \tag{3.4}$$

where c is the constant which appears in Lemma 2.2, see also Lemma 3.2 in [RSY$_2$] for $n = 2$. On the other hand, Lemma 2.17 applied to $M(f\Gamma_t)$ with $E = f\big(\overline{B^n(t)}\big)$ and $F = \overline{\mathbf{R}^n} \setminus f\big(B^n(1/e)\big)$ yields

$$M(f\Gamma_t) \ge \frac{\omega_{n-1}}{\left(\log \dfrac{2\lambda_n}{\delta\delta_t}\right)^{n-1}}, \tag{3.5}$$

and the result follows by (3.3)–(3.5) and the fact that $q\big(f(x), f(0)\big) \le \delta_t$ for $|x| = t$.

3.6. Corollary. *Let $\mathscr{F}$ be a family of Q-homeomorphisms $f\colon D \to \mathbf{R}^n$, with $Q \in \mathrm{BMO}(D)$, and let $\delta > 0$. If every $f \in \mathscr{F}$ omits two points a_f and b_f in $\mathbf{R}^n$ with $q(a_f, b_f) \ge \delta$, then $\mathscr{F}$ is equicontinuous.*

3.7. Theorem. *Let $f\colon \mathbf{B}^n \to \overline{\mathbf{R}^n}$ be a Q-homeomorphism with $Q \in L^1(\mathbf{B}^n)$, $f(0) = 0$, $q\big(\overline{\mathbf{R}^n} \setminus f(\mathbf{B}^n)\big) \ge \delta > 0$ and $q\big(f(x_0), f(0)\big) \ge \delta$ for some $x_0 \in \mathbf{B}^n$. Then, for all $|x| < r = \min(|x_0|/2, 1 - |x_0|)$,*

$$|f(x)| \ge \psi(|x|) \tag{3.8}$$

where $\psi(t)$ is a strictly increasing function with $\psi(0) = 0$ which depends only on the L^1-norm of Q in $\mathbf{B}^n$, n and δ.

Proof. Given y_0 with $|y_0| < r$ choose a continuum E_1 which contains the points 0 and x_0 and a continuum E_2 which contains the point y_0 and $\partial\mathbf{B}^n$, so that $\operatorname{dist}(E_1, E_2 \cup \partial\mathbf{B}^n) = |y_0|$. More precisely, denote by L the straight line generated by the pair of points 0 and x_0 and by P the plane defined by the triple of the points 0, x_0 and y_0 (if $y_0 \in L$, then P is an arbitrary plane passing through L). Let C be the circle under intersection of P and the sphere $S^n(y_0, |y_0|) \subset B^n(|x_0|)$. Let t_0 is the tangency point to C of the ray starting from x_0 which is opposite to y_0 with respect to L (an arbitrary one of the two possible if $y_0 \in L$). Then $E_1 = [x_0, t_0] \cup \gamma(0, t_0)$ where $\gamma(0, t_0)$ is the shortest arc of C joining 0 and t_0, and $E_2 = [y_0, i_0] \cup S^n$ where $S^n = \partial\mathbf{B}^n$ is the unit sphere and i_0 is the point (opposite to t_0 with respect to L) of the intersection of S^n with the straight line in P which is perpendicular to L and passing through y_0.

Let Γ denote the family of paths which join E_1 and E_2. Then

$$\varrho(x) = |y_0|^{-1}\chi_{\mathbf{B}^n}(x) \in \operatorname{adm}\Gamma$$

and hence,

$$M(f\Gamma) \le \int \varrho^n(x)Q(x)\,dm(x) \le |y_0|^{-n}\int_{\mathbf{B}^n} Q(x)\,dm(x) = \frac{\|Q\|_1}{|y_0|^n}. \tag{3.9}$$

The ring domain $A' = f\bigl(\mathbf{B}^n \backslash (E_1 \cup E_2)\bigr)$ separates the continua $E_1' = f(E_1)$ and $E_2' = \overline{\mathbf{R}^n}\backslash f(\mathbf{B}^n \backslash E_2)$, and since

$$q(E_1') \ge q\bigl(f(x_0), f(0)\bigr) \ge \delta, \quad q(E_2') \ge q\bigl(\overline{\mathbf{R}^n}\backslash f(\mathbf{B}^n)\bigr) \ge \delta$$

and

$$q(E_1', E_2') \le q\bigl(f(y_0), f(0)\bigr)$$

it follows that

$$M\bigl(f(\Gamma)\bigr) \ge \lambda\bigl(q\bigl(f(y_0), f(0)\bigr)\bigr) \tag{3.10}$$

where $\lambda(t) = \lambda_n(\delta, t)$ is a strictly decreasing positive function with $\lambda(t) \to \infty$ as $t \to 0$, see [Vä$_1$, 12.7]. Hence, by (3.9) and (3.10),

$$|f(y_0)| > q\bigl(f(y_0), f(0)\bigr) \ge \psi(|y_0|)$$

where

$$\psi(t) = \lambda^{-1}\left(\frac{\|Q\|_1}{t^n}\right)$$

has the required properties.

3.11. Remark. In view of Theorem 2.19 and Corollary 2.21, Theorem 3.7 is valid for $Q(x)$-qc mappings with $Q \in L^{n-1}(\mathbf{B}^n)$, and Theorem 3.1 and Corollary 3.6 are valid for $Q(x)$-qc mappings with $Q \in \mathrm{BMO}(\mathbf{B}^n)$.

4. Removability of isolated singularities

Here and in Theorems 4.1 and 4.5, we describe two different and unrelated cases where isolated singularities are removable.

4.1. Theorem. *Let $f\colon \mathbf{B}^n \setminus \{0\} \to \mathbf{R}^n$ be a Q-homeomorphism. If*

$$\limsup_{r\to 0} \frac{1}{|B^n(r)|}\int_{B^n(r)} Q(x)\,dm(x) < \infty, \tag{4.2}$$

then f has an extension to $\mathbf{B}^n$ which is a Q-homeomorphism.

Proof. As the modulus of a family of paths which pass through the origin vanishes, it suffices to show that f has a continuous extension on $\mathbf{B}^n$. Suppose that this is not the case. Since f is a homeomorphism, $\overline{\mathbf{R}^n} \setminus f(\mathbf{B}^n \setminus \{0\})$ consists of two connected compact sets F_1 and F_2 in $\overline{\mathbf{R}^n}$ where F_1 contains the cluster set $E = C(0,f)$ of f at 0. Here F_1 is a nondegenerate continuum and using an arbitrary Möbius transformation we may assume that $F_1 \subset \mathbf{R}^n$. Now $U = F_1 \cup f(\mathbf{B}^n \setminus \{0\})$ is a neighborhood of E. Thus there exists $\delta > 0$ such that all balls $B_z = B^n(z,\delta)$, $z \in F_1$, are contained in U. Let $V = \cup B_z$.

Now, choose a point $y \in F_1$ such that $\operatorname{dist}(y, \partial V) = \delta$, and a point $z \in B_y \setminus F_1$. Next, choose a path $\beta\colon [0,1] \to B_y$ with $\beta(0) = y$, $\beta(1) = z$ and $\beta(t) \in B_y \setminus F_1$ for $t \in (0,1]$. Let $\alpha = f^{-1} \circ \beta$. For $r \in \bigl(0, |f^{-1}(z)|\bigr)$, let α_r denote the connected component of the curve $\alpha(I) \setminus B^n(r)$, $I = [0,1]$, which contains the point $f^{-1}(z) = \alpha(1)$, and let Γ_r denote the family of all paths joining α_r and the point 0 in $\mathbf{B}^n \setminus \{0\}$. Then the function $\varrho(x) = 1/r$ if $x \in B^n(r) \setminus \{0\}$ and $\varrho = 0$ otherwise is in $\operatorname{adm}\Gamma_r$, and by (4.2),

$$\limsup_{r\to 0} \int_{B^n(r)\setminus\{0\}} Q(x)\varrho^n(x)\,dm(x) = \Omega_n \limsup_{r\to 0} \frac{1}{|B^n(r)|}\int_{B^n(r)\setminus\{0\}} Q(x)\,dm(x) < \infty. \tag{4.3}$$

On the other hand, if we let Γ_r' denote the family of all paths joining two continua $f(\alpha_r)$ and E in $B_y \setminus E$, then $\Gamma_r' \subset f(\Gamma_r)$, and thus

$$M(\Gamma_r') \le M(f\Gamma_r). \tag{4.4}$$

Evidently, $\operatorname{dist}(f(\alpha_r), E) \to 0$, and the diameter of $f(\alpha_r)$ increases as $r \to 0$, and as both $f(\alpha_r)$ and E are subsets of a ball, $M(f\Gamma_r) \to \infty$ as $r \to 0$. This together with (4.3) and (4.4) contradicts the modulus inequality (2.20).

4.5. Theorem. *Let $f\colon \mathbf{B}^n \setminus \{0\} \to \mathbf{R}^n$ be a Q-homeomorphism with $Q \in \operatorname{BMO}(\mathbf{B}^n \setminus \{0\})$. Then f has a $Q(x)$-homeomorphic extension to $\mathbf{B}^n$.*

Proof. Fix $t \in (0, e^{-2})$ and let $A(t)$, Γ_t and ϱ be as in Remark 2.15. Then, by Lemma 2.17,

$$\frac{\omega_{n-1}}{\left(\log \dfrac{2\lambda_n}{\delta\delta_t}\right)^{n-1}} \le M(f\Gamma_t) \le \int_{A(t)} Q\varrho^n\,dm, \tag{4.6}$$

where $\delta = q\bigl(f\bigl(\partial B^n(e^{-1})\bigr)\bigr)$ and $\delta_t = q\bigl(f\bigl(\partial B^n(t)\bigr)\bigr)$. Since isolated singularities are removable for BMO functions, see [RR], we may assume that Q is defined in $\mathbf{B}^n$ and that $Q \in \operatorname{BMO}(\mathbf{B}^n)$. Thus, by Lemma 3.2 in [RSY$_2$] for $n = 2$ and Lemma 2.2 for $n \ge 3$

$$\int_{A(t)} Q(x)\varrho^n\,dm \le \frac{c}{(\log\log 1/t)^{n-1}}. \tag{4.7}$$

Since here c depends only on n, $\|Q\|_*$ and $Q_1 = Q_{B^n(1/e)}$, we obtain by (4.6)–(4.7) that $\delta_t \to 0$ as $t \to 0$, and hence that $\lim_{x\to 0} f(x)$ exists.

4.8. Corollary. *If* $f\colon \mathbf{R}^n \to \mathbf{R}^n$ *is a BMO-qc mapping, then* f *has a homeomorphic extension to* $\overline{\mathbf{R}^n}$ *and, in particular,* $f(\mathbf{R}^n) = \mathbf{R}^n$.

5. Boundary behavior

For the boundary behavior some regularity of the boundary is needed for which the following notation is used.

We say that a domain D in $\mathbf{R}^n$ is a *BMO extension domain* if every $u \in \mathrm{BMO}(D)$ has an extension to $\mathbf{R}^n$ which belongs to $\mathrm{BMO}(\mathbf{R}^n)$. It was shown in [GO] and [Jo] that a domain D is a BMO extension domain if and only if D is a *uniform domain*, i.e., for some a and $b > 0$, each pair of points $x_1, x_2 \in D$ can be joined by a rectifiable arc $\gamma \subset D$ such that

$$l(\gamma) \le a \cdot |x_1 - x_2| \tag{5.1}$$

and, for all $x \in \gamma$,

$$\min_{i=1,2} l\bigl(\gamma(x_i, x)\bigr) \le b \cdot \mathrm{dist}(x, \partial D) \tag{5.2}$$

where $l(\gamma)$ is the Euclidean length of γ, $\gamma(x_i, x)$ is the part of γ between x_i and x.

The uniform domains were introduced in [MS] and their various characterizations can be found in [Ge$_2$], [Ma], [Mar], [Vä$_2$] and [Vu]. It was shown in [MS], p. 387, that uniform domains are invariant under quasiconformal mappings of $\mathbf{R}^n$. In particular, every domain which is bounded by a quasisphere, i.e., the image of $\partial\mathbf{B}^n$ under a qc automorphism of $\mathbf{R}^n$, is uniform. Note that a bounded convex domain is uniform. We also write $u \in \mathrm{BMO}(\overline{D})$ if u has a BMO extension to an open set $U \subset \mathbf{R}^n$ such that $\overline{D} \subset U$. Domains D in $\mathbf{R}^n$ for which every $u \in \mathrm{BMO}(D)$ admits such extension can be characterized as relatively uniform domains, see e.g. [Go$_2$].

A domain $D \subset \mathbf{R}^n$ is called a *quasiextremal distance domain* or a *QED domain* if there is $K \ge 1$ such that, for each pair of disjoint continua E and F in D,

$$M\bigl(\Gamma(E, F; \overline{\mathbf{R}^n})\bigr) \le K \cdot M\bigl(\Gamma(E, F; D)\bigr). \tag{5.3}$$

It is known that every uniform domain D is a QED domain and there exist QED domains which are not uniform, see [GM], pp. 189 and 194. Every QED domain D is *quasiconvex*, i.e., (5.1) holds for all x_1 and $x_2 \in D \setminus \{\infty\}$, see Lemma 2.7 in [GM], p. 184. Hence every QED domain D is *locally connected* at ∂D, i.e., every point $x \in \partial D$ has an arbitrarily small neighborhood U such that $U \cap D$ is connected, cf. also Lemma 2.11 in [GM], p. 187, and [HK$_1$], p. 190. Note that every Jordan domain D in $\mathbf{R}^n$ is locally connected at ∂D, see [Wi], p. 66.

We say that ∂D is *strongly accessible* if, for nondegenerate continua E and F in $\overline{D}$,

$$M\big(\Gamma(E,F;D)\big) > 0, \tag{5.4}$$

and that ∂D is *weakly flat* if, for nondegenerate continua E and F in $\overline{D}$ with $E \cap F \neq \emptyset$,

$$M\big(\Gamma(E,F;D)\big) = \infty. \tag{5.5}$$

These properties are clearly invariant under qc mappings of $\overline{\mathbf{R}^n}$ and they are closely related to properties P_1 and P_2 by Väisälä in [Vä$_1$, 17.5] as well as to the notions of quasiconformal flatness and quasiconformal accessibility by Näkki in [Nä] and by Herron and Koskela in [HK$_1$].

We shall show below that weak flatness implies strong accessibility. The converse does not hold as in the example of a disk minus a cut. Note that the conditions (5.4) and (5.5) automatically hold if E and F are inside D, see e.g. Lemma 1.15 in [Nä], p. 16, and [Vä$_1$, 10.12], but not if E and F are in ∂D. It is known that for a QED domain the inequality (5.3) holds for each pair of disjoint continua E and F in $\overline{D}$, see Theorem 2.8 in [HK$_2$], p. 173, cf. Lemma 6.11 in [MV], p. 35. The latter property of QED domains also implies (5.3) for nondegenerate intersecting continua E and F in $\overline{D}$. Hence QED domains and, consequently, uniform domains have weakly flat boundaries, cf. Lemma 3.1 in [HK$_1$], p. 196.

5.6. Lemma. *If the boundary of a domain D in $\overline{\mathbf{R}^n}$, $n \geq 2$, is weakly flat, then it is strongly accessible.*

Proof. Let E and F be nondegenerate continua in $\overline{D}$. Without loss of generality we may assume that $E \cap F = \emptyset$, that ∞ lies outside of $E \cup F$ and that $E \subset \partial D$, see Lemma 1.15 in [Nä], p. 16. Take $0 < \varepsilon < \frac{1}{2}\operatorname{dist}(E,F)$. Then $E_\varepsilon \cap F_\varepsilon = \emptyset$ where $E_\varepsilon = \{x \in D : \operatorname{dist}(x,E) < \varepsilon\}$ and $F_\varepsilon = \{x \in D : \operatorname{dist}(x,F) < \varepsilon\}$. Since each path in $\Gamma(E,E;D)$ contains a subpath which belongs to $\Gamma(E,E_\varepsilon;D)$, it follows, see e.g. [Fu], p. 178, cf. [AB], p. 115, that $M\big(\Gamma(E,E;D)\big) \leq M\big(\Gamma(E,E_\varepsilon;D)\big)$. In view of the weak flatness of ∂D, $M\big(\Gamma(E,E;D)\big) = \infty$. Therefore $M\big(\Gamma(E,E_\varepsilon;D)\big) = \infty$. By the Lindelöf principle, see e.g. [Ku], p. 54, the open subsets E_ε and F_ε of D can be covered by countable collections of open and, consequently, closed balls B inside E_ε and F_ε, respectively. Thus, by countable subadditivity of the modulus we can find a couple of closed balls, $B_0 \subset E_\varepsilon$ and $B_0^* \subset F_\varepsilon$, $B_0 \cap B_0^* = \emptyset$, such that

$$M\big(\Gamma(E,B_0;D)\big) > 0 \tag{5.7}$$

and

$$M\big(\Gamma(F,B_0^*;D)\big) > 0, \tag{5.8}$$

see Theorem 1 in [Fu], p. 176. Also

$$M\big(\Gamma(B_0, B_0^*; D)\big) > 0, \tag{5.9}$$

see e.g. Lemma 1.15 in [Nä].

Now, we use the idea of Näkki which appears in the proof of Theorem 1.16 in [Nä]. Let $\varrho \in \operatorname{adm} \Gamma(E, B_0^*; D)$. If

$$\int_\gamma \varrho\, ds \geq \frac{1}{3} \quad \text{or} \quad \int_{\gamma'} \varrho\, ds \geq \frac{1}{3} \tag{5.10}$$

for all rectifiable paths $\gamma \in \Gamma(E, B_0; D)$ and $\gamma' \in \Gamma(B_0, B_0^*; D)$, respectively, then $3\varrho \in \operatorname{adm} \Gamma(E, B_0; D)$ or $3\varrho \in \operatorname{adm} \Gamma(B_0, B_0^*; D)$, and hence by (5.7) and (5.9)

$$\int_D \varrho^n\, dm \geq 3^{-n} \min\Big(M\big(\Gamma(E, B_0; D)\big), M\big(\Gamma(B_0, B_0^*; D)\big)\Big) > 0. \tag{5.11}$$

If (5.11) does not hold for a couple of such paths γ and γ', then

$$\int_\alpha \varrho\, ds > \frac{1}{3} \tag{5.12}$$

for every rectifiable path $\alpha \in \Gamma(\gamma, \gamma'; R_0)$ where R_0 is a ring $r_0 < |x - c_0| < r_0'$, c_0 and r_0 are the center and the radius of B_0, respectively, and $r_0' = r_0 + \operatorname{dist}(B_0, B_0^* \cup \partial D)$, i.e., $3\varrho \in \operatorname{adm} \Gamma(\gamma, \gamma'; R_0)$, and hence

$$\int_D \varrho^n\, dm \geq 3^{-n} c_n \log \frac{r_0'}{r_0} > 0, \tag{5.13}$$

see [Vä$_1$, 10.12]. Thus, by (5.11) and (5.13)

$$M\big(\Gamma(E, B_0^*; D)\big) > 0. \tag{5.14}$$

If $I = \operatorname{Int} B_0^* \cap F \neq \emptyset$, then by Lemma 1.15 in [Nä] $M\big(\Gamma(B_0, \bar{I}; D)\big) > 0$, and arguing as above (take $\bar{I} = B_0^* \cap F$ instead of B_0^*) we obtain $M\big(\Gamma(E, \bar{I}; D)\big) > 0$ and hence by monotonicity of the modulus

$$M\big(\Gamma(E, F; D)\big) > 0. \tag{5.15}$$

If $B_0^* \cap F = \partial B_0^* \cap F \neq \emptyset$, then by subadditivity of the modulus we obtain (5.8), (5.9) and (5.14) for another ball $B_0' \subset \operatorname{Int} B_0^*$ with $B_0' \cap F = \emptyset$. Finally, if $B_0^* \cap F = \emptyset$, then repeating the above arguments (as in the proof of (5.14), replace B_0 and B_0^* by B_0^* and F, respectively) we again obtain (5.15) by (5.8) and (5.14).

5.16. Lemma. *Let D be a domain in $\mathbf{R}^n$, $n \geq 2$, which is locally connected at ∂D and let $f\colon D \to D' \subset \mathbf{R}^n$ be a Q-homeomorphism onto D' with $Q \in \mathrm{BMO}(\overline{D})$. If $\partial D'$ is strongly accessible, then f has a continuous extension $\tilde{f}\colon \overline{D} \to \overline{D}'$.*

Proof. Let $x_0 \in \partial D$. As BMO functions and Q-homeomorphisms are Möbius invariant, we may assume that $x_0 = 0$ and that $\partial \mathbf{B}^n \cap D \neq \emptyset$. We will show that the cluster set $E = C(0, f)$ of f at 0 is a point, which will prove that $f(x)$ has a limit at x_0.

Since D is locally connected at 0, E is a continuum. Suppose that E is nondegenerate. For $t \in (0, 1/e)$, let D_t denote the component of $B^n(t) \cap D$ for which $0 \in \overline{D}_t$. Note, that D_t is well defined, since D is locally connected at 0, and that $D_t \subset D_{t'}$ for $t < t'$. For $t \in (0, e^{-2}]$, let Γ_t denote the family of path joining $\overline{D_t}$ and the set $S = D \cap \partial B^n(1/e)$ in $D_{1/e} \setminus \overline{D}_t$. As in Lemma 2.2, we let $A(t)$ denote the spherical ring $t < |x| < 1/e$. Then the function $\varrho(x)$ defined in Remark 2.15 is admissible for Γ_t, and hence

$$M(f\Gamma_t) \leq \int_D Q(x)\varrho^n(x)\,dm(x). \tag{5.17}$$

If $Q \in \mathrm{BMO}(\overline{D})$, we may apply (3.3) in [RSY$_2$] for $n = 2$ and (2.3) for $n \geq 3$, and get

$$\int_D Q(x)\varrho^n(x)\,dm(x) \leq \int_{A(t)} Q(x)\varrho^n(x)\,dm(x) \to 0 \tag{5.18}$$

as $t \to 0$. On the other hand

$$M(f\Gamma_t) \geq M\big(\Gamma(\overline{f(S)}, E; D')\big). \tag{5.19}$$

Now, $\partial D'$ is strongly accessible, $f(S)$ contains a nondegenerate continuum and E is nondegenerate. Therefore, the right-hand side in (5.19) is positive contradicting (5.17) and (5.18). This shows that the cluster set of f at every point of ∂D is degenerate and thus f has a continuous extension on $\overline{D}$.

5.20. Lemma. *Let D be a domain in $\mathbf{R}^n$ and f a Q-homeomorphism of D onto a domain D' in $\mathbf{R}^n$ with $Q \in L^1(D)$. Suppose that D is locally connected at ∂D. If $\partial D'$ is weakly flat, then $C(x_1, f) \cap C(x_2, f) = \emptyset$ for every two distinct points x_1 and x_2 in ∂D.*

Proof. With no loss of generality we may assume that the domain D is bounded. For $i = 1, 2$, let E_i denote the cluster sets $C(x_i, f)$ and suppose that $E_1 \cap E_2 \neq \emptyset$. Write $d = |x_1 - x_2|$. Since D is locally connected in ∂D, there are neighborhoods U_i of x_i, such that $W_i = D \cap U_i$ is connected and $U_i \subset B^n(x_i, d/3)$,

$i = 1, 2$. Then the function $\varrho(x) = 3/d$ if $x \in D \cap B^n\big((x_1+x_2)/2, d\big)$ and $\varrho(x) = 0$ elsewhere is admissible for the family $\Gamma = \Gamma(\overline{W}_1, \overline{W}_2; D)$. Thus,

$$M(f\Gamma) \le \int_D Q(x)\varrho^n(x)\,dm(x) \le \frac{3^n}{d^n}\int_D Q(x)\,dm(x) < \infty. \tag{5.21}$$

On the other hand

$$M = M\big(\Gamma(E_1, E_2; D)\big) \le M(f\Gamma). \tag{5.22}$$

But as $\partial D'$ is weakly flat, and E_i, $i = 1, 2$, are nondegenerate continua in $\overline{D'}$ with non-empty intersection, $M = \infty$, contradicting (5.21). The assertion follows.

5.23. Corollary. *Let D, D', f and Q be as in Lemma* 5.20. *Then f^{-1} has a continuous extension to $\overline{D'}$.*

5.24. Corollary. *Let E be a nondegenerate continuum in $\mathbf{B}^n$ and $Q \in L^1(\mathbf{B}^n \setminus E)$. Then there exists no Q-homeomorphism of $\mathbf{B}^n \setminus E$ onto $\mathbf{B}^n \setminus \{0\}$. If $Q \in L^{n-1}(\mathbf{B}^n \setminus E)$, then there exists no $Q(x)$-qc mapping of $\mathbf{B}^n \setminus E$ onto $\mathbf{B}^n \setminus \{0\}$.*

5.25. Corollary. *Let $f\colon D \to D' \subset \mathbf{R}^n$ be a Q-homeomorphism onto D' with $Q \in \mathrm{BMO}(\overline{D})$. If D locally connected at ∂D and $\partial D'$ is weakly flat, then f has a homeomorphic extension $\tilde f\colon \overline{D} \to \overline{D'}$.*

5.26. Theorem. *Let $f\colon D \to D'$ be a Q-homeomorphism between QED domains D and D' with $Q \in \mathrm{BMO}(\overline{D})$. Then f has a homeomorphic extension $\tilde f\colon \overline{D} \to \overline{D'}$.*

This and the next theorem extend the known Gehring–Martio results, see [GM], p. 196, and [MV], p. 36, from qc mappings to Q-homeomorphisms with $Q \in \mathrm{BMO}(\overline{D})$ and to BMO-qc mappings, respectively.

5.27. Theorem. *Let $f\colon D \to D'$ be a BMO-qc mapping between uniform domains D and D'. Then f has a homeomorphic extension $\tilde f\colon \overline{D} \to \overline{D'}$.*

5.28. Corollary. *Let $f\colon D \to D'$ be a BMO-qc mapping between bounded convex domains D and D'. Then f has a homeomorphic extension $\tilde f\colon \overline{D} \to \overline{D'}$.*

5.29. Corollary. *If D is a domain in $\mathbf{R}^n$ which is locally connected at ∂D and if D is not a Jordan domain, then D cannot be mapped onto $\mathbf{B}^n$ by a Q-homeomorphism with $Q \in \mathrm{BMO}(\overline{D})$.*

5.30. Corollary. *If a domain D in $\mathbf{R}^n$ is uniform but not Jordan, then there is no BMO-qc mapping of D onto $\mathbf{B}^n$.*

In 7.2 below we show that for every $n \ge 3$ there is a bounded uniform domain in $\mathbf{R}^n$ which is a topological ball and not Jordan.

6. Mapping problems

In Section 4, we showed that there are no BMO-qc mappings of $\mathbf{R}^n$ onto a proper subset of $\mathbf{R}^n$, nor BMO-qc mappings of a punctured ball onto a domain that has two nondegenerate boundary components. We may consider the following two questions.

(a) Are there any proper subsets of $\mathbf{R}^n$ that can be mapped BMO-quasiconformally onto $\mathbf{R}^n$?

(b) Are there any nondegenerate continua E in $\mathbf{B}^n$ such that $\mathbf{B}^n \setminus E$ can be mapped BMO-quasiconformally onto $\mathbf{B}^n \setminus \{0\}$?

In [RSY$_2$] we showed that the answer to these questions is negative if $n = 2$. The proofs were based on the Riemann Mapping Theorem and on the existence of a homeomorphic solution to the Beltrami equation

$$w_{\bar z} = \mu(z) w_z$$

for measurable functions μ with $\|\mu\|_\infty \le 1$ which satisfy

$$\frac{\bigl(1 + |\mu(z)|\bigr)}{\bigl(1 - |\mu(z)|\bigr)} \le Q(z)$$

a.e. for some BMO function Q. One may modify Questions (a) and (b) by replacing the words "BMO-quasiconformally" by "by a Q-homeomorphism". Below, we provide a negative answer to Questions (a) and (b) in some special cases when $n > 2$.

We say that a proper subdomain D of $\mathbf{R}^n$ is an *L^1-BMO domain* if $u \in L^1(D)$ whenever $u \in \mathrm{BMO}(D)$. Evidently, D is an L^1-BMO domain, if D is a bounded uniform domain. By [Sta], pp. 106–107, cf. [Go$_1$], p. 69, D is an L^1-BMO domain if and only if $k_D(\,\cdot\,, x_0) \in \mathrm{L}^1(D)$ where k_D is the *quasihyperbolic metric* on D,

$$k_D(x, x_0) = \inf_\gamma \int_\gamma \frac{ds}{d(y, \partial D)} \tag{6.1}$$

where ds denotes the Euclidean length element, $d(y, \partial D)$ the Euclidean distance from $y \in D$ to ∂D, and the infimum is taken over all rectifiable paths γ joining x to x_0 in D. L^1-BMO domains are not invariant under quasiconformal mappings of $\mathbf{R}^n$, however, they are invariant under quasi-isometries, see [Sta], pp. 119 and 112.

In particular, every John domain is an L^1-BMO domain, see Theorem 3.14 in [Sta], p. 115. A domain $D \subset \mathbf{R}^n$ is called a *John domain* if there exist $0 < \alpha \le \beta < \infty$ and a point $x_0 \in D$ such that, for every $x \in D$, there is a rectifiable path γ: $[0, l] \to D$ parametrized by arclength such that $\gamma(0) = x$, $\gamma(l) = x_0$, $l \le \beta$ and

$$d\bigl(\gamma(t), \partial D\bigr) \ge \frac{\alpha}{l} \cdot t \tag{6.2}$$

for all $t \in [0, l]$. A John domain need not be uniform but a bounded uniform domain is a John domain, see [MS], p. 387. Note also that John domains are invariant under qc mappings of $\mathbf{R}^n$, see [MS], p. 388. A convex domain D is a John domain if and only if D is bounded. For characterizations of John domains, see [He], [MS] and [NV].

Hölder domains are also L^1-BMO domains. A domain $D \subset \mathbf{R}^n$ is said to be a *Hölder domain* if there exist $x_0 \in D$, $\delta \geq 1$ and $C > 0$ such that

$$k_D(x, x_0) \leq C + \delta \cdot \log \frac{d(x_0, \partial D)}{d(x, \partial D)} \tag{6.3}$$

for all $x \in D$. It is known that D is a Hölder domain if and only if the quasihyperbolic metric $k_D(x, x_0)$ is exponentially integrable in D, see [SS]. Thus, a Hölder domain is also an L^1-BMO domain.

6.4. Theorem. *Let D be a domain in $\mathbf{R}^n, D \neq \mathbf{R}^n$, $n \geq 2$, and $f\colon D \to \mathbf{R}^n$ a Q-homeomorphism. If there exist a point $b \in \partial D$ and a neighborhood U of b such that $Q|_{D\cap U} \in L^1$, then $f(D) \neq \mathbf{R}^n$.*

Proof. The statement is trivial if D is not a topological ball. Suppose that D is a topological ball. By the Möbius invariance, we may assume that $b = 0$ and that $\infty \in \partial D$. Let $r > 0$ be such that $B^n(r) \subset U$. Then Q is integrable in $B^n(r) \cap D$. Choose two arcs E and F in $B^n(r/2) \cap D$ each having exactly one end point in ∂D such that $0 < \operatorname{dist}(E, F) < r/2$. Such arcs exist. Indeed, since ∂D is connected and 0 and ∞ belong to ∂D, the sphere $\partial B^n(r/2)$ meets ∂D and contains a point x_0 which belongs to D. Then one can take E as a maximal line segment in $(0, x_0] \cap D$ with one end point at x_0 and the other one in ∂D, and F as a circular arc in the maximal spherical cap in $\partial B^n(r/2) \cap D$ which is centered at x_0, so that F has one end point in ∂D and the other one in D.

Now, let Γ denote the family of all paths which join E and F in D. Then $\varrho(x) = \operatorname{dist}(E, F)^{-1}$ if $x \in B^n(r) \cap D$ and $\varrho(x) = 0$ otherwise is admissible for Γ. Then by (1.2)

$$M(f\Gamma) \leq \int_D Q\varrho^n \, dm \leq \frac{1}{\operatorname{dist}(E, F)^n} \int_{B^n(r)\cap D} Q \, dm < \infty. \tag{6.5}$$

On the other hand, if $f(D) = \mathbf{R}^n$, then $f(E)$ and $f(F)$ meet at ∞ and $f\Gamma$ is the family of paths joining $f(E)$ and $f(F)$ in $\mathbf{R}^n$. Thus $M(f\Gamma) = \infty$. Contradiction showing that $f(D) \neq \mathbf{R}^n$.

As a consequence of Theorem 6.4, we have the following corollaries which say that a proper subdomain D of $\mathbf{R}^n$ having a nice boundary at least at one point of ∂D cannot be mapped BMO-quasiconformally onto $\mathbf{R}^n$.

6.6. Corollary. *Let D be a domain in $\mathbf{R}^n, D \neq \mathbf{R}^n$, $n \geq 2$, and let $f\colon D \to \mathbf{R}^n$ be a Q-homeomorphism with $Q \in \mathrm{BMO}(D)$. If there exists a point $b \in \partial D$ and a neighborhood U of b such that $D \cap U$ is an* L^1-BMO *domain or, in particular, if $\partial(D \cap U)$ is a quasisphere, then $f(D) \neq \mathbf{R}^n$.*

6.7. Remark. Theorem 6.4 implies in particular that, if a BMO-qc mapping f of D is onto $\mathbf{R}^n$, then either $D = \mathbf{R}^n$ or the domain D cannot be (even locally at any boundary point) convex, uniform, John or Hölder.

By the techniques which are used in the proof of Theorem 6.4, one can establish the following theorem which gives partial answers to (b).

6.8. Theorem. *Let E be a nondegenerate continuum in $\mathbf{B}^n$, $D = \mathbf{B}^n \setminus E$, and $f\colon D \to \mathbf{R}^n$ a Q-homeomorphism. If there exist a point $x_0 \in \partial D \cap \mathbf{B}^n$ and a neighborhood U of x_0 such that $Q|_{D\cap U} \in L^1$, then $f(D)$ is not a punctured topological ball.*

6.9. Corollary. *Let E be a nondegenerate continuum in $\mathbf{B}^n$ and $D = \mathbf{B}^n \setminus E$. If there exist a point $x_0 \in \partial D \cap \mathbf{B}^n$ and a neighborhood U of x_0 such that $U \setminus E$ is an* L^1-BMO *domain or, in particular, if $\partial(U \setminus E)$ is a quasisphere, then D cannot be mapped BMO-quasiconformally onto $\mathbf{B}^n \setminus \{0\}$.*

6.10. Remark. The condition $Q|_{D\cap U} \in L^1$ which appears in Theorems 6.4 and 6.8 holds for $Q \in \mathrm{BMO}(D)$ if $k_{D\cap U} \in L^1$ and $|\partial D \cap U| > 0$, see [Sta]. Note that the latter property is impossible for convex, uniform, QED as well as for John domains, see [Ma], p. 204, [GM], p. 189, and [MV], p. 33.

7. Some examples

We say that a domain D in $\mathbf{R}^n$, $n \geq 2$, is a *quasiball*, respectively, *BMO-quasiball* if there exists a homeomorphism of D onto $\mathbf{B}^n$ which is qc, respectively, BMO-qc. We say that a set S in $\overline{\mathbf{R}^n}$ is a *quasisphere*, respectively, *BMO-quasisphere* if there exists a qc mapping, respectively, BMO-qc mapping f of $\overline{\mathbf{R}^n}$ onto itself such that $f(S) = \partial \mathbf{B}^n$.

The following example shows that there is a BMO-quasicircle γ which is not a quasicircle.

7.1. Example. Consider the curve $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ where $\gamma_1 = [0, \infty]$, $\gamma_2 = [-\infty, -1/e]$ and

$$\gamma_3 = \left\{ t e^{i\pi/\log 1/t} : 0 < t < 1/e \right\}.$$

Clearly, γ does not satisfy Ahlfors's three points condition, and hence it is not a quasicircle. However, γ is a BMO-quasicircle. Indeed, the map $f\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ which is identity in $\overline{\mathbf{C}} \setminus \mathbf{B}^2$ and is given for $|z| < 1$ by

$$f(re^{i\theta}) = \begin{cases} r \exp i(\theta \log 1/r), & \text{if } 0 \le \theta \le \dfrac{\pi}{\log 1/r}, \\ r \exp i\pi \left(1 + \dfrac{1 - \theta/\pi \log 1/r}{1 - 2\log 1/r}\right), & \text{if } \dfrac{\pi}{\log 1/r} \le \theta < 2\pi \end{cases}$$

is $Q(z)$-qc with $Q(re^{i\theta}) = \max(1, \log 1/r)$ which is BMO-qc in $\overline{\mathbf{C}}$ and maps γ onto $\overline{\mathbf{R}}$.

Note that $\mathbf{R}^n$ is a topological ball which cannot be mapped by a BMO-qc mapping onto $\mathbf{B}^n$, see Corollary 4.8. In view of Corollary 5.30, the following example shows that, for every $n \ge 3$, there exists a proper subdomain of $\mathbf{B}^n$ which is a topological ball but not a BMO-quasiball.

7.2. Example. Let $B = \mathbf{B}^n \setminus C^n(\varepsilon)$ where $C^n(\varepsilon)$ is a cone in $\mathbf{B}^n$ with vertex $v = \partial\mathbf{B}^n \cap \{x_n = 1\}$ and base $\mathbf{B}^n(\varepsilon) \cap \{x_n = 0\}$, $0 < \varepsilon < 1$. For $n \ge 3$, the domain B is uniform. Evidently B is a topological ball. However, the boundary of B is not homeomorphic to the sphere S^{n-1} because the point v splits ∂B into two components.

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