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Hajłasz–Sobolev type spaces and p-energy on the Sierpinski gasket

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Abstract. We study Hajłasz–Sobolev type spaces on metric spaces that depend on quasi-distances; in particular, we may take the quasi-distance to be the power σ of the metric with $\sigma > 1$, if the metric space is highly irregular or porous. We take the Sierpinski gasket in $\mathbf{R}^2$ as an example, and show that the Hajłasz–Sobolev type space is non-trivial for $1 < \sigma < \beta_p/p$ with β_p characterizing the intrinsic property of the Sierpinski gasket. This work was strongly motivated by [8], and generalizes the result in [9] to any $1 < p < \infty$.

1. Hajłasz–Sobolev type spaces

Let F be a non-empty set and d be a metric on F. Let $q(x,y)$ be a *quasi-distance* on F (cf. [14]), that is q: $F \times F \to [0,\infty]$ satisfies

(1) $q(x,y) = 0$ if and only if $x = y$;
(2) $q(x,y) = q(y,x)$ for all $x, y \in F$;
(3) there exists a constant $1 \le c_1 < \infty$ such that, for all $x, y, z \in F$,

$$q(x,y) \le c_1\big(q(x,z) + q(z,y)\big).$$

Let μ be a Borel measure on the metric space (F,d). Let $1 \le p \le \infty$. We denote by $L^p(\mu) := L^p(F,\mu)$ the usual space of all p-integrable real-valued functions on F with respect to μ, with the norm

$$\|f\|_p := \left(\int_F |f(x)|^p \, d\mu(x)\right)^{1/p}$$

(with the obvious modification when $p = +\infty$). Motivated by [5], we say that a function $f \in L^p(\mu)$ belongs to a *Hajłasz–Sobolev type space* $M^p(\mu)$, if there exists a non-negative function $g \in L^p(\mu)$, termed an *upper gradient* of f, such that

$$|f(x) - f(y)| \le q(x,y)\big(g(x) + g(y)\big) \tag{1.1}$$

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for μ-almost all $x, y \in F$ with $0 < q(x,y) < r_0$ and some $r_0 \in (0,\infty]$. The *norm* of $f \in M^p(\mu)$ is defined by

$$\|f\|_{M^p(\mu)} := \|f\|_p + \inf_g \|g\|_p,$$

where the infimum is taken for all g satisfying (1.1). It is not hard to see that $M^p(\mu)$ is a Banach space for $1 \le p < \infty$ (the proof is similar to that in [5] or [7]). Observe that different values on r_0 for (1.1) holding give equivalent spaces.

Note that $q(x,y) = d(x,y)^\sigma$ is a quasi-distance on F for any $0 < \sigma < \infty$. The case $\sigma = 1$ was addressed in [5], and it was shown that $M^p(\mu)$ is the usual Sobolev space $W^{1,p}(F)$ if F is an open domain with Lipschitz boundary in $\mathbf{R}^n$ and μ is the Lebesgue measure. In [9], it was extended to the case $\sigma > 1$ when F is a fractal in the Euclidean setting, and was demonstrated that for $p = 2$, $M^p(\mu)$ is *non-trivial* when $1 < \sigma < \frac{1}{2}\beta$ and is trivial when $\sigma > \frac{1}{2}\beta$, if F is the Sierpinski gasket in $\mathbf{R}^n$, where $\beta = \log(n+3)/\log 2$ is the *walk dimension* of F (for Hajłasz–Sobolev spaces on fractals, see also [6], [16]). (We say that $M^p(\mu)$ is *trivial* if $M^p(\mu)$ contains only constant functions. In this connection, see [2], [3], [1]. Note that $M^p(\mu)$ is always trivial if F is an open set in $\mathbf{R}^n$ and $q(x,y) = |x-y|^\sigma$ with $\sigma > 1$, and nothing needs to be discussed under this circumstance. But if F is irregular (e.g. highly porous), the situation is considerably different, and $M^p(\mu)$ may be non-trivial, see [9] and below.) Whilst in this paper we will generalize the result in [9] to the non-Euclidean setting on one hand, we mainly give an example, on the other hand, that $M^p(\mu)$ is non-trivial for *any* $1 < p < \infty$ and $q(x,y) = d(x,y)^\sigma$ with $\sigma > 1$ in a certain range. We take F to be the Sierpinski gasket in $\mathbf{R}^2$. Our example is motivated by [8]. As a by-product, we also answer the question raised in [8] of what is the *domain* of the p*-energy*. We thank R. S. Strichartz for sending [8] to our attention.

If $q(x,y) = d(x,y)^\sigma$ $(0 < \sigma < \infty)$ and μ is a *doubling measure*, that is μ satisfies, for some $c_2 > 0$,

$$\mu\big(B(x,2r)\big) \le c_2\, \mu\big(B(x,r)\big) \tag{1.2}$$

for all $x \in F$ and all $0 < r < \infty$, where $B(x,r) = \{y \in F : d(y,x) < r\}$ is a ball in F, then $M^p(\mu)$ may be characterized as follows: for $f \in L^p(\mu)$ with $1 < p < \infty$, we have that $f \in M^p(\mu)$ if and only if $\tilde{f} \in L^p(\mu)$, where

$$\tilde{f}(x) := \sup_{0<r<r_0} \frac{1}{\mu\big(B(x,r)\big)} \int_{B(x,r)} \frac{|f(x)-f(y)|}{q(x,y)}\, d\mu(y), \quad x \in F, \tag{1.3}$$

see also [9] (we always assume that $|f(x)-f(y)|/q(x,y) = 0$ if $x = y$). To see this, let $f \in M^p(\mu)$. Then, we have from (1.1) that

$$\begin{aligned}\tilde{f}(x) &\le \sup_{0<r<r_0} \frac{1}{\mu\big(B(x,r)\big)} \int_{B(x,r)} \big(g(x)+g(y)\big)\, d\mu(y) \\ &= g(x) + \sup_{0<r<r_0} \frac{1}{\mu\big(B(x,r)\big)} \int_{B(x,r)} g(y)\, d\mu(y) \in L^p(\mu),\end{aligned}$$

since

$$M(g)(x) := \sup_{0<r<r_0} \frac{1}{\mu\big(B(x,r)\big)} \int_{B(x,r)} g(y)\, d\mu(y) \in L^p(\mu), \tag{1.4}$$

due to the doubling condition (1.2) (see for example [7]). Conversely, let $\tilde f \in L^p(\mu)$. Fix $x, y \in F$ such that $0 < r := d(x,y) < \frac{1}{2} r_0$. Then we see that, using (1.2),

$$\begin{aligned}
|f(x) - f(y)| &\le \frac{1}{\mu\big(B(x,r)\big)} \int_{B(x,r)} \big(|f(x) - f(z)| + |f(z) - f(y)|\big)\, d\mu(z) \\
&\le \frac{1}{\mu\big(B(x,r)\big)} \int_{B(x,r)} \frac{r^\sigma}{d(z,x)^\sigma} |f(x) - f(z)|\, d\mu(z) \\
&\quad + \frac{1}{\mu\big(B(x,r)\big)} \int_{B(y,2r)} \frac{(2r)^\sigma}{d(z,y)^\sigma} |f(z) - f(y)|\, d\mu(y) \\
&\le r^\sigma \left(\tilde f(x) + \frac{2^\sigma \mu\big(B(y,2r)\big)}{\mu(B(x,r))} \tilde f(y) \right) \\
&\le C\, r^\sigma \big(\tilde f(x) + \tilde f(y)\big) \\
&= C\, q(x,y) \big(\tilde f(x) + \tilde f(y)\big),
\end{aligned}$$

proving that $f \in M^p(\mu)$ if $\tilde f \in L^p(\mu)$. Here and in the sequel, we denote by C the general constant whose value may be different at a different occurrence. The function $\tilde f$ defined as in (1.3) is the upper gradient of f (multiple a constant). In what follows we will focus on a class of Hajłasz–Sobolev type spaces where $q(x,y) = d(x,y)^\sigma$ and $1 < \sigma < \infty$, and we denote this space by $M^p_\sigma(\mu)$.

For $1 \le p < \infty$ and $0 < \sigma < \infty$, we say that $f \in \mathrm{Lip}(\sigma, p, \infty)(\mu)$ if $f \in L^p(\mu)$ and

$$\begin{aligned}
W_{\sigma,p}(f)^p &:= \sup_{0<r<r_0} r^{-\sigma p} \int_F \left\{ \frac{1}{\mu\big(B(x,r)\big)} \int_{B(x,r)} |f(x) - f(y)|^p\, d\mu(y) \right\} d\mu(x) \\
&< \infty. \qquad (1.5)
\end{aligned}$$

The norm of $f \in \mathrm{Lip}(\sigma, p, \infty)(\mu)$ is defined by

$$\|f\|_{\mathrm{Lip}(\sigma,p,\infty)(\mu)} = \|f\|_p + W_{\sigma,p}(f).$$

It is easy to see that $\mathrm{Lip}(\sigma, p, \infty)(\mu)$ is a Banach space for $1 \le p < \infty$ and $0 < \sigma < \infty$ (cf. [10], [11]). By (1.1), we see that

$$M^p_\sigma(\mu) \subset \mathrm{Lip}(\sigma, p, \infty)(\mu)$$

if μ is a *doubling measure*. The converse is also true if F is a smooth domain in $\mathbf{R}^n$ and μ is the Lebesgue measure, see [9]. However, if F is irregular, the converse may not be true. But for an α*-regular* measure μ, the space $M^p_\sigma(\mu)$ is arbitrarily close to $\mathrm{Lip}(\sigma, p, \infty)(\mu)$. We say that a measure is α*-regular* if there exists a constant $c_3 > 0$ such that

$$c_3^{-1} r^\alpha \le \mu\big(B(x,r)\big) \le c_3\, r^\alpha \tag{1.6}$$

for all $x \in F$ and all $0 < r < r_0$ (some $r_0 > 0$). It is not hard to see that if μ is α-regular, then

$$W_{\sigma,p}(f)^p \simeq \sup_{m\ge1} 2^{m(\alpha+\sigma p)} \int_F \int_{B(x,c_0 2^{-m})} |f(x)-f(y)|^p \, d\mu(y)\, d\mu(x), \tag{1.7}$$

for any fixed $c_0 > 0$.

Proposition 1.1 *Let* $1 < p < \infty$ *and* $0 < \sigma < \infty$, *and let* $0 < \alpha < \infty$. *Assume that* μ *is* α*-regular. Then*

$$\mathrm{Lip}(\sigma, p, \infty)(\mu) \subset M^p_{\sigma-\theta}(\mu)$$

for any $0 < \theta < \sigma$.

Proof. See [9]. □

Proposition 1.1 says that $M^p_\sigma(\mu)$ is slightly smaller than the *Besov space* $\mathrm{Lip}(\sigma, p, \infty)(\mu)$ if μ is α-regular.

2. Examples

In this section we show that $M^p_\sigma(\mu)$ is non-trivial for any $1 < p < \infty$ and $\sigma(> 1)$ in a certain range, if the underlying metric space is irregular. We take the Sierpinski gasket in $\mathbf{R}^2$ for an example. The proof is quite technical.

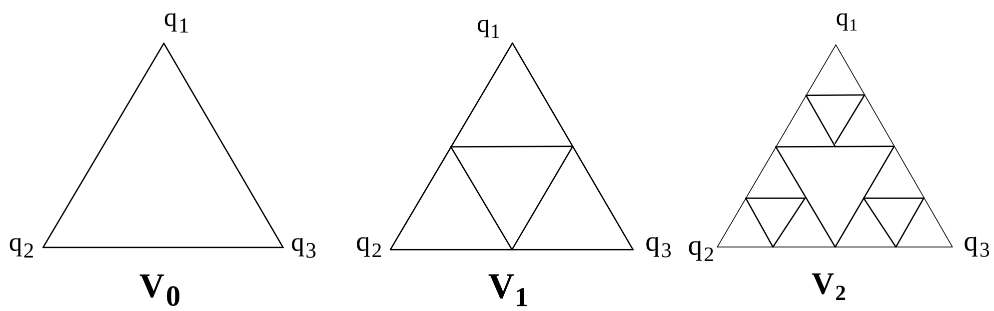

Figure 1.

Let F be the Sierpinski gasket in $\mathbf{R}^2$, that is, F is the unique non-empty compact subset of $\mathbf{R}^2$ determined by

$$F = \bigcup_{i=1}^{3} \phi_i(F),$$

where $\phi_i(x) = \frac{1}{2}(q_i + x)$, $x \in \mathbf{R}^2$ $(1 \le i \le 3)$, and q_1, q_2, q_3 are the three vertices of an equilateral triangle in $\mathbf{R}^2$. Alternatively, we may view the Sierpinski gasket F as the closure of $V_* = \bigcup_{m=1}^{\infty} V_m$ under the Euclidean metric, where $V_m = \bigcup_{i=1}^{3} \phi_i(V_{m-1})$, $m \ge 1$, and $V_0 = \{q_1, q_2, q_3\}$, see Figure 1. For $p = 2$, Kigami [12] constructed a *local regular Dirichlet form* on F by using the difference scheme. Jonsson [10] identified the *domain* of this Dirichlet form with a Besov space. Recently, Herman, Peirone and Strichartz [8] have extended Kigami's result to the case $1 < p < \infty$. Here we briefly describe the main result in [8] that will lead to our example. For $1 < p < \infty$, let E_p: $\mathbf{R}^3 \to [0, \infty]$ be given by

$$E_p(a_1, a_2, a_3) = |a_1 - a_2|^p + |a_2 - a_3|^p + |a_3 - a_1|^p, \quad a_1, a_2, a_3 \in \mathbf{R},$$

and define

$$E_p^{(m)}(f) := \sum_{|\omega|=m} E_p\big(f\big(\phi_\omega(q_1)\big), f\big(\phi_\omega(q_2)\big), f\big(\phi_\omega(q_3)\big)\big), \quad m \ge 1, \tag{2.1}$$

for any f: $F \to \mathbf{R}$, where the summation is taken over all words ω of length m, and $\phi_\omega(q_1) = \phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_m}(q_1)$ for the word $\omega = i_1 i_2 \cdots i_m$ $(i_k \in \{1, 2, 3\}$ for $1 \le k \le m)$. Let A_p: $\mathbf{R}^3 \to [0, \infty]$ be a function satisfying (among other properties)

$$c_4^{-1} E_p(a) \le A_p(a) \le c_4 E_p(a) \tag{2.2}$$

for some positive constant c_4 and for all $a := (a_1, a_2, a_3) \in \mathbf{R}^3$; in particular, A_p solves the *renormalization problem*: Given $a \in \mathbf{R}^3$ and letting

$$\begin{aligned} A_p^{(2)}(a, b) &:= A_p(a_1, b_2, b_3) + A_p(b_1, a_2, b_3) + A_p(b_1, b_2, a_3) \\ &\qquad \text{for } b = (b_1, b_2, b_3) \in \mathbf{R}^3, \end{aligned} \tag{2.3}$$

we have that there exists a number r_p such that

$$\min_{b \in \mathbf{R}^3} A_p^{(2)}(a, b) = r_p A_p(a) \quad \text{for all } a \in \mathbf{R}^3. \tag{2.4}$$

Such a function A_p was shown to exist in [8]. Moreover, the number r_p is unique (independent of the choice of A_p) and satisfies

$$2^{1-p} \le r_p \le 2^{p-1}\Big(1 + \sqrt{1 + 2^{3-1/(p-1)}}\Big)^{1-p} < 3 \cdot 2^{-p}, \tag{2.5}$$

see Lemma 3.8 in [8]. We mention that $r_p = \frac{3}{5}$ for $p = 2$. The function A_p may or may not be unique on F; what is important is that the renormalization factor r_p is unique which reflects the intrinsic properties of the Sierpinski gasket F. Now, for any $f\colon F \to \mathbf{R}$ we define the p-*energy* $\mathscr{E}(f)$ of f on F as the limit of

$$\mathscr{E}_m(f) = r_p^{-m} \sum_{|\omega|=m} A_p\big(f(\phi_\omega(q_1)), f(\phi_\omega(q_2)), f(\phi_\omega(q_3))\big), \quad m \geq 1, \tag{2.6}$$

that is,

$$\mathscr{E}(f) = \lim_{m\to\infty} \mathscr{E}_m(f). \tag{2.7}$$

Note that (2.7) makes sense since

$$\{\mathscr{E}_m(f)\}_m$$

is increasing in m for any function f, due to the renormalization problem. Note that

$$c_5^{-1}\, \mathscr{E}_m(f) \leq r_p^{-m}\, E_p^{(m)} \leq c_5\, \mathscr{E}_m(f) \tag{2.8}$$

for all $m \geq 1$ and all $f\colon F \to \mathbf{R}$, where $c_5 > 0$. Let

$$\mathscr{D}(\mathscr{E}) = \{f \in C(F) : \mathscr{E}(f) < \infty\}, \tag{2.9}$$

termed the *domain* of the p-energy, where $C(F)$ denotes the space of all continuous functions on F with the usual supremum norm. It was shown that $\mathscr{D}(\mathscr{E})$ is dense in $C(F)$, see [8]. The space $\mathscr{D}(\mathscr{E})$ will provide a *critical exponent* $\beta_p := \log_2(3r_p^{-1})$ (some $r_p > 0$) that determines whether or not a Hajłasz–Sobolev type space $M_\sigma^p(\mu)$ $(1 < p < \infty)$ is non-trivial. To see this, we first identify $\mathscr{D}(\mathscr{E})$ with a Besov space.

Theorem 2.1. *Let μ be the $\alpha := \log_2 3$-dimensional Hausdorff measure on F. Then*

$$W_{\beta_p/p,p}(f)^p \simeq \mathscr{E}(f) \tag{2.10}$$

for all $f \in C(F)$, where $W_{\beta_p/p,p}(f)$ is defined as in (1.5) and $\beta_p = \log_2(3r_p^{-1})$. Thus

$$\mathscr{D}(\mathscr{E}) = \mathrm{Lip}(\beta_p/p, p, \infty)(\mu), \tag{2.11}$$

where $\mathscr{D}(\mathscr{E})$ is defined as in (2.9).

Remarks. 1. When $p = 2$, we have that $r_p = \frac{3}{5}$ and so $\beta_p = \log_2 5$, the *walk dimension* of the Sierpinski gasket.

2. If μ is α-regular and $\sigma > \alpha/p$, then $\mathrm{Lip}(\sigma, p, \infty)(\mu)$ is embedded into the Hölder space with exponent $\sigma - \alpha/p$ on F, that is,

$$|f(x) - f(y)| \le C\,|x - y|^{\sigma-\alpha/p} W_{\sigma,p}(f) \tag{2.12}$$

for all $f \in \mathrm{Lip}(\sigma, p, \infty)(\mu)$, where C is independent of x, y and f; see for example a direct proof in [4]. Thus

$$\mathrm{Lip}(\beta_p/p, p, \infty)(\mu) \subset C(F),$$

since $\beta_p/p > \alpha/p$ (due to $r_p < 1$).

3. By Theorem 2.1, the domain of the p-energy coincides with $\mathrm{Lip}(\beta_p/p, p, \infty)(\mu)$ if μ is the Hausdorff measure. For other measures this may not be true.

Proof. The proof given here is motivated by [10] (see also [15], [17]) but with some modifications. We first show that

$$W_{\beta_p/p,p}(f)^p \le C\,\mathscr{E}(f) \tag{2.13}$$

for all $f \in \mathscr{D}(\mathscr{E})$. To see this, let $f \in \mathscr{D}(\mathscr{E})$. Let

$$f_k(x) = \begin{cases} \frac{1}{3}\big[f\big(\phi_\omega(q_1)\big) + f\big(\phi_\omega(q_2)\big) + f\big(\phi_\omega(q_3)\big)\big], & \text{if } x \in \phi_\omega(F)\backslash\phi_\omega(V_0), \\ f(x), & \text{if } x \in \phi_\omega(V_0) \end{cases} \tag{2.14}$$

for $|\omega| = k$ ($k \ge 1$). Since F is compact and f is continuous on F, the piecewise constant function f_k converges to f pointwise on F as $k \to \infty$. If we can show that, for some $c_0 > 0$ (e.g. $c_0 = \sqrt{3}/2$),

$$2^{(\alpha+\beta_p)m} \int_F \int_{|y-x|<c_0 2^{-m}} |f_{m+k}(y) - f_{m+k}(x)|^p \, d\mu(y)\, d\mu(x) \le C\,\mathscr{E}(f) \tag{2.15}$$

for all integers $m, k \ge 1$, where C is independent of f, then (2.13) follows by letting $k \to \infty$ in (2.15) and using Fatou's lemma, and (1.7). It remains to prove (2.15). For two words ω and τ with $|\tau| = |\omega| = m$, we denote by $\tau \underset{m}{\sim} \omega$ if

$\phi_\tau(F)\cap\phi_\omega(F)\neq\emptyset$ (we allow that $\tau=\omega$). Note that

$$\begin{aligned}
I_{m+k}(f) &:= \int_F\int_{|y-x|<c_0 2^{-m}} |f_{m+k}(y)-f_{m+k}(x)|^p\,d\mu(y)\,d\mu(x)\\
&\le \sum_{|\omega|=m}\sum_{\tau\underset{m}{\sim}\omega}\int_{\phi_\omega(F)}\int_{\phi_\tau(F)} |f_{m+k}(y)-f_{m+k}(x)|^p\,d\mu(y)\,d\mu(x)\\
&\le \sum_{|\omega|=m}\sum_{\tau\underset{m}{\sim}\omega}\int_{\phi_\omega(F)}\int_{\phi_\tau(F)} 2^{p-1}\big(|f_{m+k}(y)-f(x_{\omega,\tau})|^p\\
&\quad + |f(x_{\omega,\tau})-f_{m+k}(x)|^p\big)\,d\mu(y)\,d\mu(x)\\
&\le 8\cdot 2^{p-1}\cdot 3^{-m}\sum_{|\omega|=m}\int_{\phi_\omega(F)}|f_{m+k}(x)-f(x_{\omega,\tau})|^p\,d\mu(x),
\end{aligned}\tag{2.16}$$

where we have used the fact that $\mu\big(\phi_\omega(F)\big)=3^{-m}$ and $\#\{\tau:\tau\underset{m}{\sim}\omega\}\le 4$ for $|\omega|=m$, $m\ge 1$, and where $x_{\omega,\tau}$ is some point in $\phi_\omega(V_0)$ (in fact $x_{\omega,\tau}$ is the unique intersection point of two sets $\phi_\omega(V_0)$ and $\phi_\tau(V_0)$). Noting that $\mu(x)=0$ for any single point $x\in F$, it follows from (2.14) that

$$\begin{aligned}
&\int_{\phi_\omega(F)}|f_{m+k}(x)-f(x_{\omega,\tau})|^p\,d\mu(x)\\
&\quad=\sum_{|\tau|=k}\int_{\phi_{\omega\cdot\tau}(F)}\left|\frac13\sum_{j=1}^3 f(\phi_{\omega\cdot\tau}(q_j))-f(x_{\omega,\tau})\right|^p d\mu(x)\\
&\quad=3^{-(m+k)}\sum_{|\tau|=k}\left|\frac13\sum_{j=1}^3\big(f(\phi_{\omega\cdot\tau}(q_j))-f\big(x_{\omega,\tau}\big)\big)\right|^p\\
&\quad\le 3^{-(m+k)-1}\sum_{j=1}^3\sum_{|\tau|=k}\big|f\big(\phi_{\omega\cdot\tau}(q_j)\big)-f\big(x_{\omega,\tau}\big)\big|^p
\end{aligned}$$

which combines with (2.16) to give that

$$I_{m+k}(f)\le 2^{p+2}\cdot 3^{-(2m+k)-1}\sum_{j=1}^3\sum_{|\omega|=m,\,x_{\omega,\tau}\in\phi_\omega(V_0)}\sum_{|\tau|=k}\big|f\big(\phi_{\omega\cdot\tau}(q_j)\big)-f(x_{\omega,\tau})\big|^p.\tag{2.17}$$

Let q_j and $\tau:=i_1i_2\cdots i_k$ be fixed, and set $x_k=\phi_{\omega\cdot\tau}(q_j)$ and $x_0:=x_{\omega,\tau}=\phi_\omega(q_0)$ for some $q_0\in V_0$. We let $x_l=\phi_{\omega\cdot i_1i_2\cdots i_l}(q_0), 1\le l\le k-1$, and obtain a sequence of points $\{x_l\}_{l=0}^k$ (some of points may be the same). Repeatedly using the elementary inequality $|a+b|^p \le 2^{p-1}(|a|^p+|b|^p)$ for any $a,b\in\mathbf{R}$ and $1\le p<\infty$, we see that

$$\begin{aligned}\big|f\big(\phi_{\omega\cdot\tau}(q_j)\big)-f(x_{\omega,\tau})\big|^p &= \big|f(x_k)-f(x_0)\big|^p\\ &\le \sum_{l=1}^{k} 2^{(p-1)l}|f(x_l)-f(x_{l-1})|^p\\ &\le \sum_{l=1}^{k} 2^{(p-1)l}\,\widehat{E}^{(\omega,p)}_{i_1,\dots,i_{l-1}}(f),\end{aligned}$$

where

$$\widehat{E}^{(\omega,p)}_{i_1,\dots,i_{l-1}}(f)=\sum_{i=1}^{3}E_p\big(f\circ\phi_{\omega\cdot i_1i_2\cdots i_{l-1}\cdot i}(q_1),f\circ\phi_{\omega\cdot i_1i_2\cdots i_{l-1}\cdot i}(q_2),f\circ\phi_{\omega\cdot i_1i_2\cdots i_{l-1}\cdot i}(q_3)\big).$$

Thus

$$\begin{aligned}\sum_{|\tau|=k}\big|f\big(\phi_{\omega\cdot\tau}(q_j)\big)-f(x_{\omega,\tau})\big|^p &\le \sum_{i_1,i_2,\dots,i_k}\sum_{l=1}^{k}2^{(p-1)l}\,\widehat{E}^{(\omega,p)}_{i_1,\dots,i_{l-1}}(f)\\ &=\sum_{l=1}^{k}2^{(p-1)l}\cdot 3^{k-(l-1)}\sum_{i_1,\dots,i_{l-1}}\widehat{E}^{(\omega,p)}_{i_1,\dots,i_{l-1}}(f).\end{aligned}\tag{2.18}$$

Observe that

$$\begin{aligned}\sum_{|\omega|=m}\sum_{i_1,\dots,i_{l-1}}\widehat{E}^{(\omega,p)}_{i_1,\dots,i_{l-1}}(f) &= \sum_{|\omega|=m+l}E_p\big(f(\phi_\omega(q_1)\big),f\big(\phi_\omega(q_2)\big),f\big(\phi_\omega(q_3)\big)\big)\\ &=E_p^{(m+l)}(f)\\ &\le c_5\,r_p^{m+l}\,\mathscr{E}_{m+l}(f)\\ &\le c_5\,r_p^{m+l}\,\mathscr{E}(f)\end{aligned}\tag{2.19}$$

by using (2.8) and the monotonicity of $\mathscr{E}_m(f)$ in m. Combining (2.17), (2.18) and (2.19), we have that

$$\begin{aligned}I_{m+k}(f) &\le C\,3^{-2m}\sum_{l=1}^{k}2^{(p-1)l}\cdot 3^{-l}r_p^{m+l}\mathscr{E}(f)\\ &\le Cr_p^m\cdot 3^{-2m}\mathscr{E}(f)\sum_{l=1}^{\infty}2^{(p-1)l}3^{-l}r_p^l\\ &\le Cr_p^m\cdot 3^{-2m}\mathscr{E}(f)=C\,2^{-m(\alpha+\beta_p)}\mathscr{E}(f),\end{aligned}$$

where we have used the fact that

$$\sum_{l=1}^{\infty} 2^{(p-1)l} 3^{-l} r_p^l < \infty$$

since $2^{p-1}3^{-1}r_p < 1$ by virtue of (2.5). Therefore, (2.15) follows.

We next show that

$$\mathscr{E}(f) \le C W_{\beta_p/p,p}(f)^p \tag{2.20}$$

for all $f \in \mathrm{Lip}(\beta_p/p, p, \infty)(\mu)$. By (2.8), it is sufficient to show that

$$\mathscr{E}_p^{(m)}(f) := r_p^{-m} \sum_{|\omega|=m} \sum_{u,v \in \phi_\omega(V_0)} |f(u) - f(v)|^p \le C W_{\beta_p/p,p}(f)^p \tag{2.21}$$

for all $f \in \mathrm{Lip}(\beta_p/p, p, \infty)(\mu)$ and all $m \ge 1$. Let $f \in \mathrm{Lip}(\beta_p/p, p, \infty)(\mu)$. By Remark 2 above we see that f is continuous on F. Noting that

$$|f(u) - f(v)|^p \le 2^{p-1}\big(|f(u) - f(x)|^p + |f(x) - f(v)|^p\big),$$

we have that

$$|f(u) - f(v)|^p \le \frac{2^{p-1}}{\mu\big(\phi_\omega(F)\big)} \int_{\phi_\omega(F)} \big(|f(u) - f(x)|^p + |f(x) - f(v)|^p\big)\, d\mu(x).$$

It follows from (2.21) that

$$\mathscr{E}_p^{(m)}(f) \le 6 \cdot 2^{p-1}\, r_p^{-m} \sum_{|\omega|=m} \sum_{u \in \phi_\omega(V_0)} \frac{1}{\mu(\phi_\omega(F))} \int_{\phi_\omega(F)} |f(u) - f(x)|^p \, d\mu(x). \tag{2.22}$$

Now let $x \in \phi_\omega(F)$ and $u \in \phi_\omega(V_0)$ be fixed. There exists a point $p_0 \in V_0$ such that $u = \phi_\omega(p_0)$. We take i_0 such that $\phi_{i_0}(p_0) = p_0$. Set

$$S_0 = \phi_\omega(F), \quad S_1 = \phi_{\omega \cdot \underbrace{i_0 \cdot i_0 \cdot \dots \cdot i_0}_{k \text{ times}}}(F), \quad S_2 = \phi_{\omega \cdot \underbrace{i_0 \cdot i_0 \cdot \dots \cdot i_0}_{2k \text{ times}}}(F), \dots,$$

where k is an integer to be determined below. It is easy to see that $u \in S_j$ for each $j \ge 0$, and the sequence of the sets $\{S_j\}$ shrinks to the single point u. For each $x := x_{\omega,\tau} \in S_0$, $x_j \in S_j$ and each $l \ge 1$,

$$\begin{aligned} |f(u) - f(x)|^p &\le 2^{p-1}(|f(u) - f(x_l)|^p + |f(x_l) - f(x_{\omega,\tau})|^p) \\ &\le 2^{p-1}|f(u) - f(x_l)|^p + \sum_{j=1}^{l} 2^{(p-1)(j+1)} |f(x_j) - f(x_{j-1})|^p. \end{aligned}$$

Integrating the above inequality with respect to each $x_j \in S_j$ $(0 \le j \le l)$ and then dividing by $\mu(S_0)\mu(S_1)\cdots\mu(S_l)$, we obtain that

$$\begin{aligned}\frac{1}{\mu\big(\phi_\omega(F)\big)}\int_{\phi_\omega(F)}|f(u)-f(x)|^p\,d\mu(x) &\le \frac{2^{p-1}}{\mu(S_l)}\int_{S_l}|f(u)-f(x_l)|^p\,d\mu(x_l)\\ &+\sum_{j=1}^{l}2^{(p-1)(j+1)}\frac{1}{\mu(S_{j-1})\mu(S_j)}\\ &\times\int_{S_{j-1}}\int_{S_j}|f(x_j)-f(x_{j-1})|^p\,d\mu(x_j)\,d\mu(x_{j-1}).\end{aligned}\tag{2.23}$$

Noting that $\mu(S_j)=3^{-(m+kj)}$ for each $j\ge 0$ and

$$\begin{aligned}&\int_{S_{j-1}}\int_{S_j}|f(x_j)-f(x_{j-1})|^p\,d\mu(x_j)\,d\mu(x_{j-1})\\ &\qquad\le\int_{S_0}\int_{|\xi-\eta|\le 2^{-(m+(j-1)k)}}|f(\xi)-f(\eta)|^p\,d\mu(\xi)\,d\mu(\eta),\end{aligned}$$

we have from (2.22) and (2.23) that

$$\begin{aligned}\mathscr{E}_p^{(m)}(f)\le 6\cdot 2^{p-1}r_p^{-m}\sum_{|\omega|=m}\sum_{u\in\phi_\omega(V_0)}\bigg\{&\frac{2^{p-1}}{\mu(S_l)}\int_{S_l}|f(u)-f(x_l)|^p\,d\mu(x_l)\\ &+\sum_{j=1}^{l}2^{(p-1)(j+1)}3^{(2m+(2j-1)k)}\\ &\times\int_{\phi_\omega(F)}\int_{|\xi-\eta|\le 2^{-(m+(j-1)k)}}|f(\xi)-f(\eta)|^p\,d\mu(\xi)\,d\mu(\eta)\bigg\}.\end{aligned}\tag{2.24}$$

Letting $l\to\infty$, we have that the first term on the right-hand side in (2.24) tends to zero since f is continuous and

$$\frac{1}{\mu(S_l)}\int_{S_l}|f(u)-f(x_l)|^p\,d\mu(x_l)\to 0 \text{ as } l\to\infty,$$

and the second term is less than

$$\begin{aligned}&Cr_p^{-m}\sum_{j=1}^{\infty}2^{(p-1)(j+1)}3^{(2m+(2j-1)k)}\int_F\int_{|\xi-\eta|\le c_0 2^{-(m+jk)}}|f(\xi)-f(\eta)|^p\,d\mu(\xi)\,d\mu(\eta)\\ &\quad\le C\,3^{2m}\,r_p^{-m}\sum_{j=1}^{\infty}2^{(p-1)j}3^{2jk}2^{-(m+jk)(\alpha+\beta_p)}W_{\beta_p/p,p}(f)^p\\ &\quad= C\,3^{2m}r_p^{-m}\,W_{\beta_p/p,p}(f)^p\sum_{j=1}^{\infty}2^{(p-1)j}3^{2jk}\,(3^{-2}r_p)^{m+jk} \qquad (2.25)\\ &\quad= CW_{\beta_p/p,p}(f)^p\sum_{j=1}^{\infty}2^{(p-1)j}r_p^{jk},\end{aligned}$$

since $2^{-(\alpha+\beta_p)} = r_p \cdot 3^{-2}$. Since $r_p < 1$, we take k so large that $r_p^k < 2^{-(p-1)}$, and so

$$\sum_{j=1}^{\infty} 2^{(p-1)j} r_p^{jk} < \infty.$$

Therefore,

$$\mathscr{E}_p^{(m)}(f) \le C W_{\beta_p/p,p}(f)^p$$

for all $f \in \mathrm{Lip}(\beta_p/p, p, \infty)(\mu)$ and all $m \ge 1$, proving (2.21). The other statement is obvious. □

Corollary 2.1. *Let $\beta_p = \log_2(3r_p^{-1})$ as above. Then the space $\mathrm{Lip}(\bar\beta/p, p, \infty)(\mu)$ defined on the Sierpinski gasket in $\mathbf{R}^2$ contains only constant functions if $\bar\beta > \beta_p$.*

Proof. By (2.24), (2.25), we see that

$$\mathscr{E}_p^{(m)}(f) \le C W_{\bar\beta/p,p}(f)^p 2^{-m(\bar\beta-\beta_p)}$$

for all $m \ge 1$ and all $f \in \mathrm{Lip}(\bar\beta/p, p, \infty)(\mu)$. Thus we have that

$$\mathscr{E}(f) = \lim_{m\to\infty} \mathscr{E}_m(f) \le C \lim_{m\to\infty} \mathscr{E}_p^{(m)}(f) = 0,$$

giving that $f =$const. □

Theorem 2.2. *Let F be the Sierpinski gasket in $\mathbf{R}^2$ and μ be the α-dimensional Hausdorff measure on F, where $\alpha = \log_2 3$. Let $1 < p < \infty$. Then there exists some $r_p \in [2^{1-p}, 3\cdot 2^{-p})$ such that the Hajłasz–Sobolev space $M_\sigma^p(\mu)$ is dense in $C(F)$ for all $\sigma < p^{-1}\log_2(3r_p^{-1})$; in particular, the space $M_\sigma^p(\mu)$ is non-trivial if $1 < \sigma < \log_4 5$ when $p = 2$. Moreover, $M_\sigma^p(\mu)$ is trivial if $\sigma \ge 1 + 1/p$.*

Proof. By Theorem 2.1, the space $\mathrm{Lip}(\sigma, p, \infty)(\mu)$ is dense in $C(F)$ if

$$0 < \sigma \le \frac{\beta_p}{p} = p^{-1}\log_2(3r_p^{-1}).$$

Since μ is α-regular, we see from Proposition 1.1 and (2.5) that there exists some $r_p \in [2^{1-p}, 3\cdot 2^{-p})$ such that $M_\sigma^p(\mu)$ is dense in $C(F)$ if $0 < \sigma < p^{-1}\log_2(3r_p^{-1})$. By Corollary 2.1, the space $\mathrm{Lip}(\sigma, p, \infty)(\mu)$ is trivial if $\sigma \ge 1 + 1/p > \beta_p/p$ (due to $r_p \ge 2^{1-p}$). Thus the fact that $M_\sigma^p \subset \mathrm{Lip}(\sigma, p, \infty)$ implies that $M_\sigma^p(\mu)$ is also trivial if $\sigma \ge 1 + 1/p$. □

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