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AUTOMORPHISM GROUPS OF SCHOTTKY TYPE

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Abstract. A group H of (conformal/anticonformal) automorphisms of a closed Riemann surface S of genus $g \geq 2$ is said of Schottky type if there is a Schottky uniformization of S for which it lifts. We observe that H is of Schottky type if and only if it leaves invariant a collection of pairwise disjoint simple loops which disconnect S into genus zero surfaces. Moreover, in the case that H is a cyclic group (either generated by a conformal or an anticonformal automorphism) we provide a simple to check necessary and sufficient condition in order for it to be of Schottky type.

1. Introduction

Assume we have a collection of $2g > 0$ pairwise disjoint simple loops in the Riemann sphere $\widehat{\mathbf{C}}$, say $C_1, C'_1, \dots, C_g$ and C'_g, bounding a common region $\mathscr{D}$ of connectivity $2g$, and that there are loxodromic transformations $A_1, \dots, A_g$ so that $A_j(C_j) = C'_j$ and $A_j(\mathscr{D}) \cap \mathscr{D} = \emptyset$, for each $j = 1, 2, \dots, g$. The group G, generated by $A_1, \dots, A_p$, is a Schottky group of genus g. The collection of loops C_1, $C'_1, \dots, C_g$ and C'_g, is called a *fundamental system of loops* of G with respect to the *Schottky generators* $A_1, \dots, A_g$. Every set of g generators of a Schottky group of genus g is a set of Schottky generators, that is, has associated a fundamental system of loops [C]. In [M1] it is shown that a purely loxodromic Kleinian group isomorphic to a free group of rank g is a Schottky group of genus g. The trivial group is defined as the Schottky group of genus zero. If we denote by Ω the region of discontinuity of a Schottky group G of genus g, then the quotient $S = \Omega/G$ turns out to be a closed Riemann surface of genus g. The reciprocal is valid by the retrosection theorem [Ko2] (see [B] for a modern proof using quasiconformal deformation theory). A triple $(\Omega, G, P : \Omega \to S)$ is called a *Schottky uniformization* of a closed Riemann surface S if G is a Schottky group with Ω as its region of discontinuity and $P\colon \Omega \to S$ is a holomorphic regular covering with G as covering group. Schottky uniformizations correspond to the lowest planar regular coverings of S and also they correspond to geometrically finite hyperbolic structures on handlebodies, with inner injectivity radii bounded below by a positive value, having S as conformal border. In this note, by an automorphism of a Riemann surface S we mean either a conformal or an anticonformal automorphism. A group of automorphisms which only contains conformal automorphisms will be said to be a

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conformal group of automorphisms. A group H of automorphisms of S is said of *Schottky type* if there is a Schottky uniformization of S, say $(\Omega, G, P : \Omega \to S)$, so that H lifts, that is, for every $h \in H$ there is an automorphism $\hat{h} : \Omega \to \Omega$ for which $h \circ P = P \circ \hat{h}$. The main problem we are interested in is to decide when a given group H of automorphisms of S is of Schottky type. As in genus 0 this is trivial and in genus 1 this is completely known [H4], we restrict ourselves to the situation of genus at least 2. For some particular classes of groups (for instance, conformal cyclic, conformal abelian, conformal dihedral, and some anticonformal cyclic groups) there are complete answers for the above lifting problem (see for instance, [HC], [H1], [H2], [H4], [H5], [H6], [H8], [RZ1] and [RZ2]). In this note we provide a necessary and sufficient condition for a group of automorphisms to be of Schottky type. In the particular case of cyclic groups of automorphisms generated by an anticonformal involution, we also provide a condition which is simple to check for it to be of Schottky type, completing the work done in [HC].

This note is organized as follows. In Section 2 we provide a necessary and sufficient condition for a given finite group of automorphisms to be of Schottky type (see Theorem 1). Such a condition relays on the existence of certain collection of pairwise disjoint simple loops on the surface, invariant under the group under consideration, which dissects the surface into genus zero surfaces. In Section 3 we restrict to the case of conformal automorphisms and we recall an already known necessary condition (condition (A)), which turns out to be sufficient in many cases, for instance for the cyclic case (see Theorems 2 and 3). In Section 4 we restrict to groups containing anticonformal automorphisms and we mainly consider the cyclic case. Necessary and sufficient conditions (simpler to check than the ones given in Theorem 1) are given (see Theorems 5 and 6). In Section 5 we provide the definition of Klein–Schottky pairings, needed in the necessary and sufficient conditions for the anticonformal cyclic case. In Sections 6 and 7 we give the proof of Theorem 6. In Section 8 we provide, as a consequence of Theorem 1, a method to construct all Schottky type groups of automorphisms.

2. A necessary and sufficient condition

To give an answer to the Schottky lifting problem, we need the following definition. Let us consider a closed Riemann surface S of genus $g \geq 2$ together a group H of automorphisms of it (then a finite group by Hurwitz). A collection of pairwise disjoint simple loops on S, say $L_1, \dots, L_k \subset S$, is called a *Schottky system of loops of* H if

(1) each connected component of $S - \bigcup_{j=1}^{n} L_j$ is a genus zero bordered surface; and

(2) the collection of loops $\{L_1, \dots, L_k\}$ is invariant under the action of H.

We have the following necessary and sufficient condition for a group of automorphisms to be of Schottky type.

Theorem 1. *A group H of automorphisms of a closed Riemann surface S of genus $g \geq 2$ is of Schottky type if and only if there is a Schottky system of loops of H.*

Remark 1. (i) In the case of genus $g = 1$, Theorem 1 is still valid with $k = 1$ and replacing the property of "invariant under H" by the property of "homotopically invariant under H".

(ii) If the group H is of Schottky type, then Theorem 1 asserts the existence of a Schottky system of loops for H, say $L_1, \ldots, L_k \subset S$. Let us consider a component X of $S - \bigcup_{j=1}^{k} L_j$, and let H_X be the stabilizer of X in H. We may identify X with a subset of the Riemann sphere bounded by a finite collection of pairwise disjoint simple loops and H_X with a finite subgroup of the extended Möbius group (the group of conformal and anticonformal automorphisms of $\widehat{\mathbf{C}}$). One may use this information to give a topological classification of all the possible geometrically finite Kleinian groups we may obtain by the lifting process of H under Schottky uniformizations. In the cyclic conformal case it is done in [H9]. In the last section we provide a method to obtain all Schottky type automorphisms. In this way, Theorem 1 may be used to obtain some of the results in [MMZ] from a planar point of view.

(iii) As said before, a Schottky group of genus g defines a geometrically finite complete hyperbolic structure on a handlebody of genus g with injectivity radii bounded below by a positive value and vice-versa. In particular, we may interpret the notion of a Schottky type group of automorphisms as follows. Assume we have given a pair (S, H), where S is a closed orientable surface of genus g and H is some finite group of its homeomorphisms. As a consequence of Nielsen's realization problem [Ke], we may give to S the structure of a Riemann surface so that, up to homotopy, H is a group of (conformal/anticonformal) automorphisms. The surface S may be thought as the conformal boundary of a handlebody of genus g; such a handlebody is not unique and corresponds to the different Schottky uniformizations of the Riemann surface S. We may ask for the existence of one of these handlebodies for which the group H extends continuously as a group of hyperbolic isometries. The existence of such a handlebody is equivalent to the existence of a Schottky uniformization of S for which H lifts.

(iv) Let V_g be a handlebody of genus g, $\mathrm{Out}(F_g)$ be the group of outer automorphisms of the free group F_g of rank g, $\mathrm{Diff}^+(V_g)$ be the group of orientation preserving homeomorphisms of V_g and $\Psi\colon \mathrm{Diff}^+(V_g) \to \mathrm{Out}(F_g)$ be the natural homomorphism. Assume we are given a finite group H and a homomorphism $\eta\colon H \to \mathrm{Out}(F_g)$. In [MZ], [MMZ], [Z] the following question was studied and solved: *Is there an imbedding $\phi\colon H \to \mathrm{Diff}^+(V_g)$ so that $\Psi\phi = \eta$?* This problem is related to ours by the fact that H solves positively the above if and only if $\phi(H)$ restricted to the border S of V_g is of Schottky type. Unfortunately, in order to use their ideas to solve our problem, we need to check at all possible homomorphisms $\eta\colon H \to \mathrm{Out}(F_g)$ for the (finite) group of (conformal) automorphisms of S, which seems to be not a good strategy. This problem makes our work different and not a consequence of the above papers.

A simple consequence of Theorem 1 is the following reducibility necessary condition on a Schottky type group of automorphisms.

Corollary 1. *If a group H of automorphisms of a closed Riemann surface S is not reducible, then it cannot be of Schottky type.*

2.1. Proof of Theorem 1. Assume we have a group H of automorphisms of a closed Riemann surface S of genus $g \geq 2$.

Let us assume we have a collection of simple loops $L_1, \dots, L_k \subset S$, so that:

(1) the connected components of $S - (L_1 \cup L_2 \cup \cdots \cup L_k)$ consists of genus zero bordered surfaces; and
(2) the collection of loops $\{L_1, \dots, L_k\}$ is invariant under the action of H.

Condition (1) ensures $k \geq g$ and that we are able to find a subcollection $\mathscr{L} \subset \{L_1, \dots, L_k\}$ consisting on g homologically independent loops. The same condition ensures that the normalizer in the free homotopy class of $\mathscr{L}$ is generated by the total collection of loops. Now condition (2) ensures that any Schottky uniformization of S defined by $\mathscr{L}$ has the required lifting property.

Reciprocally, assume H is of Schottky type and let $(\Omega, G, P : \Omega \to S)$ be a Schottky uniformization for which H lifts. As consequence of the uniformization theorem, in S we have a hyperbolic metric for which H acts as group of isometries. Let us consider a simple closed geodesic $L_1 \subset S$ of smallest hyperbolic length with the property that it lifts to a loop on Ω. It follows that for each $h \in H$ either: (i) $h(L_1) = L_1$, or (ii) $h(L_1) \cap L_1 = \emptyset$. In fact, assume that $h(L_1) \cap L_1 \neq \emptyset$ and $h(L_1) \neq L_1$. As h lifts to the Schottky uniformization we have that both L_1 and $h(L_1)$ lift to loops on Ω. Both of them have the same hyperbolic length as L_1. Let us choose respective liftings $\widehat{L_1}$ and $\widehat{h(L_1)}$ so that they intersect. The planarity of Ω asserts that the number of intersection points is even. Let us orient $\widehat{L_1} \subset \Omega$ in counterclockwise order. We now fix an intersection point $p \in \widehat{L_1} \cap \widehat{h(L_1)}$. Let us start from p and follow $\widehat{L_1}$, in the given orientation, until we arrive for first time to a second intersection point, say $q \neq p$. These two points divide $\widehat{L_1}$ (respectively, $\widehat{h(L_1)}$) into two pairwise disjoint arcs A_1 and A_2 (respectively, B_1 and B_2). We may assume that the hyperbolic length of A_1 (respectively, of B_1) is at most half the hyperbolic length of L_1. We may then consider the simple loop $\hat{L} = A_1 \cup B_1 \cup \{p, q\}$. We have that the hyperbolic length of $\hat{L}$ is at most the hyperbolic length of L_1 and that $\hat{L}$ is not a simple closed geodesic. Also, $\hat{L}$ projects on S to a simple closed curve L, which is not geodesic. We consider the unique simple closed geodesic homotopic to L, say N. We then have that N lifts to a loop on the above Schottky uniformization and has strictly smaller hyperbolic length than L_1, a contradiction.

Let us consider the collection of translates geodesics of L_1 under the group H, say $L_1, \dots, L_r$. We have that the collection of connected components of $S-(L_1 \cup \cdots \cup L_r)$ is invariant under H. If some of such components, say X, has positive genus, then we may find a simple closed geodesic $L_{r+1} \subset X$ of smallest hyperbolic length with the property that it lifts to a loop on Ω. As for the case of L_1, we have that for each $h \in H$ either: (i) $h(L_{r+1}) = L_{r+1}$, or (ii) $h(L_{r+1}) \cap L_{r+1} = \emptyset$ and $h(L_{r+1})$ is disjoint from $L_1, \dots, L_r$. We now consider the translates under H of the geodesic L_{r+1}, say $L_{r+1}, \dots, L_{r+s}$. The collection of connected components of $S-(L_1 \cup \cdots \cup L_{r+s})$ still invariant under the action of H. As the genus of S is finite, we may proceed with the above argument a finite number of times until we get that each connected component (of the complement of the respective collection of geodesics) has genus zero. The final collection of loops obtained with such a procedure is the desired one. ▫

2.2. Some generalities of Schottky type groups of automorphisms. We end this section with some generalities on Schottky type groups of automorphisms. First, we need the following definition. Let $q>0$ be any odd integer number. An extended Möbius transformation η which is conjugated to a transformation of the form

$$\widehat{\eta}(z) = \frac{e^{k\pi i/q}}{\bar{z}},$$

where $k \in \{1,3,\dots,q-2\}$ is odd and relatively prime with q, is called an *imaginary elliptic transformation.* In this way, η^2 is an elliptic transformation of order q and η^q is an imaginary reflection. If $k=1$, then we say that η is a geometric imaginary elliptic transformation. In any case, if η is an imaginary reflection of order $2q$, then $\widehat{\mathbf{C}}/\eta$ is a real projective plane with exactly one branch value of order q. When $q=1$, we are in the presence of imaginary reflections and $\widehat{\mathbf{C}}/\eta$ is a real projective plane without branch values.

Let us assume we have a Schottky type group H of automorphisms of a Riemann surface S of genus $g \geq 2$. Theorem 1 provides the existence of a Schottky system of loops for H. To check the existence of such a collection of loops is in general not so easy to get. It is for that reason one would like to have conditions, which should be easy to check, ensuring the existence of a Schottky system of loops. Let $(\Omega, G, P\colon \Omega \to S)$ be a Schottky uniformization of S for which H lifts. We have the following facts.

(1) As the region of discontinuity Ω of a Schottky group is known to be a domain of type O_{AD} [AS], we have that for each $h \in H$, the lifted automorphism $\hat{h}$ should be either the restriction of (i) an extended Möbius transformation if h is anticonformal or (ii) a Möbius transformation if h is conformal. In particular, the group $\widehat{G}$, generated by all the lifted transformations $\hat{h}$, for all $h \in H$, contains G as a finite index normal subgroup. If $\widehat{G}^+$ denotes the subgroup of $\widehat{G}$ consisting of only the Möbius transformations, then we have, as $\widehat{G}^+$ contains G as a finite index

normal subgroup, that $\widehat{G}^+$ turns out to be a geometrically finite function group. Geometrically finite function groups have been classified by B. Maskit [M3].

(2) As $\widehat{G}$ is a finite extension of G and G contains no parabolic transformations, then neither $\widehat{G}$ does.

(3) If $\hat{h} \in \widehat{G}^+$ is an elliptic transformation, then we know from [H3] that either (i) both fixed points of $\hat{h}$ belong to the region of discontinuity Ω of G or (ii) there is a loxodromic transformation in G commuting with $\hat{h}$.

(4) If $\hat{h} \in \widehat{G}^+$ is an elliptic transformation, $\mathrm{Fix}(\hat{h}) = \{a, b\} \subset \Omega$ and there is some $\hat{t} \in \widehat{G}$ so that $\hat{t}(a) = b$, then the non-existence of parabolics in $\widehat{G}$ ensures that $\hat{t}(b) = a$. It follows that: (i) if $\hat{t} \in \widehat{G}^+$, then $\hat{t}^2 = I$; and (ii) if $\hat{t} \notin \widehat{G}^+$, then $\hat{t}$ is imaginary elliptic.

3. The conformal situation

In the case of conformal groups H a much simpler necessary condition, called condition (A), was obtained in [H4]; this condition (A) can be obtained from the arguments done at the end of last section.

3.1. Condition (A). Let S be a closed Riemann surface and H a finite group of its conformal automorphisms. If $a \in S$ is a fixed point of some $h \in H - \{I\}$, then we denote by $R(h, a) \in (-\pi, \pi]$ the rotation number of h about a, and we denote by $H(a)$ the stabilizer subgroup of a in H. We say that H satisfies the *condition* (A) if the set of all fixed points of the non-trivial elements of H can be put into pairs satisfying the following properties.

(A1) If $\{p, q\}$ is such a pair, then $p \neq q$, $H(p) = H(q) = H_{\{p,q\}}$ and, for each $h \in H_{\{p,q\}}$ of order greater than two, $R(h, p) = -R(h, q)$.
(A2) If $\{p, q\}$ and $\{r, t\}$ are two such pairs, then either $\{p, q\} \cap \{r, t\} = \emptyset$ or $\{p, q\} = \{r, t\}$.
(A3) If $\{p, q\}$ is a pair and $t \in H$ is so that $t(p) = q$, then t has order two.
(A4) If p is a fixed point of some non-trivial element of H, then there is another fixed point q so that $\{p, q\}$ is one of the above pairs.

A pairing of H satisfying (A1)–(A4) is called a *Schottky pairing* of H.

Theorem 2 ([H4]). *Condition* (A) *is a necessary condition for a group H of conformal automorphisms of a closed Riemann surface to be of Schottky type.*

3.2. Condition (A*). The same as condition (A), but replacing (A3) with the following:

(A3*) for each pair $\{p, q\}$ there is no transformation $t \in H$ so that $t(p) = q$.

Remark 2. (1) If either (i) the order of H is odd or (ii) H is a cyclic group, then (A3) cannot happen, in particular, condition (A) turns out to be condition (A*) in this situation.

(2) If H satisfies condition (A*), then S/H cannot have signature $(0,3;m,n,t)$, that is, it is not of genus zero with exactly three branch values (see Corollary 2). An easy way to see that in this case H cannot be of Schottky type is the following. Since every function group of such signature is a Fuchsian group of the first kind [Kr], it contains no finite index Schottky subgroups. This observation and Riemann–Hurwitz's formula permits us to obtain that the order of every Schottky type group of conformal automorphisms of a surface of genus $g \geq 2$ has upper bound equal to $12(g-1)$ in contrast to Hurwitz's bound $84(g-1)$.

(3) If H is an Abelian group and there is a pair $\{p,q\}$ in a Schottky pairing which is permuted by some involution, then $H_{\{p,q\}}$ is necessarily a cyclic group of order 2.

(4) Given a Schottky pairing for a group H, we may obtain a new Schottky pairing with the following extra symmetrical property [H4]:

If $\{p,q\}$ is a pair, then for all $h \in H$ we have that $\{h(p),h(q)\}$ is again a pair.

(5) If H is a group of conformal automorphisms that satisfies condition (A), then every subgroup $K < H$ satisfies condition (A).

(6) Condition (A) trivially holds in the following cases: (i) H acts freely, i.e., no element of $H-\{I\}$ has fixed points; (ii) H is a cyclic group of order 2; (iii) H is a dihedral group.

As a consequence of part (1) and (2) of the above remark and Theorem 1 we have the following easy fact.

Corollary 2. *If H is a group of conformal automorphisms of a closed Riemann surface S so that S/H is the Riemann sphere with exactly three branch values, then H is not reducible.*

Remark 3. The above fact tells us the difficulty of obtaining an explicit example of both a Schottky group and an algebraic curve representing the same conformal class of Riemann surface. This type of problem has been carried out (numerically) in [H7].

Theorem 3 ([H2], [H5], [H6], [H8]). *Condition* (A) *turns out to be necessary and sufficient in the class of* (i) *Abelian groups,* (ii) *dihedral groups,* (iii) *the alternating groups $\mathscr{A}_4$ and $\mathscr{A}_5$ and* (iv) *the symmetric group $\mathscr{S}_4$.*

In [RZ2] the above theorem is proved using 3-dimensional methods in the case of dihedral and Abelian groups. Also, a general necessary condition for a finite group to be of Schottky type is given in [RZ1].

Corollary 3. *If H is either* (i) *a freely acting Abelian group,* (ii) *a cyclic group of order* 2*; or* (iii) *a dihedral group, then H is of Schottky type.*

Remark 4. Condition (A) is not in general sufficient as was seen in [H8].

4. The anticonformal situation

Let us now consider a group H of automorphisms of a closed Riemann surface S, containing necessarily anticonformal ones. We denote by H^+ its index two subgroup consisting of the conformal automorphisms. We start with the following easy fact.

Theorem 4. *If H is an Abelian group, then H^+ is of Schottky type.*

Proof. Choose $\tau \in H - H^+$. Let $h^+ \in H^+$ be different from the identity. As an involution has an even number of fixed points, we only need to see how to construct pairings, satisfying condition (A), for h^+ of order at least 3. Assume then that h^+ has order bigger than 2. Let $a \in S$ be a fixed point of h^+. We have that $\tau(a)$ is also a fixed point of h^+ and $R\big(h^+, \tau(a)\big) = R\big(\tau \circ h^+ \circ \tau^{-1}, \tau(a)\big) = -R(h^+, a)$. It follows that, as $R\big(h^+, \tau(a)\big) \in (-\pi, \pi)$, that $\tau(a) \neq a$ and that the pairings of type $\big(a, \tau(a)\big)$ will give us a Schottky pairing for the Abelian group H^+. As H^+ is Abelian group we have, as consequence of Theorem 3, that H^+ is of Schottky type. □

4.1. The cyclic case. If H is a cyclic group of order 2, say generated by the anticonformal involution $\tau\colon S \to S$, then H is of Schottky type. In the case that τ is a reflection (that is, has fixed points), this fact was already known to Koebe [Ko1]. In the case that τ is an imaginary reflection (that is, has no fixed points), this follows from the fact that the topological action of such an involution is rigid. Some results in the case when τ is an imaginary reflection have been obtained in [HM]. Not much is known in the general anticonformal case as it is in the conformal situation. In [HC] we have considered the cyclic case. The following summarizes the results obtained there.

Theorem 5 ([HC]). *Let S be a closed Riemann surface and $\psi\colon S \to S$ be an anticonformal automorphism of order $2p$.*

(i) *If $p = 2, 3$, then ψ is of Schottky type.*
(ii) *If S/ψ has non-empty border, then ψ is of Schottky type.*
(iii) *If no non-trivial power of ψ has fixed points, then ψ is of Schottky type.*

Situation (ii) of the above takes care of the case when S/ψ has non-empty border. Situations (i) and (iii) take care of some of the complementary cases. The following, which is the subject of the rest of this paper (except for the last section), complete the situation in the cyclic non-orientable case.

Theorem 6. *Let S be a closed Riemann surface and $\psi\colon S \to S$ be an anticonformal automorphism of order $2p$ so that S/ψ has no boundary.*

(i) *ψ is of Schottky type if and only if ψ has a Klein–Schottky pairing.*
(ii) *If p is a prime, then ψ is always of Schottky type.*

In Section 5 we give the definition of a Klein–Schottky pairing and in Sections 6 and 7 we give the proof of Theorem 6 (in those sections the sufficiency part corresponds to Theorem 7 and the necessity part corresponds to Theorem 8).

5. Klein–Schottky pairings

In the rest of this section we fix a closed Riemann surface S, of positive genus g, an anticonformal automorphism $\psi\colon S \to S$ of order $2p$, where p is a positive integer, so that S/ψ has empty boundary. If we set $\tau = \psi^p$ and $\phi = \psi^2$, then we have that:

(i) τ is either an imaginary reflection (if p is odd) or a conformal involution (if p is even); and
(ii) ϕ is a conformal automorphism of order p.

Let us denote by H the cyclic group generated by ψ, by H^+ the index two subgroup generated by ϕ, by $R = S/\phi$ and by $Q\colon S \to R$ be the p-fold holomorphic branched covering induced by the action of ϕ. Theorem 4 asserts the existence of a Schottky pairing for the cyclic group H^+. If γ is the genus of R and r is the number of pairs in a Schottky pairing for H^+, then we have that R is an orbifold of genus γ with exactly $2r$ branch values (as a consequence of part (1) of Remark 2). Let us denote by $\delta\colon R \to R$ the anticonformal involution induced by ψ.

Lemma 1. *The anticonformal involution δ is an imaginary reflection. In particular,* (i) *the non-trivial powers of ψ only have isolated fixed points, and* (ii) *the odd powers ψ^{2k-1} have no fixed points.*

Proof. As $S/\psi = R/\delta$ has no boundary, then δ cannot be a reflection. As each odd power of ψ induces δ, we obtain part (ii). Now part (i) is clear since even powers are conformal and odd powers have no fixed points. ▫

5.1. Loops and arcs. Let us consider a simple loop $L \subset R$, disjoint from the branching locus. We say that L *lifts to a loop* on S if the lifting of L consists exactly of p pairwise disjoint simple loops. Otherwise, we say that L *lifts to an arc* on S.

5.2. Oriented and non-oriented pairs of Schottky pairings. As a consequence of Theorem 4 we have that $\phi = \psi^2$ satisfies condition (A) and, in particular, we have the existence of Schottky pairings for ϕ.

Assume $\bigl\{\{a_1, b_1\}, \dots, \{a_r, b_r\}\bigr\}$ is one of such Schottky pairings. By part (1) of Remark 2, we have $Q(a_j) \neq Q(b_j)$. There are two possibilities:

(i) $\delta\bigl(Q(a_j)\bigr) = Q(b_j)$; or
(ii) $\delta\bigl(Q(a_j)\bigr) \neq Q(b_j)$.

In case (i) we say that the pair $\{a_j, b_j\}$ is a *non-oriented pair* and in case (ii) we say it is an *oriented pair*. As the involution δ has no fixed points, we have that the number of oriented pairs in any Schottky pairing is necessarily even.

5.3. Cylinders. A *cylinder* $C \subset S$ is a closed subset homeomorphic to the set $\left\{z \in \mathbf{C} : \frac{1}{2} \leq |z| \leq 2\right\}$. Any simple loop on the interior of C homotopic to the loop corresponding to $\{|z| = 1\}$ is called a *waist* of C.

5.4. Petals. A simple loop $L \subset R$, either (i) containing no branch values or (ii) containing $Q(a)$ and $Q(b)$, where $\{a, b\}$ is a non-oriented pair of some Schottky pairing of ϕ, but containing no other branch value of Q, is called a *petal* if it is possible to find a cylinder $C \subset R$ for which L is a waist and so that:

(i) the only branch values of Q in the closure of C are those contained in L;
(ii) each of the two boundary loops of C, say A and B, lifts to a loop on S;
(iii) $\delta(L) = L$; and
(iv) $\delta(C) = C$ (in particular, $\delta(A) = B$).

A set of petals $L_1, \dots, L_k$ are called disjoint if they are disjoint as loops. In this case, we may choose their respective cylinders to be pairwise disjoint.

Remark 5. If we have a petal L, with respective cylinder C, then we have that a connected component P of $Q^{-1}(C)$ is a genus zero surface with boundary. The stabilizer of P in H^+ is either trivial (if the petal L has no branch values on it) or it is exactly $H^+(a) = H^+(b)$, where $\{a, b\}$ is the non-oriented pair for which $Q(a), Q(b) \in L$.

5.5. Klein–Schottky pairing. As observed in Section 5.2, any Schottky pairing for ϕ can be written as a collection

$$(*) \qquad \big\{\{a_1, b_1\}, \dots, \{a_s, b_s\}, \{c_1, d_1\}, \dots, \{c_{2t}, d_{2t}\}\big\}$$

so that the pairs $\{a_j, b_j\}$ are the non-oriented ones and the pairs $\{c_j, d_j\}$ are the oriented ones. We say that the Schottky pairing $(*)$ is a *Klein-Schottky pairing* of $\langle\psi\rangle$ if:

(1) $s \leq \gamma + 1$; and
(2) if $s > 0$, then it is possible to find a collection of pairwise disjoint petals $L_1, \dots, L_s \subset R$, so that:
  (2.1) the loop L_j contains $Q(a_j)$ and $Q(b_j)$;
  (2.2) $\bigcup_{j=1}^{\gamma+1} L_j$ divides R into two bordered surfaces R_1 and R_2, which are permuted by δ, and so that $Q(c_k), Q(d_k) \in R_1$ and $Q(c_{t+k}), Q(d_{t+k}) \in R_2$, for $k = 1, \dots, t$.

Lemma 2. *A Schottky pairing without non-oriented pairs is a Klein–Schottky pairing. If the genus of S/ϕ is even, then any Schottky pairing with at most one non-oriented pair is a Klein–Schottky pairing.*

Proof. The first statement is clear by the definition. Now we assume the genus of S/ϕ is even and that we have a Schottky pairing with exactly one non-oriented pair, say

$$\big\{\{a_1, b_1\}, \{c_1, d_1\}, \dots, \{c_{2t}, d_{2t}\}\big\},$$

so that the pair $\{a_1, b_1\}$ is the non-oriented one and the pairs $\{c_j, d_j\}$ are the oriented ones.

We have the following facts:

(i) A simple loop $N \subset R$ which bounds a topological disc $\Delta_N \subset R$, containing in its interior the branch values $Q(c_j)$ and $Q(d_j)$ and whose closure is disjoint from all other branch values, must lift a loop on S;

(ii) A diving simple loop $M \subset R$, disjoint from the branch locus, which does not separate $Q(c_j)$ and $Q(d_j)$ (for all j) also lifts to a loop on S. This is consequence of the fact that such a loop will be homotopic to the product of commutators and loops as in (i).

It is clear from the topology of the action of an imaginary reflection on an even genus surface the existence of a dividing loop $L \subset R$ satisfying that: (a) L is invariant under δ, (b) $Q(a_1), Q(b_1) \subset L$, and (c) L is disjoint from all other branch values. As a consequence of the above facts we have that the loop L is necessarily a petal. This petal satisfies the conditions for the above Schottky pairing being a Klein–Schottky pairing. □

Proposition 1. *If p is a prime, then there is a Klein–Schottky pairing for $\langle\psi\rangle$.*

Proof. Let us consider first the case $p \geq 3$. In this case we have that τ is an imaginary reflection. We may write the set of fixed points of ϕ as:

$$\mathrm{Fix}(\phi) = \big\{x_1, x_2 = \tau(x_1), x_3, x_4 = \tau(x_3), \ldots, x_{2r-1}, x_{2r} = \tau(x_{2r-1})\big\},$$

so that (the rotation numbers) $R(\phi^+, x_{2j-1}) \in (0, +\pi)$. If r is even, say $r = 2s$, then the collection

$$\big\{\{x_1, x_{2r}\}, \{x_3, x_{2r-2}\}, \ldots, \{x_{2s-1}, x_{2s+2}\}, \{x_{2s+1}, x_{2s}\}\big\}$$

is a Schottky pairing with no non-oriented pairs, then a Klein–Schottky pairing as consequence of Lemma 2. If r is odd, say $r = 2s-1$, then we consider the collection

$$\big\{\{x_{4s-3}, x_{4s-2}\}, \{x_1, x_{4s-4}\}, \{x_3, x_{4s-6-4}\}, \ldots, \{x_{2s-1}, x_{2s+2}\}, \{x_{2s+1}, x_{2s}\}\big\}.$$

The above is a Schottky pairing with exactly one non-oriented pair, given by $(\{x_{4s-3}, x_{4s-2}\})$. If S/ϕ has even genus (that is, if S has even genus by Riemann–Hurwitz formula), then we have a Klein–Schottky pairing as consequence of Lemma 2.

Let us now consider the case that S/ϕ has odd genus. We may draw two pairwise non-dividing disjoint simple loops $L_1, L_2 \subset R$, each one invariant under δ, none of them containing branch values, both together dividing R into two components, say R_1 and R_2 (which are permuted by δ). On R_1 we may draw a dividing simple closed loop L_3 so that L_3 divides R_1 into two surfaces, say $R_{1,1}$ and $R_{1,2}$, so that $R_{1,1}$ contains no branch values, $R_{1,2}$ contains all branched values on R_1 and $R_{1,2}$ is homeomorphic to a three-holed sphere. The loop L_3 lifts to a loop on S since it is product of commutators. Just by a simple modification on the loop L_1, we may assume that the branched values contained on $R_{1,2}$ are: $Q(x_1)$, $Q(x_3)$, $Q(x_5),\dots,Q(x_{2s-3})$, $Q(x_{2s})$, $Q(x_{2s+2}),\dots,Q(x_{4s-4})$ and $Q(x_{4s-3})$. We may draw pairwise disjoint simple loops $N_1,\dots,N_{s-1}$, where

(i) N_j bounds a topological closed disc $\Delta_j \subset R_{1,2}$ containing the points $Q(x_{2j-1})$ and $Q(x_{2(r-j)})$, and
(ii) $\Delta_j \cap \Delta_k = \emptyset$, for $j \neq k$.

As we know that each loop N_j lifts to a loop on S, we only need to consider the case that R has genus 1 and exactly two branched values which are permuted by δ, that is, S has genus the prime value p. As the topological action of a cyclic group of order a prime p, with exactly two fixed points, on a genus p Riemann surface is rigid, a petal as needed is easy to obtain (see Figure 1 for $p=3$).

In the case $p=2$ we have that ϕ is a conformal involution, then it has an even number of fixed points. Now we may use the same arguments as in the above case. □

6. Klein–Schottky pairings: A sufficient condition

Theorem 7. *Let S be a closed Riemann surface and $\psi\colon S \to S$ an anticonformal automorphism of order $2p$ so that S/ψ has no boundary. If $\langle\psi\rangle$ has a Klein–Schottky pairing, then ψ is of Schottky type.*

The above together Proposition 1 gives us as a consequence part (ii) of Theorem 6.

Corollary 4. *Let S be a closed Riemann surface and $\psi\colon S \to S$ an anticonformal automorphism of order $2p$ with p a prime. If S/ψ has no boundary, then ψ is of Schottky type.*

Proof of Theorem 7. Let γ be the genus of $R = S/\phi$ and $\delta\colon R \to R$ be the imaginary reflection induced by ψ.

Let $\bigl\{\{a_1,b_1\},\dots,\{a_s,b_s\},\{c_1,d_1\},\dots,\{c_{2t},d_{2t}\}\bigr\}$ be a Klein–Schottky pairing for ϕ, so that the pairs $\{a_j,b_j\}$ are the non-oriented ones and the pairs $\{c_j,d_j\}$ are the oriented ones. In particular, $\gamma = 2m+s-1$ for some non-negative integer m.

In the case that our Klein–Schottky pairing has no non-oriented pairs, the arguments are the same as for the case when there is no branching values [HC]. This is consequence of the fact that a simple loop $N \subset R$ which bounds a topological disc $\Delta_N \subset R$, containing in its interior the branch values $Q(c_j)$ and $Q(d_j)$ and whose closure is disjoint from all other branch values, must lift a loop on S.

We assume from now on that $s>0$. We may also assume (after reindexing) that $\delta\bigl(Q(c_j)\bigr) = Q(c_{t+j})$, $\delta\bigl(Q(d_j)\bigr) = Q(d_{t+j})$, for $j=1,\dots,t$. As $\{a_j,b_j\}$

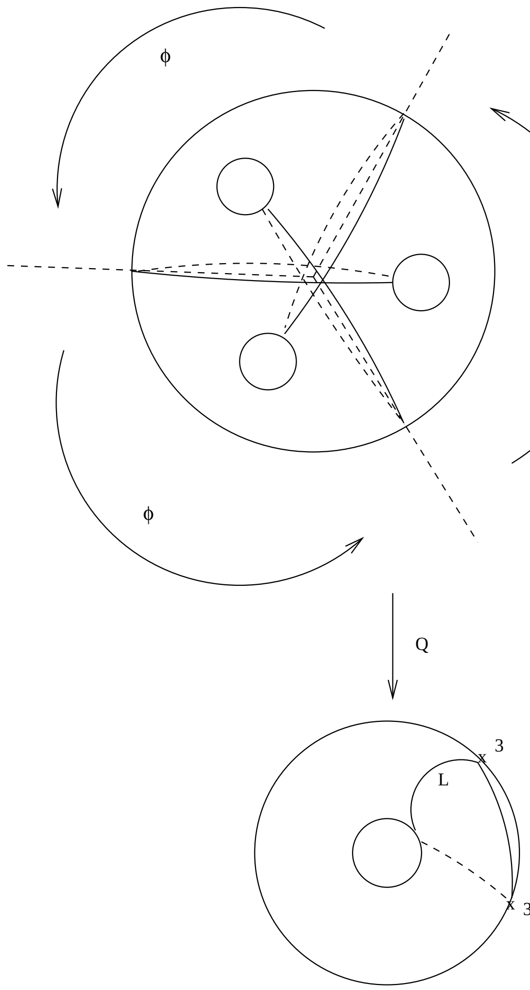

Figure 1. The topological action in case $p=3$ and a petal L.

is non-oriented, we have $\delta\bigl(Q(a_j)\bigr) = Q(b_j)$, for $j=1,\dots,s$. By the definition of Klein–Schottky pairings, we have the existence of a set of disjoint petals $Z_1,\dots,Z_s$, so that Z_j contains the points $Q(a_j)$ and $Q(b_j)$. We have a collection of pairwise disjoint cylinders $C_1,\dots,C_s$, where the waist of C_j is Z_j and its boundary loops are $Z_{j,1}$ and $Z_{j,2}=\delta(Z_{j,1})$. The cylinders C_j can be assumed such that no other branch value than $Q(a_j)$ and $Q(b_j)$ are contained on it. Let us denote by R_1 and $R_2=\delta(R_1)$ the two components of $R-(C_1\cup\dots\cup C_s)$. We have that exactly $2t$ branch values must belong to R_1 and the other $2t$ branch values belong to R_2. We may draw a dividing simple loop $L_{00}\subset R_1$ so that it divides R_1 into two components, say $R_{1,1}$ and $R_{1,2}$, so that $R_{1,1}$ is a surface of genus m with exactly one border (the curve L_{00}) and $R_{1,2}$ is a genus zero surface with $1+s$ boundary loops (L_{00}, $Z_{1,1}$, $Z_{2,1},\dots,Z_{s,1}$) containing all $2t$ branch values on R_1. We may assume without problem that these branched values are given by $Q(c_1)$, $Q(d_1)$, $Q(c_2)$, $Q(d_2),\dots,Q(c_t)$, $Q(d_t)$. We may now construct:

(i) a collection of (oriented) homologically independent pairwise disjoint simple loops $L_1,\dots,L_m$ on $R_{1,1}$; and
(ii) a collection of (oriented) pairwise disjoint simple loops $M_1,\dots,M_t$ on $R_{1,2}$, so that M_j bound a disc containing exactly the two branch values of order p given by $Q(c_j), Q(d_j)$.

As consequence of Lemma 3 below we may assume that each loop $L_1,\dots,L_m$ lifts to a loop on S. Let U be the bordered orbifold bounded by the loops $Z_{1,1}$, $Z_{2,1},\dots,Z_{s,1}$, L_{00}, $M_1,\dots,M_t$. We have that on U there are no branch values. As $Z_{1,1}$, $Z_{2,1},\dots,Z_{s,1}$, L_{00}, $M_1,\dots,M_s$ all lift to a loop, we have that the connected components of $Q^{-1}(U)$ consist of exactly genus zero bordered surfaces homeomorphic to U. The collection of loops on S, obtained as liftings by Q of the following collection

$$\big\{L_1,\dots,L_m,\delta(L_1),\dots,\delta(L_m),M_1,\dots,M_t,\delta(M_1),\dots,\delta(M_t),\\ Z_{1,1},Z_{2,1},\dots,Z_{s,1},Z_{s,2},L_{00}\big\}$$

(a) is invariant under the action of ψ; and
(b) divides the surface S into genus zero surfaces.

Now Theorem 1 asserts that ψ is of Schottky type. □

Lemma 3. *In the above proof, each of the loops $M_1,\dots,M_t$ lifts to a loop on S and the loops $L_1,\dots,L_m$ can be chosen so that each of them lifts to a loop on S.*

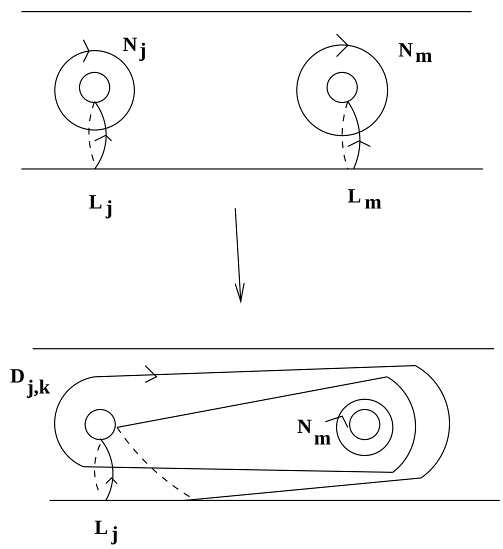

Figure 2.

Proof. The idea is the following. We first construct a collection of (oriented) loops satisfying (i) and (ii). Clearly, each loop M_j lifts to a loop since it goes around exactly the two branch values $Q(c_j), Q(d_j)$, which satisfies the rotation number property of condition (A). As the loop L_{00} is free homotopic to commutators and the covering is Abelian, it also lifts to a loop. Choose m homologically independent (oriented) simple loops $N_1, \dots, N_m$ on $R_{1,1}$ so that N_j intersects only the loop L_j at one point. We have that at least one of the simple loops L_j or N_j or $L_j^{k_j} \cdot N_j$ ($k_j \in \{1, 2, 3, \dots, p-1\}$) must lift to a loop. We replace L_j by the correct one. □

Remark 6. In the above construction of loops L_j and N_j, we may assume that each of the loops $L_1, \dots, L_m$, $N_1, \dots, N_{m-1}$ lifts to a loop. In fact, by the above lemma, we have that each of the loops $L_1, \dots, l_m$ lifts to a loop. Assume one of the loop N_j lifts to an arc. We may assume that the loop N_m lifts to an arc. Let $j \in \{1, 2, \dots, m-1\}$ be so that the loops N_j lifts to an arc. Choosing orientations in a suitable manner, we have that, for each $k = 0, 1, \dots, p$, there is a simple loop free homotopic to $D_{j,k} = N_j \cdot (N_m^{-1} \cdot L_j) \cdot N_m^{-1})^k$, which is disjoint from all loops N_i and L_i, for $i \in \{1, \dots, m-1\} - \{j\}$, and intersects exactly at one point with L_j (see Figure 2). Since we are dealing with a cyclic covering, we may choose a suitable value of k so that $D_{j,k}$ lifts to a loop. We replace N_j with $S_{j,k}$ and also replace the loop L_m by a suitable one in order to have the new loops $L_1, \dots, L_m$, $N_1, \dots, N_m$, with the starting intersection conditions. Applying this argument for each $j \in \{1, 2, \dots, m-1\}$ we end with a collection $L_1, \dots, L_m$, $N_1, \dots, N_m$ so that at least each $L_1, \dots, L_{m-1}$, $N_1, \dots, N_{m-1}$ lifts to a loop. If we have that L_m also lifts to a loop, then we are done. In the other case, we can replace the pair L_m and L_m respectively by L_m^k and L_m^{-1} (see Figure 3), for a suitable value of k, to get the desired collection of curves.

7. Klein–Schottky pairings: A necessary condition

Theorem 8. *Let S be a closed Riemann surface and $\psi\colon S \to S$ an anticonformal automorphism of order $2p$, so that S/ψ has no boundary. If ψ is of Schottky type, then ψ has a Klein–Schottky pairing.*

Proof. As consequence of Theorem 1 we have the existence of a Schottky system of loops $L_1, \dots, L_k \subset S$ of $H = \langle \psi \rangle$. We may assume that: (i) such a system of loops is minimal in the sense that no strictly smaller subcollection is a Schottky system of loops of H, and (ii) no fixed point of a non-trivial element of $H^+ = \langle \phi = \psi^2 \rangle$ is contained in some of the above loops. Since we are assuming that S/H has no border, we have as consequence of Lemma 1, that the odd powers of ψ have no fixed points.

Let $x \in S$ be a fixed point of ϕ^l, where the stabilizer of x in H^+ is generated by ϕ^l. Let $X \subset S - \bigcup_{j=1}^{k} L_j$ be the component containing x and let $H_X < H$ be its stabilizer in H. Let us denote by ${H_X}^+$ its subgroup of conformal

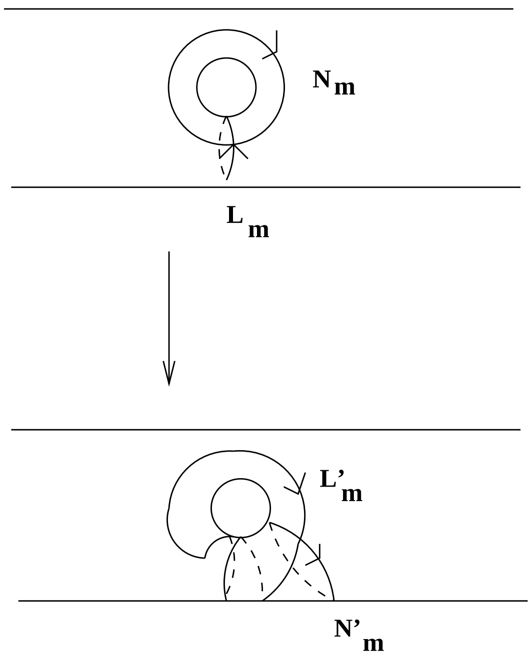

Figure 3.

automorphisms. Clearly, $\phi^l \in {H_X}^+$. As X has genus zero, ϕ^l may only have at most two fixed points on X, one of them being x.

Claim. *ϕ^l has exactly two fixed points on X and H_X^+ is generated by ϕ^l.*

In fact, assume we have exactly one fixed point on X for ϕ^l. It follows that there is a loop $L \subset \{L_1, \dots, L_k\}$ which belongs to the boundary of X and which is invariant under ϕ^l. As each element $h \in H_X$ commutes with ϕ^l, we have that $h(x) = x$. As no odd power of ψ has fixed points, this implies that $H_X = \langle \phi^l \rangle = H_X^+$. In particular, we may delete L and its H-translates to obtain a Schottky system of loops of H, a contradiction to the minimality of the Schottky system of loops. It follows that on X we should have exactly two fixed points of ϕ^l, one of them is x and that ${H_X}^+$ is generated by ϕ^l. This ends the proof of our fact.

The above claim asserts that we are able to obtain a collection of pairs $\{x, y\}$, where x, y run over all pairs of fixed points of a (conformal) generator ϕ^l of ${H_X}^+$ for all possible components X of $S - \bigcup_{j=1}^{k} L_j$ for which we have ${H_X}^+ \neq \{I\}$. Such a collection is in fact a Schottky pairing for ϕ just by construction.

Let us consider a component X as above and the corresponding two fixed points $x, y \in X$ of the conformal automorphism ϕ^l generating ${H_X}^+ \neq \{I\}$. We may assume that the rotation number of ϕ^l at x has the form $2\pi/r$ (in particular, the order of ϕ^l is r, a divisor of p).

Let us first assume that $H_X = {H_X}^+$. As consequence of Riemann–Hurwitz's formula, and the minimality of the Schottky system of loops, we have that X is a sphere with r boundary loops, cyclically permuted by ϕ^l. The quotient X/H_X is a disc with two branched values; the projections of x and y. In this way, we see that the pair $\{x,y\}$ is an oriented pair.

Let us now assume that $H_X \neq {H_X}^+$. As H_X is cyclic, we have some odd power of ψ, say ψ^{2t-1} that generates H_X. We may assume that such odd power is such that $\psi^{2(2t-1)} = \phi^l$. As ψ^{2t-1} has no fixed points, we should have that it permutes x and y. By Riemann–Hurwitz's formula and minimality of the Schottky system of loops, we have that X/H_X is a Möbius band with exactly two branch values; the projections of x and y. In this way, we see that the pair $\{x,y\}$ is a no-oriented pair. Moreover, the above asserts that such a pair has a petal (whose cylinder corresponds to X/H_X^+). This also asserts that such a number of Möbius bands cannot be bigger that $1+\gamma$, where γ is the genus of S/H^+, in particular, the collection of non-oriented pairs of the above Schottky pairing have the required properties for a Klein–Schottky pairing. □

8. Construction of Schottky type groups

In this section we provide, as consequence of Theorem 1, a method to construct all those Kleinian groups containing a Schottky group as a normal subgroup of finite order (called in the rest of this section a Schottky extension group), or equivalently, how to construct groups of automorphisms of Schottky type. We first start with a group K containing such a Schottky subgroup and find some regions and subgroups of (extended) Möbius transformations.

8.1. Some admissible regions and some groups. Let us assume we have a non-elementary Schottky extension group K. Let G be a Schottky group which is a normal subgroup, then of finite index. Let Ω be the region of discontinuity of K and set $S = \Omega/G$, $H = K/G$. We have a Schottky uniformization $(\Omega, G, P: \Omega \to S)$ of S for which the group H lifts to the group K and a surjective homomorphism

$$\Theta\colon K \to H,$$

whose kernel is the Schottky group G, satisfying

$$\Theta(k) \circ P = P \circ k, \quad \text{for all } k \in K.$$

As a consequence of Theorem 1, we have the existence of a Schottky system of loops for H, say

$$\mathscr{F} = \{L_1, \dots, L_k\},$$

where $g \le k \le 3g-3$ (we may assume that $L_1, \dots, L_g$ are homologically independent) corresponding to the above Schottky uniformization, that is, each loop in $\mathscr{F}$ lifts to loops under $P\colon \Omega \to S$. Let us denote by $\widehat{\mathscr{F}}$ the collection of loops obtained by lifting all loops in $\mathscr{F}$ under P.

If $L \in \mathscr{F}$ and $\hat{L} \in \widehat{\mathscr{F}}$ are so that $P(\hat{L}) = L$, and the respective stabilizers are given by cyclic groups

$$\begin{cases} K(\hat{L}) = \{k \in K : k(\hat{L}) = \hat{L}\}, \\ H(L) = \{h \in H : h(L) = L\}, \end{cases}$$

then we have that

$$\begin{cases} P\colon \hat{L} \to L \text{ is a homeomorphism,} \\ \Theta\bigl(K(\hat{L})\bigr) = H(L); \\ \Theta\colon K(\hat{L}) \to H(L) \text{ is an isomorphism.} \end{cases}$$

Similarly, if R is a connected component of $S - \mathscr{F}$ and $\widehat{R}$ is a connected component of $\Omega - \widehat{\mathscr{F}}$ so that $P(\widehat{R}) = R$, and the respective stabilizers are given by finite order Möbius groups

$$\begin{cases} K(\widehat{R}) = \{k \in K : k(\widehat{R}) = \widehat{R}\}, \\ H(R) = \{h \in H : h(R) = R\}, \end{cases}$$

then we have that

$$\begin{cases} P\colon \widehat{R} \to R \text{ is a homeomorphism,} \\ \Theta\bigl(K(\widehat{R})\bigr) = H(R); \\ \Theta\colon K(\widehat{R}) \to H(R) \text{ is an isomorphism.} \end{cases}$$

Let us now consider a loop $\hat{L} \in \widehat{\mathscr{F}}$ and denote by $\widehat{R}_1$ and $\widehat{R}_2$ the two connected components of $\Omega - \widehat{\mathscr{F}}$ having $\hat{L}$ as common border (as consequence of Jordan's theorem).

Remark 7. If we have that $P(\widehat{R}_1) = R = P(\widehat{R}_2)$, then $L = P(\hat{L})$ is a non-dividing loop on S so that R is at both sides of L.

We have two possibilities:

(1) $K(\widehat{R}_1) \cap K(\widehat{R}_2) = K(\hat{L})$, in which case we say that $\hat{L}$ is *extraible*; or

(2) $K(\widehat{R}_1) \cap K(\widehat{R}_2)$ has index two in $K(\hat{L})$, in which case we say that $\hat{L}$ is *non-extraible*.

Lemma 4. *In case* (2) *we have that* $K(\hat{L})$ *is a dihedral group generated by the cyclic group* $K(\widehat{R}_1) \cap K(\widehat{R}_2)$ *and a conformal involution* k_L *that permutes* R_1 *with* R_2.

Proof. In case (2) we have the existence of some transformation $k_L \in K(\hat{L}) - K(\widehat{R}_1) \cap K(\widehat{R}_2)$ so that $\widehat{R}_2 = k_L(\widehat{R}_1)$ and $k_L^2 \in K(\widehat{R}_1) \cap K(\widehat{R}_2)$. We have that $k_L^2 = I$. In fact, we have that k_L is a finite order elliptic transformation that preserves a loop L and interchanges both topological discs bounded by it. It follows that both fixed points of k_L should be inside L and, in particular, it must be of order 2. In this way we have that $K(\hat{L})$ is the dihedral group generated by the cyclic group $K(\widehat{R}_1) \cap K(\widehat{R}_2$ and the involution k_L. □

Remark 8. (i) If the group H has odd order, then (as the Schottky group G has no non-trivial finite order transformations) we have that all loops in $\widehat{\mathscr{F}}$ are extraible.

(ii) If $L \subset \partial(\widehat{R})$ is a border loop of some component $\widehat{R}$ of $\Omega - \widehat{\mathscr{F}}$, then we have that L is non-extraible if and only if $h(L)$ is non-extraible for every $h \in K(\widehat{R})$.

We proceed to construct a special domain inside Ω as follows.

8.1.1. *Construction of the first domain.* Let us take a connected component $\widehat{R}_1$ of $\Omega - \widehat{\mathscr{F}}$ and choose a boundary loop of it, say $\hat{L}_1$. If this loop is extraible, then we add to the above domain both the loop $\hat{L}_1$ and the connected component of $\Omega - \widehat{\mathscr{F}}$ at the other side of $\hat{L}_1$ and set $\widehat{R}_2$ as the new domain. If $\hat{L}_1$ is non-extraible, then set $\widehat{R}_2 = \widehat{R}_1$. Let us observe that in any of the two situations there are not two different connected components of $\Omega - \widehat{\mathscr{F}}$ inside $\widehat{R}_2$ which are equivalent under K (but it may happen that a connected component of $\Omega - \widehat{\mathscr{F}}$ inside $\widehat{R}_2$ has a non-trivial stabilizer in K).

8.1.2. *Construction of the second domain.* Let us consider a boundary loop of $\widehat{R}_2$, say $\hat{L}_2$ (different from $\hat{L}_1$ in case that $\widehat{R}_2 = \widehat{R}_1$). If $\hat{L}_2$ is extraible and the connected component of $\Omega - \widehat{\mathscr{F}}$ which is disjoint from $\widehat{R}_2$ is non-equivalent under K to a connected component of $\Omega - \widehat{\mathscr{F}}$ inside $\widehat{R}_2$, then we add to $\widehat{R}_2$ both such a connected component and the loop $\hat{L}_2$ and we set $\widehat{R}_3$ the new domain. In the complementary situation, we set $\widehat{R}_3 = \widehat{R}_2$. We have again that there are not two different connected components of $\Omega - \widehat{\mathscr{F}}$ inside $\widehat{R}_3$ which are equivalent under K.

8.1.3. *The inductive process.* We may proceed as in the previous situation to construct a sequence of domains $\widehat{R}_j$ so that $\widehat{R}_{j+1}$ contains $\widehat{R}_j$ and there are not two different connected components of $\Omega - \widehat{\mathscr{F}}$ inside $\widehat{R}_{j+1}$ which are equivalent under K. As the number of non-equivalent components of $\Omega - \widehat{\mathscr{F}}$ under K is the same number as for the non-equivalent components of $S - \mathscr{F}$ under H, we have that such a sequence is finite. The maximal of them is a domain $\widehat{R}_{S,H}$ with the following properties:

(1) $\widehat{R}_{S,H}$ is disjoint finite union of loop in $\widehat{\mathscr{F}}$ and components of $\Omega - \widehat{\mathscr{F}}$;

(2) any two different components of $\Omega - \widehat{\mathscr{F}}$ inside $\widehat{R}_{S,H}$ are not equivalent under K;

(3) any component of $\Omega - \widehat{\mathscr{F}}$ has an equivalent component (under K) inside $\widehat{R}_{S,H}$;

(4) if $\hat{L} \in \widehat{\mathscr{F}}$ is a boundary component of $\widehat{R}_{S,H}$, then either:

(4.1) there exists a conformal involution $k_{\hat{L}} \in K(\hat{L})$ so that $k_{\hat{L}}(\widehat{R}_{S,H}) \cap \widehat{R}_{S,H} = \emptyset$ and $K(\hat{L})$ is a dihedral group generated by the involution k_L and a cyclic group, which coincides with the stabilizer of $\hat{L}$ of the stabilizer in K of any of the two components of $\Omega - \widehat{\mathscr{F}}$ containing $\hat{L}$ in the border; or

(4.2) there is no a conformal involution in $k \in K$ that preserves $\hat{L}$ so that $k(\widehat{R}_{S,H}) \cap \widehat{R}_{S,H} = \hat{L}$. In this case, there exists another different boundary loop $\hat{L}' \in \widehat{\mathscr{F}}$ of $\widehat{R}_{S,H}$ and a loxodromic transformation $k_{\hat{L}} \in K$ so that

$$\begin{cases} k_{\hat{L}}(\hat{L}) = \hat{L}'; \\ k_{\hat{L}}(\widehat{R}_{S,H}) \cap \widehat{R}_{S,H} = \hat{L}'. \end{cases}$$

Let us consider two different boundary loops $\hat{L}_1$ and $\hat{L}_2$ of our region $\widehat{R}_{S,H}$, both of them satisfying the property (4.1) above. Let us assume there is a transformation $k \in K$ satisfying $k(\hat{L}_1) = \hat{L}_2$. As we are assuming the loops to be different, we have that $k \neq I$. Let us denote by $k_{L_j} \in K$ the conformal involution preserving $\hat{L}_j$ that permutes both topological discs bounded by $\hat{L}_j$, for $j = 1, 2$. We have that either k or $k_{L_2}k$ sends the component of $\Omega - \widehat{\mathscr{F}}$ contained inside $\widehat{R}_{S,H}$ having $\hat{L}_1$ as border loop to the component of $\Omega - \widehat{\mathscr{F}}$ contained inside $\widehat{R}_{S,H}$ having $\hat{L}_2$. As consequence of (2) above, we have that both components should be the same, say $\widehat{R}$, and that k must belong to either $K(\widehat{R})$ or $k_{L_2}K(\widehat{R})$.

8.2. The construction method. Let us consider a finite collection R_1, ..., R_m of admissible regions in the Riemann sphere so that $R = \bigcup_{j=1}^{m} R_j$ is connected. In this way, R is again an admissible region.

(1) Let us assume we have finite Möbius group $H_1, \dots, H_m$ so that:

(1.1) For every $h \in H_j$ we have $h(R_j) = R_j$;

(1.2) If $R_i \neq R_j$ are border related, say with common border loop L, then $H_i(L) = H_j(L)$;

(1.3) Let R_j, R_k and R_t be three different regions so that R_k and R_t are each one border related to R_j (in particular, R_k and R_t cannot be border related). Then there is no $h \in H_j$ so that $h(R_j \cap R_k) = R_j \cap R_t$.

Under the above conditions we may apply the first Klein–Maskit combination theorem [M4] to obtain that $H = \langle H_1, \dots, H_m \rangle$ is a geometrically finite function group which is a free amalgamated product of the finite order groups H_j (the amalgamations are given over the cyclic groups stabilizers of the common boundary loops of different border related regions). Moreover, if L is a boundary loop of R and let $R_j \in \{R_1, \dots, R_m\}$ be the (unique) component that contains L, then we have that $H_j(L) = H(L)$.

(2) If L is a boundary loop of R, then we assume we have a Möbius transformation h_L, so that:

(2.1) $h_L(L) = L'$ is a boundary loop of R;

(2.2) $h_L(R) \cap R = L'$;

(2.3) $h_{L'} = h_L^{-1}$;

(2.4) if $L = L'$, then $K(L) = \langle h_L, H(L) \rangle$ is either a dihedral group or $\mathbf{Z}_2$;

(2.5) if $L \neq L'$ and

(2.5.1) there is an element $h \in H$ so that $h(L) = L'$, then $h^{-1}h_L$ has order 2, permutes both topological discs bounded by L and the group $K(L) = \langle h^{-1}h_L, H(L) \rangle$ is either a dihedral group or $\mathbf{Z}_2$ which does not depend on the choice of h;

(2.5.2) if there is no element $h \in H$ so that $h(L) = L'$, then we set $K(L) = H(L)$; and

(2.6) h_L conjugates $H(L)$ onto $H(L')$.

(3) If we have two different boundary loops L_1 and L_2 of R and some $h \in H$ so that $h(L_1) = L_2$, then h conjugates $K(L_1)$ onto $K(L_2)$.

If we have that $L' \neq L$, then conditions (2.1) and (2.2) and the fact that the boundary loops of R are pairwise disjoint assert that the transformation h_L is a loxodromic transformation.

If $L' = L$, then we have from (2.2) and (2.3) that $h_L \neq I$ and $h_L^2 = I$; this makes clear why condition (2.4) makes sense.

With respect to conditions (2.5) and (3), let us observe that the geometrical construction of H by use of the first Klein–Maskit combination theorem asserts that if there are two boundary loops of R, say L_1 and L_2 and there is some $h \in H$ so that $h(L_1) = L_2$, then either (i) L_1 and L_2 belong to the same domain R_j and $h \in H_j$ or (ii) there are two border related regions R_j and R_i so that L_1 is in the border of R_j, L_2 is in the border of R_i and there exist $h_j \in H_j$ and $h_i \in H_i$ so that $h = h_i h_j$.

Condition (2.6) is necessary as consequence of the decomposition done in the previous section in order to get a Schottky extension group.

Let us consider the group K generated by the function group H and the transformations h_L, where L runs over all boundary loops of R. We may now apply the second Klein–Maskit combination theorem [M4] to obtain that K is a geometrically finite Kleinian group. Moreover, its region of discontinuity will be necessarily connected. As a consequence of Selberg's lemma K contains a finite index torsion free normal subgroup G. In particular, G will be purely loxodromic finitely generated geometrically finite Kleinian group with connected region of discontinuity. It follows that G is a Schottky group as a consequence of the classification of function groups [M2], [M3]. In this way, the class of groups constructed by the above procedure are Schottky extension groups. The reciprocal is consequence of the decomposition done in the previous section.

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