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MEROMORPHIC FUNCTIONS AND ALSO THEIR FIRST TWO DERIVATIVES HAVE THE SAME ZEROS

Lian-Zhong Yang

Shandong University, School of Mathematics & System Sciences
Jinan, Shandong, 250100, P.R. China; lzyang@sdu.edu.cn

Abstract. In this paper, we present a uniqueness theorem of meromorphic functions which together with their first two derivatives have the same zeros, this generalizes a result of C. C. Yang. As applications, we improve a result of L. Köhler, and answer a question of Hinkkanen in more weak conditions for meromorphic functions of hyper-order less than one, and we supply examples to show that the order restriction is sharp.

1. Introduction

In this paper a meromorphic function will mean meromorphic in the finite complex plane. We say that two meromorphic functions f and g share a finite value a IM (ignoring multiplicities) when $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share the value a CM (counting multiplicities). We say that f and g share ∞ CM provided that $1/f$ and $1/g$ share 0 CM. It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna Theory, as found in [9] and [18].

The Nevanlinna Four Values Theorem says that if two meromorphic functions f and g share four distinct values CM, then $f\equiv g$ or f is a Möbius transformation of g (see [14]). The condition that f and g share four distinct values CM has been weakened to the condition that f and g share two values CM and share the other two values IM by Gundersen(see [6] and [7]), as well as by Mues [12] and Wang [16]. When a meromorphic function f and one of its derivatives $f^{(k)}$ share values, Frank and Weissenborn proved that if f and $f^{(k)}$ $(k\ge 1)$ share two distinct finite values CM, then $f\equiv f^{(k)}$ (see [3]). In a special case, it is known that if an entire function f and its first derivative f' share two finite values CM, then $f\equiv f'$ (see [15]). This result has been generalized to the case that f and f' share two values IM by Gundersen and by Mues–Steinmetz independently (see [4] and [13]). If an entire function f of finite order and its two derivatives $f^{(n)}$,

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$f^{(n+1)}$ $(n \geq 1)$ share a finite value $a \neq 0$ CM, then $f \equiv f'$ (see [8]), the case $n = 1$ is due to Jank–Mues–Volkmann (see [10]). For entire functions of infinite order, it is impossible for f, $f^{(n)}$ and $f^{(n+1)}$ to share a finite value $a \neq 0$ CM (see [19]).

We are concerned with the uniqueness questions that arise when two meromorphic functions and also their first two derivatives have the same zeros.

Let f be a meromorphic functions. It is known that the order $\sigma(f)$ and the hyper-order $\sigma_2(f)$ of f are defined by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \quad \sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

In 1976, Yang proved the following theorem in [17].

Theorem A. *Let f and g be nonconstant entire functions. If f and g satisfy the following conditions*:

(a) f *and* g *share* 0 CM, *and all the zeros of* f *are simple zeros.*
(b) f' *and* g' *share* 0 CM.
(c) $\max\{\sigma_2(f), \sigma_2(g)\} < 1$.

Then f and g satisfy one of the following three cases:

(i) $f = cg$, *for a constant* $c \neq 0$.
(ii) $f = e^{h(z)}$, $g = e^{ah(z)+b}$, *where* $a \neq 0$, b *are constants, and* $h(z)$ *is an entire function of order less than one.*
(iii) $f = a(e^{\mu(z)} - 1)$, $g = b(1 - be^{-\mu(z)})$, *where* $a \neq 0$, $b \neq 0$ *are constants, and* $\mu(z)$ *is an entire function of order less than one.*

In 1981, Gundersen proved the next result in [5].

Theorem B. *Let f and g be nonconstant entire functions. If $f^{(j)}$ and $g^{(j)}$ $(j = 0, 1)$ share* 0 CM, *then f and g satisfy one of the following relations*:

(i) $f = cg$, *for a constant* $c \neq 0$.
(ii) $f = e^{h(z)}$, $g = e^{ah(z)+b}$, *where* $a \neq 0$, b *are constants, and* $h(z)$ *is an entire function.*
(iii) $f = \exp\{\int u(z)\,dz/(1 - e^{-v(z)})\}$, $g = \exp\{\int u(z)\,dz/(e^{v(z)} - 1)\}$, *where* $u(z)$ *and* $v(z)$ *are entire functions.*

For meromorphic functions f and g, we know that $f^{(j)}$ and $g^{(j)}$ share the value 0 and ∞ CM for each non-negative integer j whenever f and g satisfy one of the following four cases:

(i) $f = cg$, for a constant $c \neq 0$,
(ii) $f = e^{az+b}$, $g = e^{cz+d}$, a, b, c, d $(ac \neq 0)$ are constants,
(iii) $f = a(1 - be^{cz})$, $g = d(e^{-cz} - b)$, a, b, c, d are nonzero constants,
(iv) $f = a/(1 - be^{\alpha})$, $g = a/(e^{-\alpha} - b)$, a, b are nonzero constants and α is a nonconstant entire function.

In 1984, A. Hinkkanen asked the following question in [1].

Hinkkanen's problem. *Does there exist a positive integer n such that two meromorphic functions f and g satisfy one of the above four cases* (i)–(iv) *when $f^{(j)}$ and $g^{(j)}$ share the values* 0 *and* ∞ CM *for $j=0,1,\dots,n$?*

It is known that the answer to the above problem is positive and $n=6$ solves the problem by a result of Köhler. For meromorphic functions of finite order, Köhler proved the following theorem in [11].

Theorem C. *Let f and g be nonconstant meromorphic functions of finite order. If $f^{(j)}$ and $g^{(j)}$ share the value* 0 *and* ∞ CM *for $j=0,1,2$, then f and g satisfy one of the four cases* (i)–(iv) *mentioned above.*

Recently, Yi surmised, in a seminar at Shandong University, that the conditions of Theorem C may be weakened based on Theorem A and Theorem B.

Let f and g be meromorphic functions, a be a finite value. If $f(z)-a=0$ when $g(z)-a=0$ and the order of each zero z_0 of $f(z)-a$ is greater than or equal to the order of the zero z_0 of $g(z)-a$, in other words, $(f(z)-a)/(g(z)-a)$ does not have a pole at a zero of $g(z)-a$, we will denote this by

$$g(z)-a=0 \overset{\text{CM}}{\longrightarrow} f(z)-a=0.$$

It is obvious that f and g share the value a CM if and only if

$$g(z)-a=0 \overset{\text{CM}}{\longrightarrow} f(z)-a=0 \quad \text{and} \quad f(z)-a=0 \overset{\text{CM}}{\longrightarrow} g(z)-a=0.$$

In this paper, we first present a functional equation (Lemma 1 of Section 2) which is obtained from meromorphic functions satisfying certain properties, and then by using this functional equation, we give a result that proves Theorem C still holds when the condition that f'' and g'' share 0 CM is replaced by

$$g''(z)=0 \overset{\text{CM}}{\longrightarrow} f''(z)=0. \tag{1.1}$$

This confirms Yi's surmise and shows that the answer to Hinkkanen's problem is positive for meromorphic functions of finite order if $n=1$ and the additional condition (1.1) holds. We also generalize this result to the meromorphic functions of hyper-order less than one. Examples show that the order restriction is sharp.

2. Some lemmas

Lemma 1. *Let f and g be nonconstant meromorphic functions. Suppose that there exist two entire functions α, β and a nonzero meromorphic function ϕ such that*

$$f=e^{\alpha}g, \quad f'=e^{\beta}g' \quad \text{and} \quad f''=\phi g''. \tag{2.1}$$

Then α, β and ϕ satisfy the following functional equation

$$\{{\alpha'}^2-\alpha'\beta'+\alpha''\}e^{\beta-\alpha}+\{\alpha''-{\alpha'}^2+\alpha'\beta'\}\frac{e^{\beta}}{\phi}-\{\alpha''+{\alpha'}^2\}\frac{e^{2\beta-\alpha}}{\phi}\equiv\alpha''-{\alpha'}^2.$$

Proof. From (2.1), we have

$$f'=e^{\alpha}\alpha' g+e^{\alpha}g',$$
$$f''=e^{\alpha}(\alpha''+{\alpha'}^2)g+2e^{\alpha}\alpha' g'+e^{\alpha}g'',$$

and also

$$f''=e^{\beta}\beta' g'+e^{\beta}g''.$$

Together with (2.1), we have the following linear system

$$\begin{cases} e^{\alpha}\alpha' g+(e^{\alpha}-e^{\beta})g'=0,\\ e^{\beta}\beta' g'+(e^{\beta}-\phi)g''=0,\\ e^{\alpha}(\alpha''+{\alpha'}^2)g+2\alpha' e^{\alpha}g'+(e^{\alpha}-\phi)g''=0.\end{cases}$$

Since $\{g,g'g''\}\not\equiv\{0,0,0\}$ we know from the above linear system that

$$\begin{vmatrix} e^{\alpha}\alpha' & e^{\alpha}-e^{\beta} & 0\\ e^{\alpha}({\alpha'}^2+\alpha'') & 2\alpha' e^{\alpha} & e^{\alpha}-\phi\\ 0 & e^{\beta}\beta' & e^{\beta}-\phi\end{vmatrix}\equiv 0,$$

and we have

$$\{{\alpha'}^2-\alpha'\beta'+\alpha''\}e^{\beta-\alpha}+\{\alpha''-{\alpha'}^2+\alpha'\beta'\}\frac{e^{\beta}}{\phi}-\{\alpha''+{\alpha'}^2\}\frac{e^{2\beta-\alpha}}{\phi}\equiv\alpha''-{\alpha'}^2.$$

Lemma 1 is thus proved.

Lemma 2. *Let α and β be nonconstant entire functions of order less than one. If $\beta-\alpha$ is not a constant, then*

$${\alpha'}^2-\alpha'\beta'+\alpha''\not\equiv 0,\quad \alpha''-{\alpha'}^2+\alpha'\beta'\not\equiv 0.$$

Proof. By the assumptions of Lemma 2, we have $\alpha'(\alpha'-\beta')\not\equiv 0$. If $\alpha''\equiv 0$, then Lemma 2 follows. If $\alpha''\not\equiv 0$, then α' is not a constant. Since $\sigma(\alpha')<1$, α' must have zeros. Now we suppose that

$${\alpha'}^2-\alpha'\beta'+\alpha''\equiv 0,$$

then

$$\alpha'-\beta'+\frac{\alpha''}{\alpha'}\equiv 0.$$

This is impossible, because α' has zeros, and so

$${\alpha'}^2-\alpha'\beta'+\alpha''\not\equiv 0.$$

By the same reasoning, we have

$$\alpha''-{\alpha'}^2+\alpha'\beta'\not\equiv 0.$$

Lemma 2 is proved.

Lemma 3. *Let $Q(z)$ be a nonzero entire function of order less than one, p and q $(pq \neq 0)$ be constants. If each zero of $Q^2 + qQ'$ is a zero of $Q^2 + pQ'$ (ignoring multiplicities), then $p = q$ or $Q' \equiv 0$.*

Proof. Suppose that $p \neq q$ and $Q' \not\equiv 0$. Let z_0 be a zero of order m of $Q^2 + qQ'$, then z_0 is a zero of order n of $Q^2 + pQ'$, where m and n are positive integers. It is obvious that $Q(z_0) = Q'(z_0) = 0$ and $n = m$.

Now let $H(z) = Q^2/Q'$. Then from the conditions of Lemma 3, we know that a zero of $H(z) + q$ is a zero of $H(z) + p$. Hence $H(z) + q$ has no zeros.

If Q is a polynomial, then $H(z) + q$ must have zeros. This is a contradiction.

Now we assume that Q is a transcendental entire function, and we denote by $h_1(z)$ the canonical product of the common zeros $\{z_k\}$ of Q' and Q, where the multiplicities of the common zeros are counted with respect to Q' (we take $h_1(z) \equiv 1$ if Q' and Q have no common zeros). Let

$$h_2(z) = \frac{Q'}{h_1}.$$

Then h_2 and h_1 are entire functions of order less than one.

From the definition of H, we know that the poles of H are the zeros of Q' which are not the zeros of Q, hence $(H + q)h_2$ is an entire function of order less than one, and $(H + q)h_2$ has no zeros. Since $(H + q)h_2$ is of order less than one, we obtain that $(H + q)h_2$ is a nonzero constant. Set $(H + q)h_2 = A$, where $A \neq 0$ is a constant. We have

$$Q^2 + qQ' = Ah_1. \tag{2.2}$$

From (2.2), we have

$$T(r, h_1) = 2T(r, Q) + S(r, Q), \tag{2.3}$$

$$A = \frac{Q^2}{h_1} + qh_2. \tag{2.4}$$

By (2.3) and (2.4)

$$T(r, h_2) \le T(r, Q^2) + T(r, h_1) + O(1) \le 4T(r, Q) + S(r, Q). \tag{2.5}$$

From (2.4), (2.5) and the Second Fundamental Theorem, we have

$$\begin{aligned} T(r, h_2) \le & \overline{N}\left(r, \frac{1}{h_2}\right) + \overline{N}\left(r, \frac{h_1}{Q^2}\right) + S(r, Q) \\ \le & \overline{N}\left(r, \frac{1}{h_2}\right) + \overline{N}\left(r, \frac{1}{Q}\right) + S(r, Q). \end{aligned} \tag{2.6}$$

Since

$$
\begin{aligned}
T(r,h_2) &= T\left(r,\frac{h_1}{Q^2}\right)+O(1)\geq N\left(r,\frac{h_1}{Q^2}\right)+O(1)\\
&\geq N\left(r,\frac{1}{Q}\right)+\overline{N}\left(r,\frac{1}{Q}\right)+O(1).
\end{aligned}
\tag{2.7}
$$

From (2,6) and (2.7), we get

$$\overline{N}\left(r,\frac{1}{Q}\right)+N\left(r,\frac{1}{Q}\right)\leq\overline{N}\left(r,\frac{1}{h_2}\right)+\overline{N}\left(r,\frac{1}{Q}\right)+S(r,Q).$$

Hence

$$N\left(r,\frac{1}{Q}\right)\leq\overline{N}\left(r,\frac{1}{h_2}\right)+S(r,Q). \tag{2.8}$$

Since $Q'=h_1h_2$ and

$$N\left(r,\frac{1}{Q'}\right)\leq N\left(r,\frac{1}{Q}\right)+S(r,Q),$$

we get

$$N\left(r,\frac{1}{h_1}\right)+N\left(r,\frac{1}{h_2}\right)\leq N\left(r,\frac{1}{Q}\right)+S(r,Q). \tag{2.9}$$

Together with (2.8) and (2.9), we obtain

$$N\left(r,\frac{1}{h_1}\right)=S(r,Q)=O(\log r).$$

Thus h_1 must be a polynomial. This contradicts (2.3). We have completed the proof of Lemma 3.

Lemma 4. *Let α be an entire function. If $\alpha'\not\equiv 0$, then*

$$\alpha'^2+\alpha''\not\equiv 0,\quad \alpha'^2-\alpha''\not\equiv 0.$$

Proof. Suppose that $\alpha'^2+\delta\alpha''\equiv 0$, where $\delta=1$ or $\delta=-1$. Then $1-\delta(1/\alpha')'\equiv 0$, and we have $1/\alpha'=z/\delta+c$, where c is a constant. This contradicts the fact that α' is an entire function. Thus we have $\alpha'^2+\delta\alpha''\not\equiv 0$. Lemma 4 is thus proved.

3. A uniqueness theorem

Theorem 3.1. *Let f and g be nonconstant meromorphic functions of hyper-order less than one. If $f^{(j)}$ and $g^{(j)}$ $(j=0,1)$ share 0 and ∞* CM, *and*

$$N\left(r, \frac{f''}{g''}\right) = O(r^{\lambda}), \tag{3.1}$$

where λ is a constant satisfying $0<\lambda<1$. Then f and g satisfy one of the following four cases (i)–(iv):

(i) $f = cg$, *for a constant $c \neq 0$.*

(ii) $f = e^{h(z)}$, $g = e^{ah(z)+b}$, *where a $(a \neq 0,1)$ and b are constants, and $h(z)$ is a nonconstant entire function of order less than one.*

(iii) $f = a(e^{\mu(z)} - 1)$, $g = b(1 - e^{-\mu(z)})$, *where $a \neq 0$, $b \neq 0$ are constants, and $\mu(z)$ is a nonconstant entire function of order less than one.*

(iv) $f = a/(1 - be^{\alpha})$, $g = a/(e^{-\alpha} - b)$, *where a, b are nonzero constants and α is an entire function of order less than one.*

Proof. Since $f^{(j)}$ and $g^{(j)}$ $(j=0,1)$ share 0 and ∞ CM, and f and g are meromorphic functions of hyper order less than one, we have

$$f = e^{\alpha} g, \quad f' = e^{\beta} g' \quad \text{and} \quad f'' = \phi g'', \tag{3.2}$$

where α and β are entire functions of order less than one, and ϕ is a meromorphic function of hyper-order less than one.

If f and g are rational functions, then it is easy to see that f and g satisfy (i). If $\alpha' \equiv 0$, then f and g satisfy (i). If $\alpha' \not\equiv 0$, and $\beta' \equiv 0$, then β is a constant. Let $e^{\beta} = a$. From (3.2), we have

$$f' = ag', \quad f = ag + c,$$

where $c \neq 0$ is a constant. Since $f = e^{\alpha} g$, we obtain

$$g = \frac{c}{e^{\alpha} - a}, \quad f = \frac{c}{1 - ae^{-\alpha}}.$$

This proves f and g satisfy (iv).

Now we suppose that f and g are transcendental meromorphic functions and $\alpha'\beta' \not\equiv 0$. From (3.2) and Lemma 1, we have

$$\phi\{(\alpha'^2 - \alpha'\beta' + \alpha'')e^{\beta-\alpha} - (\alpha'' - \alpha'^2)\} \equiv e^{\beta}\{(\alpha'' + \alpha'^2)e^{\beta-\alpha} - (\alpha'' - \alpha'^2 + \alpha'\beta')\}. \tag{3.3}$$

We distinguish the following two cases.

Case 1. $\beta-\alpha$ is a constant. Let $e^{\beta-\alpha}=c$. Then $c\neq 0$ is a constant. From (3.2), we know that

$$\frac{f'}{f}=c\frac{g'}{g},\quad \alpha'=\frac{f'}{f}-\frac{g'}{g} \tag{3.4}$$

and so we have

$$\alpha'=(c-1)\frac{g'}{g}. \tag{3.5}$$

Since $\alpha'\not\equiv 0$, we have $c\neq 1$. From (3.5), we know that g has no zeros and poles. Since f and g share 0 and ∞ CM, f has no zeros and poles. Let $f=e^P$, $g=e^Q$, where P and Q are nonconstant entire functions of order less than one. From the first equation of (3.4) and

$$f'=e^PP'=fP',\quad g'=e^QQ'=gQ',$$

we obtain $P'=cQ'$, and so

$$f=e^{P(z)},\quad g=e^{aP(z)+b},$$

where $a=1/c$ and b are constants. Thus f and g satisfy (ii) in Case 1.

Case 2. $\beta-\alpha$ is not a constant. Since $\alpha'\not\equiv 0$, from Lemma 2 and Lemma 4, we have

$$\alpha'^2-\alpha''\not\equiv 0,\quad \alpha''-\alpha'^2+\alpha'\beta'\not\equiv 0,$$
$$\alpha'^2+\alpha''\not\equiv 0,\quad \alpha'^2-\alpha'\beta'+\alpha''\not\equiv 0.$$

Let

$$f_1=(\alpha'^2-\alpha'\beta'+\alpha'')e^{\beta-\alpha}-(\alpha''-\alpha'^2),$$
$$f_2=(\alpha''+\alpha'^2)e^{\beta-\alpha}-(\alpha''-\alpha'^2+\alpha'\beta').$$

From (3.3), we get

$$\phi f_1\equiv e^\beta f_2. \tag{3.6}$$

By the Second Fundamental Theorem concerning small functions, we have

$$\begin{aligned} T(r,e^{\beta-\alpha})&=\overline{N}\left(r,\frac{1}{f_2}\right)+S(r,e^{\beta-\alpha}),\\ T(r,e^{\beta-\alpha})&=\overline{N}\left(r,\frac{1}{f_1}\right)+S(r,e^{\beta-\alpha}). \end{aligned} \tag{3.7}$$

Let $\overline{N}_0(r, f_1, f_2)$ be the counting function of the common zeros (ignoring multiplicities) of f_1 and f_2. Then from (3.6), (3.1) and (3.2), we have

$$\overline{N}_0(r, f_1, f_2) = \overline{N}\left(r, \frac{1}{f_1}\right) + O(r^{\lambda}),$$

where λ is a constant satisfying $0 < \lambda < 1$. Putting this together with (3.7), we obtain

$$\overline{N}_0(r, f_1, f_2) = \{1 + o(1)\}T(r, e^{\beta-\alpha}), \quad r \notin E, \tag{3.8}$$

where E has finite linear measure.

Since α and β are entire functions of order less than one, we know that α, β and all their derivatives are small functions with respect to $e^{\beta-\alpha}$.

Let $\varrho = \max\{\sigma(\alpha), \sigma(\beta)\}$. Then $0 \le \varrho < 1$. Take a positive number ε such that $\varrho + \varepsilon < 1$. Let z_0 be a common zero of f_1 and f_2 such that

$$\alpha'(z_0)^2 - \alpha''(z_0) \neq 0, \quad \alpha''(z_0) - \alpha'(z_0)^2 + \alpha'(z_0)\beta'(z_0) \neq 0, \tag{3.9}$$

$$\alpha'(z_0)^2 + \alpha''(z_0) \neq 0, \quad \alpha'(z_0)^2 - \alpha'(z_0)\beta'(z_0) + \alpha''(z_0) \neq 0, \tag{3.10}$$

From (3.9), (3.10), $f_1(z_0) = 0$ and $f_2(z_0) = 0$, we obtain

$$\alpha'(z_0)\beta'(z_0)\big(2\alpha'(z_0) - \beta'(z_0)\big) = 0.$$

If

$$\alpha'(z)\beta'(z)\big(2\alpha'(z) - \beta'(z)\big) \not\equiv 0,$$

we have

$$\begin{aligned}
\overline{N}_0(r, f_1, f_2) \le {} & N\left(r, \frac{1}{\alpha'(z)\beta'(z)\big(2\alpha'(z) - \beta'(z)\big)}\right) + N\left(r, \frac{1}{\alpha'^2 - \alpha''}\right) \\
& + N\left(r, \frac{1}{\alpha'' - \alpha'^2 + \alpha'\beta'}\right) + N\left(r, \frac{1}{\alpha'^2 + \alpha''}\right) \\
& + N\left(r, \frac{1}{\alpha'^2 - \alpha'\beta' + \alpha''}\right) \\
= {} & O(r^{\varrho+\varepsilon}), \quad r \to +\infty, \ r \notin E.
\end{aligned}$$

This contradicts (3.8), and thus $\alpha'(z)\beta'(z)\big(2\alpha'(z) - \beta'(z)\big) \equiv 0$. Since $\alpha'(z)\beta'(z) \not\equiv 0$, we have

$$2\alpha'(z) - \beta'(z) \equiv 0. \tag{3.11}$$

From (3.11) and (3.2), we have

$$\beta = 2\alpha + c, \quad \frac{f'}{f^2} = e^c \frac{g'}{g^2},$$

hence

$$-\frac{1}{f} = -e^c \frac{1}{g} + d, \tag{3.12}$$

where c and d are constants. It is obvious that $d = 0$ contradicts $\alpha' \not\equiv 0$, and we have $d \neq 0$. From (3.12), we obtain

$$(df + 1)\left(g - \frac{e^c}{d}\right) = -\frac{e^c}{d}.$$

Thus both f and g have no poles. Let $df + 1 = e^{\mu(z)}$, where $\mu(z)$ is a nonconstant entire function of order less than one. Then we have

$$g - \frac{e^c}{d} = -\frac{e^c}{d} e^{-\mu(z)},$$

and we obtain

$$f = a(e^{\mu(z)} - 1), \quad g = b(e^{-\mu(z)} - 1),$$

where $a = 1/d$, $b = -e^c/d$ are nonzero constants. This proves that f and g satisfy (iii) in Case 2. Theorem 3.1 is thus proved.

Example 1. Let $f = (e^z - 1)^2$, $g = (e^{-z} - 1)^2$. Then $f^{(j)}$ and $g^{(j)}$ share 0 and ∞ CM for $j = 0, 1$, but

$$N\left(r, \frac{f''}{g''}\right) = O(r),$$

and f and g do not satisfy any of the four cases (i)–(iv) in Theorem 3.1.

Remark. Example 1 shows that the condition (3.1) is sharp and cannot be deleted.

4. Applications of Theorem 3.1

Theorem 4.1. *Let f and g be nonconstant meromorphic functions of hyper-order less than one. If $f^{(j)}$ and $g^{(j)}$ $(j=0,1)$ share 0 and ∞ CM, and*

$$g''(z)=0 \xrightarrow{\text{CM}} f''(z)=0. \tag{4.1}$$

then f and g satisfy one of the following four cases (A)–(D):

(A) $f=cg$, *for a constant* $c\neq 0$,
(B) $f=e^{az+b}$, $g=e^{cz+d}$, a, b, c, d $(ac\neq 0)$ *are constants,*
(C) $f=a(1-be^{cz})$, $g=d(e^{-cz}-b)$, a, b, c, d *are nonzero constants,*
(D) $f=a/(1-be^{\alpha})$, $g=a/(e^{-\alpha}-b)$, a, b *are nonzero constants and* α *is an entire function of order less than one.*

Proof. Since f and g share ∞ CM and (4.1) holds, we know that f''/g'' is an entire function, and

$$N\left(r,\frac{f''}{g''}\right)=0. \tag{4.2}$$

Hence f and g satisfy all the conditions of Theorem 3.1, from Theorem 3.1, f and g satisfy one of the four cases (i)–(iv) in Theorem 3.1.

Theorem 4.1 follows if we prove, under the conditions of Theorem 4.1, that $h(z)$ and $\mu(z)$ are linear functions in Case (ii) and in Case (iii) of Theorem 3.1.

If f and g satisfy (ii) of Theorem 3.1, then $f=e^{h(z)}$, $g=e^{ah(z)+b}$ $(a\neq 0,1)$ and

$$f'=e^{h}h',\quad f''=e^{h}({h'}^2+h''), \tag{4.3}$$

$$g'=ae^{ah+b}h',\quad g''=a^2e^{ah+b}\left({h'}^2+\frac{1}{a}h''\right). \tag{4.4}$$

Since (4.1) holds, it follows from (4.3) and (4.4) that each zero of ${h'}^2+h''/a$ must be a zero of ${h'}^2+h''$. Note that $h(z)$ is an entire function of order less than one, we obtain from Lemma 3 that $h''\equiv 0$. Hence h must be a linear function.

If f and g satisfy (iii) of Theorem 3.1, by the same arguments as above, we know that f and g satisfy (C). The proof of Theorem 4.1 is complete.

From Theorem 4.1 and the properties of functions mentioned in (B)–(D) of Theorem 4.1, we have the following corollaries.

Corollary 1. *Let f and g be nonconstant meromorphic functions of hyper-order less than one. If $f^{(j)}$ and $g^{(j)}$ $(j=0,1)$ share 0 and ∞ CM, and*

$$g''(z)=0 \xrightarrow{\text{CM}} f''(z)=0.$$

Then $f^{(k)}$ and $g^{(k)}$ share 0 and ∞ CM for any non-negative integer k.

Corollary 2. *Let f and g be nonconstant meromorphic functions of hyper-order less than one. If $f^{(j)}$ and $g^{(j)}$ $(j=0,1,2)$ share 0 and ∞* CM, *then f and g satisfy one of the four cases* (A)–(D) *in Theorem* 4.1.

Example 2. Let $f=\exp(e^z)$, $g=\exp(e^{-z})$. Then $f^{(j)}$ and $g^{(j)}$ share 0 and ∞ CM for $j=0,1,2$, but f and g do not satisfy one of the four cases (A)–(D) mentioned in Theorem 4.1.

Example 3. Let $f=(e^{2z}-1)\exp(-ie^z)$, $g=(1-e^{-2z})\exp(ie^{-z})$. Then $f^{(j)}$ and $g^{(j)}$ share 0 and ∞ CM for $j=0,1,2$, but f and g do not satisfy one of the four cases (A)–(D) in Theorem 4.1.

Example 4. Let $f=e^{z^2+z}$, $g=e^{4z^2+4z+1}$. Then $f^{(j)}$ and $g^{(j)}$ share 0 and ∞ CM for $j=0,1$, but g'' has two zeros which are not the zeros of f'', and f and g do not satisfy one of the four cases (A)–(D) in Theorem 4.1.

The above results are closely related to Hinkkanen's problem. Theorem 4.1 confirms Yi's surmise and improves Theorem C. It also shows that $n=2$ solves Hinkkanen's problem for meromorphic functions of hyper-order less than one, and the restriction on the second derivative can be weakened. Examples 2 and 3 show that the order restriction of Theorem 4.1 is sharp. Example 4 presents two entire functions which show that the condition

$$g''(z)=0 \xrightarrow{\text{CM}} f''(z)=0$$

in Theorem 4.1 can not be deleted.

By Theorem 3.1, Lemma 3 and the arguments similar to the proof of Theorem 4.1, we have the following result.

Theorem 4.2. *Let f and g be nonconstant meromorphic functions of hyper-order less than one. If $f^{(j)}$ and $g^{(j)}$ $(j=0,1)$ share 0 and ∞* CM, *$f''(z)=0$ when $g''(z)=0$ (ignoring multiplicities), and*

$$N\left(r,\frac{f''}{g''}\right)=O(r^{\lambda}),$$

where λ is a constant satisfying $0<\lambda<1$. Then f and g satisfy one of the following four cases (A)–(D):

(A) *$f=cg$, for a constant $c\neq 0$,*
(B) *$f=e^{az+b}$, $g=e^{cz+d}$, a, b, c, d $(ac\neq 0)$ are constants,*
(C) *$f=a(1-be^{cz})$, $g=d(e^{-cz}-b)$, a, b, c, d are nonzero constants,*
(D) $f=a/(1-be^{\alpha})$, $g=a/(e^{-\alpha}-b)$, where a, b are nonzero constants and α is an entire function of order less than one.

Remark. Example 4 shows that the condition "$f''(z)=0$ when $g''(z)=0$" cannot be deleted. G. Frank, X. Hua and R. Vaillancourt have recently shown that the sharp answer to Hinkkanen's problem is $n=4$, and gave an example of two meromorphic functions f and g of hyper-order equal to one such that $f^{(j)}$ and $g^{(j)}$ share 0 and ∞ CM for $j=1,2,3$, but where $f^{(4)}$ and $g^{(4)}$ do not share 0 CM. This together with Corollary 2 show that $n=2$ solves Hinkkanen's problem for meromorphic functions of hyper-order less than one, but that $n=4$ is needed to solve Hinkkanen's problem for meromorphic functions of hyper-order equal to one [2].

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