Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 30, 2005, 303–311

Complex dynamics, value distributions, and potential theory

Yûsuke Okuyama

Kanazawa University, Department of Mathematics
Faculty of Science, Kanazawa 920-1192, Japan; okuyama@kenroku.kanazawa-u.ac.jp

Abstract. Value distributions and complex dynamics are intertwined in that they involve the study of preimages and forward-images under iteration, respectively. In this article, we first show the equivalence of the dynamical exceptional set of a rational map $f\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ and both the dynamical Nevanlinna and Valiron exceptional sets. As a consequence, we establish several convergence theorems of the potentials of the averaged value distributions, which are stronger than what can be obtained from the general axiomatic potential theory.

1. Introduction

A rational map f is a holomorphic endomorphism of the Riemann sphere $\widehat{\mathbf{C}}$.

Notation 1.1. Rat denotes the set of all rational endomorphisms of $\widehat{\mathbf{C}}$. $\widehat{\mathbf{C}}$ is identified with the set of all constant maps of $\widehat{\mathbf{C}}$.

In the case that f is non-invertible, the Fatou and Julia strategy for studying the complex dynamics $(\widehat{\mathbf{C}}, f)$, which treats *forward-images* under iterations, is the separation of $\widehat{\mathbf{C}}$ into two completely invariant complementary subsets, one of which is the *Fatou set* $F(f)$, the region of normality of $\{f^k := f^{\circ k}\}$, and the other the *Julia set* $J(f)$. In other words, the restricted dynamical systems $\bigl(F(f), f\bigr)$ and $\bigl(J(f), f\bigr)$ are tame and chaotic, respectively. Consequently, the dynamical system *around* $J(f)$ has an *almost covering* feature, that is,

$$E(f) := \widehat{\mathbf{C}} - \bigcap_{U\colon \text{an open neighborhood of a point of } J(f)} \left(\bigcup_{k\in\mathbf{N}} f^k(U) \right)$$

consists of at most two points.

Definition 1.1 (dynamical exceptional set). $E(f)$ is called the *dynamical exceptional set* of f.

From this almost covering feature, a Nevanlinna theoretical study, which treats *preimages* under iterations, arises naturally.

For the Nevanlinna theory, see, for example, [3] and [5].

2000 Mathematics Subject Classification: Primary 37F10; Secondary 30D35, 31D05, 32H50.

Partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Young Scientists (B), 15740085, 2003.

Notation 1.2. The Dirac measure at $w \in \widehat{\mathbf{C}}$ is written as δ_w.

Definition 1.2 (averaged value distribution). For distinct $f, g \in \mathrm{Rat}$, put $\mu(f,g) := \sum_{f(w)=g(w)} \delta_w$, where the summation takes into account the multiplicity of each root of the equation $f = g$. The *averaged value distribution* of f for g is the probability measure $\mu(f,g)/(\deg f + \deg g)$.

The spherical area measure and the chordal distance on $\widehat{\mathbf{C}}$ are given by

$$\sigma(w) = \frac{\mathrm{d}x\,\mathrm{d}y}{\pi(1+|w|^2)^2} \quad (w = x+iy) \qquad \text{and} \quad [z,w] = \frac{|z-w|}{\sqrt{1+|z|^2}\,\sqrt{1+|w|^2}},$$

respectively. We note that they are normalized as $\sigma(\widehat{\mathbf{C}}) = 1$ and $[0,\infty] = 1$.

Definition 1.3 (dynamical Nevanlinna theory [12]). For $f, g \in \mathrm{Rat}$, the *pointwise proximity function* is defined by

$$\big(w(g,f)\big)(z) := \log \frac{1}{[g(z), f(z)]} \colon \widehat{\mathbf{C}} \to [0,\infty],$$

which is δ-subharmonic, and the *mean proximity* by

$$m(g,f) := \int_{\widehat{\mathbf{C}}} w(g,f)\,\mathrm{d}\sigma \in [0,\infty).$$

Let $\mathscr{F}$ be a *rational sequence* $\{f_k\}_{k=0}^{\infty} \subset \mathrm{Rat}$ with increasing degrees $\{d_k := \deg f_k\}$. For $g \in \mathrm{Rat}$, the dynamical *Nevanlinna* and *Valiron exceptionalities* are defined by

$$\mathrm{NE}(g;\mathscr{F}) := \liminf_{k\to\infty} \frac{m(g,f_k)}{d_k} \in [0,\infty],$$
$$\mathrm{VE}(g;\mathscr{F}) := \limsup_{k\to\infty} \frac{m(g,f_k)}{d_k} \in [0,\infty],$$

respectively.

From now on, we consider the *iteration* sequence $\{f^k\}_{k=1}^{\infty}$ of $f \in \mathrm{Rat}$ of degree $d \geq 2$.

Definition 1.4 (dynamical Nevanlinna and Valiron exceptional sets). The dynamical *Nevanlinna* and *Valiron* exceptional sets of f in $\widehat{\mathbf{C}}$ are defined by

$$E_N(f) := \big\{p \in \widehat{\mathbf{C}} \colon \mathrm{NE}(p;\{f^k\}) > 0\big\},$$
$$E_V(f) := \big\{p \in \widehat{\mathbf{C}} \colon \mathrm{VE}(p;\{f^k\}) > 0\big\},$$

respectively.

We shall use several notions from the geometric measure theory and the potential theory. For the details, see, for example, [1], [10], and [4].

Convention. We consider only weak convergences for measures.

Theorem 1 (All exceptional sets are same). *For* $f \in \mathrm{Rat}$ *of degree* ≥ 2,

$$E_N(f) = E_V(f) = E(f).$$

Remark 1.1. In [11], Theorem 1 has been already implicitly applied to the Siegel–Cremer linearizability problem of rational maps.

The proof relies on a fundamental theorem in complex dynamics proved by Lyubich [6] and independently by Freire–Lopes–Mañé [2]:

Theorem 1.1 ([6] and [2]). *Let* $f \in \mathrm{Rat}$ *be of degree* $d \geq 2$. *For every* $p \in \widehat{\mathbf{C}} - E(f)$,

$$\lim_{k\to\infty} \frac{\mu(f^k, p)}{d^k} = \lim_{k\to\infty} \frac{(f^k)^*\sigma}{d^k} =: \mu_f. \tag{1}$$

An important conclusion from Theorem 1 is a convergence theorem of the *potentials* of the averaged value distributions.

Definition 1.5 (spherical potential). For a regular measure μ on $\widehat{\mathbf{C}}$, the *potential* is defined by

$$V_\mu := \int_{\widehat{\mathbf{C}}} \log \frac{1}{[\,\cdot\,, w]}\, \mathrm{d}\mu(w) : \widehat{\mathbf{C}} \to [-\infty, \infty],$$

which is a δ-subharmonic function on $\widehat{\mathbf{C}}$.

Remark 1.2. In the potential theory, the potential is sometimes defined as $-V_\mu$, but the definition will be more convenient in our study.

The (axiomatic) potential theory implies that when positive regular measures $\{\mu_k\}$ converges to μ, then

$$\liminf_{k\to\infty} V_{\mu_k} = V_\mu \tag{2}$$

quasieverywhere on $\widehat{\mathbf{C}}$.

Theorem 2 (convergence theorem of potentials). *Let* $f \in \mathrm{Rat}$ *be of degree* $d \geq 2$ *and* $p \in \widehat{\mathbf{C}} - E(f)$. *If* p *is not a fixed point of* f, *then*

$$\liminf_{k\to\infty} V_{\mu(f^k,p)/d^k} = V_{\mu_f} \tag{3}$$

on $\widehat{\mathbf{C}}$. *If* p *is a fixed point of* f, (3) *holds on* $\widehat{\mathbf{C}} - \bigcup_{k\geq 0} f^{-k}(p)$.

We also characterize such points that the potentials actually *converge* there.

Theorem 3 (convergence of potentials and pointwise behavior). *Let* $f \in \mathrm{Rat}$ *be of degree* $d \geq 2$. *For* $p \in \widehat{\mathbf{C}} - E(f)$ *and* $q \in \widehat{\mathbf{C}}$,

$$\lim_{k \to \infty} V_{\mu(f^k, p)/d^k}(q) = V_{\mu_f}(q) \tag{4}$$

if and only if

$$\lim_{k \to \infty} \frac{1}{d^k} \log \frac{1}{[p, f^k(q)]} = 0. \tag{5}$$

Hence it follows from the classification of the Fatou components that for $p \in \widehat{\mathbf{C}} - E(f)$ and $q \in \widehat{\mathbf{C}}$, if (4) (or (5)) does not hold, then either

(i) p is periodic and $q \in \bigcup_{k \geq 0} f^{-k}(p)$,
(ii) $p, q \in J(f)$, or
(iii) p is not periodic but contained in a rotation domain and

$$q \in \mathrm{Fol}_f(p), \tag{6}$$

which is called the *foliated equivalence class* of p (cf. [8]) defined by

$$\mathrm{Fol}_f(p) := \left\{ q \in \widehat{\mathbf{C}} : \overline{\mathrm{GO}_f(p)} = \overline{\mathrm{GO}_f(q)} \right\},$$

where $\mathrm{GO}_f(p)$ is the *grand orbit* of p defined by

$$\mathrm{GO}_f(p) := \{ q \in \widehat{\mathbf{C}} : \text{for some } k, l \in \mathbf{N},\ f^k(p) = f^l(q) \}.$$

Conversely, in the case (i), (5) never holds. Concerning (ii), it is possible to find such an f that for some $p, q \in J(f)$ not satisfying (i), (5) does not hold: for example, let $f(z) = z^d$, choose such a sequence $\{a_l\}_{l=1}^{\infty} \subset \mathbf{N}$ that $\limsup_{l \to \infty} (a_{l+1} - a_l)/d^{a_l} \geq 1$, and set $p = 1$ and $q = \exp\left(2i\pi \sum_{l=1}^{\infty} 1/d^{a_l}\right)$. Then $p, q \in J(f) = \{|z| = 1\}$, $q \in \widehat{\mathbf{C}} - \bigcup_{k \geq 0} f^{-k}(p)$ and, by a direct calculation, $\limsup_{m \to \infty} (1/d^{a_m}) \log\left(1/[p, f^{a_m}(q)]\right) \geq \log d > 0$.

In the third case (iii), we have the following.

Theorem 4 (convergence of potentials and rotation domains). *Let* $f \in \mathrm{Rat}$ *be of degree* $d \geq 2$. *For* $p, q \in \widehat{\mathbf{C}}$ *such that* p *is not periodic but contained in a rotation domain and* $q \in \mathrm{GO}_f(p)$, (5) (*and* (4)) *holds.*

Hence in (iii), we have a stronger assertion that

$$q \in \mathrm{Fol}_f(p) - \mathrm{GO}_f(p) \tag{7}$$

than (6). The following is open.

Problem (a new problem on rotation domains). For some $p \in \widehat{\mathbf{C}}$ not periodic but contained in a rotation domain and $q \in \mathrm{Fol}_f(p) - \mathrm{GO}_f(p)$, is it possible that (5) does not hold?

If the answer to this problem is No, we can conclude that $\{V_{\mu(f^k,p)/d^k}\}$ always converges to V_{μ_f} on $F(f)$ except for the trivial case (i).

Acknowledgement. This work was partially done while the author was a long-term researcher of International Project Research 2003 "Complex Dynamics" of RIMS, Kyoto University. The author is very grateful to the staff of RIMS, Kyoto University for their hospitality.

He would like to express his gratitude to Professor Vincent Guedj for helpful discussions, and to Professors Peter Haissinsky and Mitsuhiro Shishikura for useful comments.

The author must particularly thank the referee, who read the paper with great care and whose careful comments helped the author refine both the contents and the presentation of this article.

2. The continuity of the potential of μ_f

In the rest of this paper, we fix $f \in \mathrm{Rat}$ of degree $d \geq 2$.

Notation 2.1. (i) $\|\cdot\|$ denotes the Euclidean norm on $\mathbf{C}^2$.

(ii) $\mathrm{d} = \partial + \overline{\partial}$ and $\mathrm{d}^c = \big(i/(2\pi)\big)(\overline{\partial} - \partial)$, hence $\mathrm{dd}^c = (i/\pi)\partial\overline{\partial}$.

(iii) π: $\mathbf{C}^2 - O \to \widehat{\mathbf{C}}$ is the canonical projection which maps (z_0, z_1) to z_1/z_0 when $z_0 \neq 0$. Here O is the origin in $\mathbf{C}^2$.

(iv) An integration $\int_{\widehat{\mathbf{C}}} \phi \, \mathrm{d}\mu$ is also written as $\langle \phi, \mu \rangle$.

Lemma 2.1 (continuity of the potential [7]). *V_{μ_f} is continuous on $\widehat{\mathbf{C}}$.*

This lemma was proved by Mañé [7], whose proof was based on a quite technical lemma. We shall give a short proof of Lemma 2.1 by the pluripotential theory rather than his technical lemma.

Proof. There exists a homogeneous polynomial map F: $\mathbf{C}^2 \to \mathbf{C}^2$ of degree d such that $\pi \circ F = f \circ \pi$ on $\mathbf{C}^2 - O$. The *escaping rate function* G^F: $\mathbf{C}^2 \to [-\infty, +\infty)$ is the limit of $\{\log \|F^k\|/d^k\}$, which is uniform on $\mathbf{C}^2 - O$ (cf. [13, Theorem 1.5]). Hence G^F is continuous on $\mathbf{C}^2 - O$ and plurisubharmonic on $\mathbf{C}^2$.

Since $\pi^*\sigma = \mathrm{dd}^c \log \|\cdot\|$, it follows from Theorem 1.1 that $\pi^*\mu_f = \mathrm{dd}^c G^F$.

Lemma 2.2 (a Stokes formula). *For $p \in \widehat{\mathbf{C}}$,*

$$\mathrm{dd}^c \log \frac{1}{[\,\cdot\,, p]} = \sigma - \delta_p, \tag{8}$$

and hence for every regular probability measure μ on $\widehat{\mathbf{C}}$,

$$\mathrm{dd}^c V_\mu = \sigma - \mu. \tag{9}$$

Hence $dd^c(\pi^* V_{\mu_f} - \log\|\cdot\| + G^F) = 0$, which concludes that $\pi^* V_{\mu_f}$ is continuous on $\mathbf{C}^2 - O$ and hence V_{μ_f} is continuous on $\widehat{\mathbf{C}}$. □

3. Proof of Theorem 1

Definition 3.1 (accumulation and convergence loci). The *accumulation* and *convergence* loci of the *averaged* value distributions of f in $\widehat{\mathbf{C}}$ are defined by

$$A(f) := \left\{p \in \widehat{\mathbf{C}} : \text{there exists } \{k_j\}_{j=0}^{\infty} \subset \mathbf{N} \text{ such that } \lim_{j\to\infty} \frac{\mu(f^{k_j}, p)}{d^{k_j}} = \mu_f\right\},$$

$$\mathrm{Conv}(f) := \left\{p \in \widehat{\mathbf{C}} : \lim_{k\to\infty} \frac{\mu(f^k, p)}{d^k} = \mu_f\right\},$$

respectively.

Theorem 1.1 means $\widehat{\mathbf{C}} - E(f) = \mathrm{Conv}(f)$. We show the following, which contains Theorem 1:

Theorem 5 (characterizations of exceptional sets). *We have*

$$E_N(f) = E_V(f) = E(f) = \widehat{\mathbf{C}} - \mathrm{Conv}(f) = \widehat{\mathbf{C}} - A(f).$$

Proof. For $p \in \widehat{\mathbf{C}}$ and $k \in \mathbf{N}$, it follows from (8) that

$$dd^c \frac{w(f^k, p)}{d^k} = \frac{(f^k)^*\sigma - \mu(f^k, p)}{d^k}.$$

Hence for every C^∞ function ϕ on $\widehat{\mathbf{C}}$,

$$\left|\left\langle \phi, \frac{(f^k)^*\sigma - \mu(f^k, p)}{d^k}\right\rangle\right| = \left|\left\langle \frac{w(f^k, p)}{d^k}, dd^c\phi\right\rangle\right| \le \left(\max_{\widehat{\mathbf{C}}}\left|\frac{dd^c\phi}{\sigma}\right|\right)\frac{m(p, f^k)}{d^k}.$$

The $\liminf$ and $\limsup$ of the right-hand side are 0 if $p \in \widehat{\mathbf{C}} - E_N(f)$ and $p \in \widehat{\mathbf{C}} - E_V(f)$, respectively, which in fact implies that $A(f) \supset \widehat{\mathbf{C}} - E_N(f)$ and $\mathrm{Conv}(f) \supset \widehat{\mathbf{C}} - E_V(f)$ since $\{w(f^k, p)/d^k\}$ is a sequence of δ-subharmonic functions on $\widehat{\mathbf{C}}$.

For the proof of the following, see [12] or [11].

Lemma 3.1 (Riesz decomposition). *For* $p, q \in \widehat{\mathbf{C}}$,

$$w(f, p)(q) = V_{\mu(f,p) - f^*\sigma}(q) + m(f, p).$$

By Lemma 3.1, for $p, q \in \widehat{\mathbf{C}}$ and $k \in \mathbf{N}$,

$$\frac{w(f^k, p)(q)}{d^k} = V_{(\mu(f^k,p) - (f^k)^*\sigma)/d^k}(q) + \frac{m(f^k, p)}{d^k}. \tag{10}$$

We integrate both sides of (10) by $\mathrm{d}\mu_f(q)$. Then by the Fubini theorem and $f_*\mu_f = \mu_f$ from (1), we have

$$\frac{1}{d^k} V_{\mu_f}(p) = \left\langle V_{\mu_f}, \frac{\mu(f^k, p) - (f^k)^*\sigma}{d^k} \right\rangle + \frac{m(f^k, p)}{d^k}. \tag{11}$$

Since V_{μ_f} is continuous on $\widehat{\mathbf{C}}$ by Lemma 2.1, (11) implies that $A(f) \subset \widehat{\mathbf{C}} - E_N(f)$ and $\mathrm{Conv}(f) \subset \widehat{\mathbf{C}} - E_V(f)$.

Finally we show that $E(f) \subset E_N(f)$. We recall the following (cf. [9]).

Lemma 3.2 (algebraic characterization of the exceptional set).

$$E(f) = \{p \in \widehat{\mathbf{C}} : \textit{periodic of period } \leq 2 \textit{ and critical of order } d - 1\}.$$

Let $p \in E(f)$. When p is of period one, by the Böttcher theorem, there exists a neighborhood U'' of p such that f is conformally conjugate to $z \mapsto z^d$ there. When p is of period two, there exist neighborhoods U of p and U' of $f(p)$ such that f^2 is conformally conjugate to $z \mapsto z^{d^2}$ on U and $f(U') \subset U$, and we put $U'' := U \cup U'$. For such U'', there exists $C > 0$ such that $\int_{U''} w(f^k, p)\, \mathrm{d}\sigma > Cd^k$ for every $k \in \mathbf{N}$. Hence $\mathrm{NE}(p; \{f^k\}) \geq C$, which implies $p \in E_N(f)$. Now the proof is completed. □

4. Proof of Theorems 2 and 3

Lemma 4.1 (convergence of potentials).

$$\lim_{k \to \infty} V_{(f^k)^*\sigma/d^k} = V_{\mu_f} \tag{12}$$

on $\widehat{\mathbf{C}}$.

Proof. Let s and t be local holomorphic sections of π on $\mathbf{C}$ and $\widehat{\mathbf{C}} - \{0\}$, respectively. Since there exists a holomorphic function c on $\mathbf{C} - \{0\}$ such that $t = cs$ on $\mathbf{C} - \{0\}$, we have $t^*F = c^d s^*F$ and $t^*G^F = s^*G^F + \log|c|$ there. Hence

$$H_k(z) := \begin{cases} s^*\left(\dfrac{1}{d^k}\log\|F^k\| - G^F\right) & \text{on } \mathbf{C}, \\ t^*\left(\dfrac{1}{d^k}\log\|F^k\| - G^F\right) & \text{on } \widehat{\mathbf{C}} - \{0\} \end{cases}$$

is a well-defined function on $\widehat{\mathbf{C}}$, and tends to 0 uniformly on $\widehat{\mathbf{C}}$ as $k \to \infty$. By $\pi^*\sigma = \mathrm{dd}^c \log\|\cdot\|$ and $\pi^*\mu_f = \mathrm{dd}^c G^F$, it follows that

$$\frac{(f^k)^*\sigma}{d^k} - \mu_f = \mathrm{dd}^c H_k$$

on $\widehat{\mathbf{C}}$. Now for every $q \in \widehat{\mathbf{C}}$,

$$V_{(f^k)^*\sigma/d^k}(q) - V_{\mu_f}(q) = \left\langle \log\frac{1}{[\cdot, q]}, \mathrm{dd}^c H_k \right\rangle = \left\langle H_k, \mathrm{dd}^c \log\frac{1}{[\cdot, q]} \right\rangle = \langle H_k, \sigma - \delta_q \rangle$$

converges to 0. □

We shall finish the proof of Theorems 2 and 3.

From Theorem 1, (10) and Lemma 4.1, it follows that for $p \in \widehat{\mathbf{C}} - E(f)$ and $q \in \widehat{\mathbf{C}}$,

$$\liminf_{k\to\infty} \frac{w(f^k,p)(q)}{d^k} = \liminf_{k\to\infty} V_{\mu(f^k,p)/d^k}(q) - V_{\mu_f}(q). \tag{13}$$

When $\liminf_{k\to\infty} V_{\mu(f^k,p)/d^k}(q) \neq V_{\mu_f}(q)$ at $q \in \widehat{\mathbf{C}}$, it follows that there exist $N \in \mathbf{N}$ and $\delta > 0$ such that for every $k \geq N$,

$$[f^k(q), p] \leq e^{-\delta d^k}. \tag{14}$$

Hence p is a fixed point of f. When $q \in \widehat{\mathbf{C}} - \bigcup_{k>0} f^{-k}(p)$, it follows from (14) that p is a superattractive fixed point of f of order $d-1$. Hence $p \in E(f)$ from Lemma 3.2, which is a contradiction. Now we have proved Theorem 2.

Again, from Theorem 1, (10) and Lemma 4.1, it follows that for $p \in \widehat{\mathbf{C}} - E(f)$ and $q \in \widehat{\mathbf{C}}$,

$$\limsup_{k\to\infty} \frac{w(f^k,p)(q)}{d^k} = \limsup_{k\to\infty} V_{\mu(f^k,p)/d^k}(q) - V_{\mu_f}(q); \tag{15}$$

(13) and (15) conclude the proof of Theorem 3. □

5. Proof of Theorem 4

It is enough to give a proof in the case of $q = p$.

Let D denote the rotation domain containing p, and S the unique analytic f-invariant circle in D containing p. Without loss of generality, we assume that $D \subset \mathbf{C}$ and the period of D equals one, and choose an analytic *linearizing* map h of $f \mid D$ that conformally maps D onto a disc or a concentric annulus centered at the origin containing the unit circle, and as S onto it. Then on D,

$$h \circ f = \lambda \cdot h,$$

where $\lambda = e^{2i\pi\alpha}$ for some $\alpha \in \mathbf{R} - \mathbf{Q}$ is called the *rotation number* of D.

Notation 5.1. $A \asymp B$ means $A/C < B < CA$ for some constant C.

Then the following uniform estimate holds: on $S \Subset D$,

$$[f^k(z), z] \asymp |f^k(z) - z| \asymp |h \circ f^k(z) - h(z)| = |\lambda^k - 1| \cdot |h(z)| = |\lambda^k - 1|,$$

where the implicit constants are independent of $k \in \mathbf{N}$ and $z \in S$. Hence

$$\limsup_{k\to\infty} \frac{1}{d^k} \log \frac{1}{[p, f^k(p)]} = \limsup_{k\to\infty} \frac{1}{d^k} \log \frac{1}{|\lambda^k - 1|}. \tag{16}$$

On the right-hand side of (16), we have already shown the following.

Theorem 5.1 (a priori bound [11]). *The rotation number λ of every rotation domain satisfies*

$$\limsup_{k\to\infty} \frac{1}{d^k} \log \frac{1}{|\lambda^k - 1|} = 0. \tag{17}$$

Combining (16) and (17) we finish the proof. □

References

[1] Federer, H.: Geometric Measure Theory. - Grundlehren Math. Wiss. 153, Springer-Verlag, New York, 1969.

[2] Freire, A., A. Lopes, and R. Mañé: An invariant measure for rational maps. - Bol. Soc. Brasil. Mat. 14:1, 1983, 45–62.

[3] Hinkkanen, A.: Sharp error term in the Nevanlinna theory. - In: Complex Differential and Functional Equations (Mekrijärvi, 2000), Univ. Joensuu Dept. Math. Rep. Ser. 5, Univ. Joensuu, Joensuu, 2003, 51–79.

[4] Klimek, M.: Pluripotential Theory. - London Math. Soc. Monogr. (N.S.) 6, Oxford Univ. Press, New York, 1991.

[5] Laine, I.: Nevanlinna Theory and Complex Differential Equations. - de Gruyter Stud. Math. 15, de Gruyter, Berlin, 1993.

[6] Ljubich, M. J.: Entropy properties of rational endomorphisms of the Riemann sphere. Ergodic Theory Dynam. Systems 3:3, 1983, 351–385.

[7] Mañé, R.: The Hausdorff dimension of invariant probabilities of rational maps. - In: Dynamical Systems, Valparaiso 1986, Lecture Notes in Math. 1331, Springer, Berlin, 1988, 86–117.

[8] McMullen, C., and D. Sullivan: Quasiconformal homeomorphisms and dynamics III: The Teichmüller space of a holomorphic dynamical system. - Adv. Math. 135, 1998, 351–395.

[9] Morosawa, S., Y. Nishimura, M. Taniguchi, and T. Ueda: Holomorphic Dynamics. - Cambridge Stud. Adv. Math. 66, Cambridge Univ. Press, Cambridge, 2000;

[10] Noguchi, J., and T. Ochiai: Geometric Function Theory in Several Complex Variables. - Transl. Math. Monogr. 80, Amer. Math. Soc., Providence, RI, 1990.

[11] Okuyama, Y.: Nevanlinna, Siegel, and Cremer. - Indiana Univ. Math. J. 53, 2004, 755–764.

[12] Sodin, M.: Value distribution of sequences of rational functions. - In: Entire and subharmonic functions, Adv. Soviet Math. 11, Amer. Math. Soc., Providence, RI, 1992, 7–20.

[13] Ueda, T.: Fatou sets in complex dynamics on projective spaces. - J. Math. Soc. Japan 46:3, 1994, 545–555.

Received 18 May 2004