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THE v-BOUNDARY OF WEIGHTED SPACES OF HOLOMORPHIC FUNCTIONS

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Abstract. We commence a geometric theory of the weighted spaces of holomorphic functions on bounded open subsets of $\mathbf{C}^n$, $\mathscr{H}_v(U)$ and $\mathscr{H}_{v_o}(U)$ by finding an upper bound for the set of extreme points of the unit ball of $\mathscr{H}_{v_o}(U)$. When U is balanced and v is radial we show that $\mathscr{H}_{v_o}(U)$ is not isometrically isomorphic to a subspace of c_o. We give a Choquet type theorem for $\mathscr{H}_{v_o}(U)$ and use it to study the centraliser of $\mathscr{H}_{v_o}(U)$.

1. Introduction

From a functional analysis point of view spaces of weighted holomorphic functions have been studied, to date, under at least four broad headings. The duality problem (see [37], [41], [42], [43], [19], [6], [38], [39]) which seeks to represent the dual and bidual of $\mathscr{H}_{v_o}(U)$ (defined below) as spaces of analytic functions. The theory of M-ideals (see [45], [46], [24]) which studies $\mathscr{H}_{v_o}(\Delta)$ as an M-ideal in its bidual. The isomorphic theory (see [28], [29], [30], [31], [32], [33], [34], [9]) which seeks to classify $\mathscr{H}_{v_o}(U)$ as a Banach space up to isomorphism. The theory of composition and multiplication operators on $\mathscr{H}_{v_o}(U)$ and its bidual $\mathscr{H}_v(U)$ (see [7], [8], [16], [17], [20], [40], [44], [15]). In this paper we examine a new aspect of weighted spaces of holomorphic functions—the isometric theory. As we shall see this new aspect leads to a deeper understanding of all the previous aspects mentioned above. The isometric theory of weighted spaces of holomorphic functions is further investigated in [11], [12] and [13].

Let U be a bounded open subset of $\mathbf{C}^n$. A continuous weight v on U is a bounded, strictly positive real valued function on U. We shall mainly concentrate on weights which converge to 0 on the boundary of U. On occasions we will restrict our attention to balanced domains. Here radial weights (weights v with the property that $v(z)=v(\lambda z)$ whenever $|\lambda|=1$) will play an important role in both our theory and examples. We will use $\mathscr{H}_v(U)$ to denote the space of all holomorphic functions f on U which have the property that $\|f\|_v := \sup_{z\in U} v(z)|f(z)| < \infty$.

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Endowed with $\|\cdot\|_v$, $\mathscr{H}_v(U)$ becomes a non-separable Banach space. A separable subspace of $\mathscr{H}_v(U)$ is got by considering all f in $\mathscr{H}_v(U)$ with the property that $|f(z)|v(z)$ converges to 0 as z converges to the boundary of U i.e. given $\varepsilon > 0$ there is a compact subset K of U such that $v(z)|f(z)| < \varepsilon$ for z in $U \setminus K$. This subspace is denoted by $\mathscr{H}_{v_o}(U)$. Thus $\mathscr{H}_v(U)$ may be regarded as all holomorphic functions on U which satisfy a growth condition of order $O\big(1/v(z)\big)$ while $\mathscr{H}_{v_o}(U)$ are those functions with a growth rate of order $o\big(1/v(z)\big)$. Under moderate conditions on U and v (see [6]), $\mathscr{H}_v(U)$ is the bidual of $\mathscr{H}_{v_o}(U)$. When this happens the dual of $\mathscr{H}_{v_o}(U)$ is denoted by $G_v(U)$. The geometry of the unit ball of $\mathscr{H}_{v_o}(U)'$ is the primary object of study of this paper. An upper bound for the set of extreme points of the unit ball of $\mathscr{H}_{v_o}(U)'$ is given by $\{\lambda v(z)\delta_z : z \in U,\ |\lambda| = 1\}$. We shall use $\mathscr{B}_v(U)$ to denote the set of $z \in U$ for which $v(z)\delta_z$ is an extreme point of the unit ball of $\mathscr{H}_{v_o}(U)'$. We study the topological properties of this set showing that the mapping which takes z to $v(z)\delta_z$ is a homeomorphism onto its range.

The set $\mathscr{B}_v(U)$ enables us to study the geometry of $\mathscr{H}_{v_o}(U)$. An example of the type of result we obtain is: if v is radial on a balanced domain then a bounded sequence $(f_k)_k$ in $\mathscr{H}_{v_o}(U)$ converges weakly to f in $\mathscr{H}_{v_o}(U)$ if and only if $(f_k)_k$ converges pointwise to f. Bonet and Wolf, [9], and Lusky, [28], [29], [30], [31], [32], [33] have shown that $\mathscr{H}_{v_o}(U)$ is isomorphic to a subspace of c_o. We will show that when v is either complete or radial this isomorphism is never an isometry. A Choquet type theorem allows us to recover the values of functions in $\mathscr{H}_{v_o}(U)$ from the values it obtains on $\overline{\mathscr{B}_v(U)}$. This allows us to examine the centraliser of weighted spaces of holomorphic functions. In [11] we shall examine the weak*-exposed and weak*-strongly exposed points of the unit ball of $\mathscr{H}_{v_o}(U)$ and in [12] and [13] we make use of the v-boundary to classify isometries between weighted spaces of holomorphic functions.

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2. Elementary theory of the v-boundary

Let U be a bounded open subset of $\mathbf{C}^n$ and $v: U \to \mathbf{R}^+$ be a continuous strictly positive weight. We use $G_v(U)$ to denote the space of linear functionals on $\mathscr{H}_v(U)$ whose restriction to the unit ball of $\mathscr{H}_v(U)$, $B_{\mathscr{H}_v(U)}$, are continuous for the compact open topology, τ_o. If we endow $G_v(U)$ with the topology induced from $\mathscr{H}_v(U)'$ then it follows from [6, Theorem 1.1(a)] that $G_v(U)'$ is isometrically isomorphic to $(\mathscr{H}_v(U), \|\cdot\|_v)$. We shall say that v converges to 0 as z converges to the boundary of U if given $\varepsilon > 0$ there is a compact subset K of U such that $v(z) < \varepsilon$ for z in $U \setminus K$.

Given a bounded open subset U of $\mathbf{C}^n$ and $v: U \to \mathbf{R}^+$ a continuous strictly positive weight we use $\mathscr{H}_{v_o}(U)$ to denote the subspace of $\mathscr{H}_v(U)$ defined by

$$\mathscr{H}_{v_o}(U) := \Big\{ f \in \mathscr{H}_v(U) : \lim_{z \to \partial U} v(z)|f(z)| = 0 \Big\}.$$

We endow $\mathscr{H}_{v_o}(U)$ with the norm induced from $\mathscr{H}_v(U)$. If we assume that $v(z)$ converges to 0 as z converges to the boundary of U, that U is balanced and v is radial, then $\mathscr{H}_{v_o}(U)$ is equal to the closure of the polynomials with respect to the norm $\|\cdot\|_v$. Bierstedt and Summers, [6, Theorem 1.1(b)] show that the condition that $B_{\mathscr{H}_{v_o}(U)}$ is τ_o-dense in $B_{\mathscr{H}_v(U)}$ is a necessary and sufficient condition to ensure that $G_v(U)$ is isometrically isomorphic to the dual of $\mathscr{H}_{v_o}(U)$. In particular, if $B_{\mathscr{H}_{v_o}(U)}$ is τ_o-dense in $B_{\mathscr{H}_v(U)}$ then as $G_v(U)$ is a dual Banach space its unit ball will have extreme points.

The purpose of this paper is to examine the geometric structure of the space $\mathscr{H}_{v_o}(U)'$. In particular, we will investigate how the geometric theory of $\mathscr{H}_{v_o}(U)'$ depends on the weight v. In the special case when $U = \Delta$ is the unit disc in $\mathbf{C}$, and $v(x) \equiv 1$, $(\mathscr{H}_v(\Delta), \|\cdot\|_v) = (\mathscr{H}^\infty(\Delta), \|\cdot\|_\infty)$, the Banach space of all bounded holomorphic functions on the unit disc. We let $L^1(\delta\Delta)$ denote the space of all integrable functions on the unit circle and H^1_o denote the space of all holomorphic functions f on the unit disc with $\sup_{0<r<1}\big(\int_0^{2\pi} |f(re^{i\theta})|\,d\theta\big) < \infty$ and $f(0) = 0$. The Banach space H^1_o can be identified with a subspace of $L^1(\delta\Delta)$. A classical result of Ando, [1], shows that $L^1(\delta\Delta)/H^1_o(\Delta)$ is the unique isometric predual of $(\mathscr{H}^\infty(\Delta), \|\cdot\|_\infty)$. Furthermore, Ando, [1], shows that the set of extreme points of the unit ball of $L^1(\delta\Delta)/H^1_o(\Delta)$ is empty.

Let us begin our description of the geometry of the unit ball of $\mathscr{H}_{v_o}(U)'$ with an upper bound on the possible extreme points this set may have. We shall use Γ to denote the set $\{\lambda \in \mathbf{C} : |\lambda| = 1\}$.

Given a Banach space E we use $\operatorname{Ext} B_E$ to denote the set of extreme points of the closed unit ball of E.

Proposition 1. *Let U be a bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight on U which converges to 0 on the boundary of U. Then the extreme points of the unit ball of $\mathscr{H}_{v_o}(U)'$ are contained in the set $\{\lambda v(z)\delta_z : z \in U,\ \lambda \in \Gamma\}$.*

Proof. The mapping $f \to fv$ is an isometric isomorphism of $\mathscr{H}_{v_o}(U)$ onto a subspace of $C(\overline{U})$ (fv tends to 0 on the boundary of U). Applying [18, Lemma V.8.6] we see that the set of extreme points of the unit ball of $\mathscr{H}_{v_o}(U)'$ is contained in the set of extreme points of the unit ball of $C(\overline{U})'$. However the set of extreme points of the unit ball of $C(\overline{U})'$ is $\{\lambda\delta_z : z \in U,\ \lambda \in \Gamma\}$. Restricting these to the image of the unit ball of $\mathscr{H}_{v_o}(U)'$ we get that the extreme points of the unit ball of $\mathscr{H}_{v_o}(U)'$ are contained in the set $\{\lambda v(z)\delta_z : z \in U,\ \lambda \in \Gamma\}$. □

In particular, if U is a bounded balanced open subset of $\mathbf{C}^n$ and v is a continuous strictly positive radial weight on U which tends to 0 on the boundary of U then the extreme points of the unit ball of $G_v(U)$ are contained in the set $\{\lambda v(z)\delta_z : z \in U,\ \lambda \in \Gamma\}$.

From the remark before [5, Proposition 1.2], [5, Theorem 1.5(c)] and [6, Theorem 1.1] (see also [38, Proposition 2.2.1]) we have the following result:

Proposition 2. *Let U be a bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight on U. Then $\mathscr{H}_{v_o}(U)$ contains all polynomials on $\mathbf{C}^n$ if and only if v extends continuously to the boundary of U with $v|_{\partial U} \equiv 0$. Furthermore, if U is balanced and v is radial, either of these equivalent conditions will imply that $G_v(U)$ is isometrically isomorphic to the dual of $(\mathscr{H}_{v_o}(U), \|\cdot\|_v)$.*

Under the conditions of Proposition 2 we have:

Theorem 3. *Let U be a bounded balanced open subset of $\mathbf{C}^n$ and v be a continuous strictly positive radial weight which converges to 0 on the boundary of U. Then $G_v(U)$ is the unique isometric predual of $\mathscr{H}_v(U)$.*

Proof. Since U is separable and the mapping $z \to v(z)\delta_z$ is continuous the set $\{v(z)\delta_z : z \in U\}$ is separable. Hence, its closed linear span, $G_v(U)$, is also separable. By Proposition 2 $G_v(U)$ is a separable dual space and so has the Radon–Nikodým Property. [21, Theorem 10] (see also [23, (b), p. 144]) implies that $G_v(U)$ is the unique isometric predual of $\mathscr{H}_v(U)$. □

Suppose that $z \in U$ is such that $v(z)\delta_z$ is not an extreme point of the unit ball of $\mathscr{H}_{v_o}(U)'$. Then $v(z)\delta_z = \frac{1}{2}(\phi_1 + \phi_2)$ for some ϕ_1, ϕ_2 in the unit ball of $\mathscr{H}_{v_o}(U)'$. As $\lambda v(z)\delta_z = \frac{1}{2}(\lambda\phi_1 + \lambda\phi_2)$ for every λ in $\mathbf{C}$ with $|\lambda| = 1$ we see that $\lambda v(z)\delta_z$ will not be an extreme point of the unit ball of $\mathscr{H}_{v_o}(U)'$ for any λ in Γ. With this observation, we give the following definition:

Definition 4. Let U be a bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight on U which converges to 0 on the boundary of U. The *v-boundary* of U, $\mathscr{B}_v(U)$, is $\{z \in U : v(z)\delta_z$ is an extreme point of the unit ball of $\mathscr{H}_{v_o}(U)'\}$.

Radial weights have radial v-boundaries.

Lemma 5. *Let U be a balanced bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight on U which converges to 0 on the boundary of U. If v is radial then $\mathscr{B}_v(U)$ is radial in the sense that $z \in \mathscr{B}_v(U)$ implies $\lambda z \in \mathscr{B}_v(U)$ for all $\lambda \in \Gamma$.*

Proof. Given $f \in \mathscr{H}_{v_o}(U)$ and $\lambda \in \Gamma$ we define f_λ by $f_\lambda(z) = f(\lambda z)$. We note that $f \in B_{\mathscr{H}_{v_o}(U)}$ if and only if $f_\lambda \in B_{\mathscr{H}_{v_o}(U)}$. Given $\phi \in \mathscr{H}_{v_o}(U)'$ we define ϕ_λ by $\phi_\lambda(f) = \phi(f_\lambda)$. It follows from the definition of the norm on $\mathscr{H}_{v_o}(U)'$, that $\|\phi_\lambda\| = \|\phi\|$.

Suppose that $z \in U$ but $z \notin \mathscr{B}_v(U)$. Then we can find ϕ_1, ϕ_2 in the unit ball of $\mathscr{H}_{v_o}(U)'$, $\phi_1 \neq \phi_2$, so that $v(z)\delta_z = \frac{1}{2}(\phi_1 + \phi_2)$. Then for $\lambda \in \Gamma$ and each

$f \in \mathscr{H}_{v_o}(U)$ we have

$$v(\lambda z)\delta_{\lambda z}(f) = v(z)f(\lambda z) = v(z)f_\lambda(z) = \tfrac{1}{2}\big(\phi_1(f_\lambda)+\phi_2(f_\lambda)\big) = \tfrac{1}{2}\big((\phi_1)_\lambda+(\phi_2)_\lambda\big)(f).$$

Therefore $v(\lambda z)\delta_{\lambda z} = \frac{1}{2}\big((\phi_1)_\lambda + (\phi_2)_\lambda\big)$. Hence $\lambda z \notin \mathscr{B}_v(U)$ and the result is proven. ▫

Definition 6. Let U be a bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight on U which converges to 0 on the boundary of U. We shall say that v is a *complete* weight if $\mathscr{B}_v(U) = U$.

We recall the following definition:

Definition 7. Let E be a complex Banach space. A point x in E is said to be an exposed point of the unit ball of E if there is $\phi \in E'$ of norm 1 such that $\operatorname{Re}\big(\phi(x)\big) = 1$ and $\operatorname{Re}\big(\phi(y)\big) < 1$ for all $y \in E$, $\|y\| \le 1$, $y \ne x$. When $E = F'$ is a dual space and the vector ϕ which *exposes* x in B_E is in F, we say that x is *weak***-exposed* and that ϕ *weak** *exposes* the unit ball of E at x.

A continuous strictly positive weight v on $B_{\mathbf{C}^n}$ is said to be unitary if $v(z) = v(Az)$ for every $n \times n$ unitary matrix A. Hence v is unitary if and only if $v(z) = v(w)$ whenever $\|z\| = \|w\|$. If $n = 1$ the concept of a unitary weight coincides with the concept of a radial weight. For $n = 2$ the weight $v(z) = (1 - \|z\|^{1+2/\pi \tan^{-1}(|z_2|/|z_1|)})$ is a radial weight which is not unitary. It is readily shown that if v is unitary then $\mathscr{B}_v(B_{\mathbf{C}^n})$ is unitary in the sense that $z \in \mathscr{B}_v(B_{\mathbf{C}^n})$ if and only if $Az \in \mathscr{B}_v(B_{\mathbf{C}^n})$ for all unitary matrices A.

In [11] we obtain the following sufficient condition for completeness of a unitary weight on $B_{\mathbf{C}^n}$.

Proposition 8. *Let $v\colon B_{\mathbf{C}^n} \to \mathbf{R}$ be a continuous strictly positive strictly decreasing unitary weight on the unit ball of $\mathbf{C}^n$ which converges to 0 on the boundary of $B_{\mathbf{C}^n}$ such that $v(x)$ is twice differentiable and $\big(\partial v(x)/\partial x_1\big)^2 - v(x)\partial^2 v(x)/\partial x_1^2 > 0$ for x of the form $(x_1, 0, \dots, 0)$ with x_1 in $(0,1)$. Then the weak*-exposed points (and hence the extreme points) of the unit ball of $\mathscr{H}_{v_o}(B_{\mathbf{C}^n})'$ is the set $\{v(z)\lambda\delta_z : \lambda \in \Gamma,\ z \in B_{\mathbf{C}^n}\}$.*

This condition allows us to show that when $\alpha > 0$, $\beta \ge 1$ each of the following weights on the unit ball of $\mathbf{C}^n$ is complete.

(a) $v_{\alpha,\beta}(z) = (1 - \|z\|^\beta)^\alpha$.
(b) $w_{\alpha,\beta}(z) = e^{-\alpha/(1-\|z\|^\beta)}$.
(c) $v(z) = \big(\log(2 - \|z\|)\big)^\alpha$.
(d) $v(z) = \big(1 - \log(1 - \|z\|)\big)^{-\alpha}$.
(e) $v(z) = \cos\big(\frac{1}{2}\pi\|z\|\big)$.
(f) $v(z) = \cos^{-1}\|z\|$.
(g) Finite products of the examples in (a) to (f).

3. Structure of the v-boundary

Lemma 9. *Let U be a bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight which converges to 0 on the boundary of U.*

(a) If $\lambda, \mu \in \Gamma$ and $z, w \in U$ then $\lambda v(z)\delta_z = \mu v(w)\delta_w$ on $\mathscr{H}_{v_o}(U)$ implies $z = w$ and $\lambda = \mu$.

(b) Let $z \in \overline{U}$. If $v(z)\delta_z = 0$ on $\mathscr{H}_{v_o}(U)$ then $z \in \partial U$.

(c) Let $\lambda, \mu \in \Gamma$ and $z, w \in \overline{U}$. If $\lambda v(z)\delta_z = \mu v(w)\delta_w$ in $\mathscr{H}_{v_o}(U)'$ then $z = w$ or z, $w \in \partial U$.

Proof. (a) If $z \neq w$ we may suppose without loss of generality that $z_1 \neq w_1$ and take

$$p(t) = \frac{2}{v(w)\mu}\frac{t_1 - z_1}{w_1 - z_1} + \frac{1}{v(z)\lambda}\frac{t_1 - w_1}{z_1 - w_1}.$$

Then $p \in \mathscr{H}_{v_o}(U)$ and

$$\lambda v(z)p(z) = 1 \neq 2 = \mu v(w)p(w).$$

(b) If $z \in U$ then $v(z) > 0$ and the constant map $p(w) \equiv 1/v(z) \in \mathscr{H}_{v_o}(U)$. But then $v(z)\delta_z(p) = 1$.

Part (c) follows from (a) and (b). □

Lemma 10. *Let U be a bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight which converges to 0 on the boundary of U. Then the map*

$$\mu\colon U \to \big(\mathscr{H}_{v_o}(U)', \sigma\big(\mathscr{H}_{v_o}(U)', \mathscr{H}_{v_o}(U)\big)\big)$$

given by $\mu(z) := v(z)\delta_z$ is a homeomorphism onto its image.

Proof. Consider $\mu\colon \overline{U} \to \big(\mathscr{H}_{v_o}(U)', \sigma\big(\mathscr{H}_{v_o}(U)', \mathscr{H}_{v_o}(U)\big)\big)$ given by $\mu(z) := v(z)\delta_z$. Define the relation $\sim$ on $\overline{U}$ by $z \sim w$ if $\mu(z) = \mu(w)$ and consider the quotient space $\overline{U}/\mu$. Consider the map $\bar{\mu}\colon \overline{U}/\mu \to \mathscr{H}_{v_o}(U)'$ given by $\bar{\mu} \circ q = \mu$, where $q\colon \overline{U} \to \overline{U}/\mu$ is the natural quotient map. (The set $\overline{U}/\mu$ may also be regarded as the one-point compactification of U.) Let us show that $\bar{\mu}$ is continuous when $\mathscr{H}_{v_o}(U)'$ is endowed with the weak*-topology. Let $z_0 \in \overline{U}$ and consider the subbasic neighbourhood of $v(z_0)\delta_{z_0}$

$$N(z_0; f, \varepsilon) := \{\phi \in \mathscr{H}_{v_o}(U)' : |\phi(f) - v(z_0)\delta_{z_0}(f)| < \varepsilon\},$$

where $f \in \mathscr{H}_{v_o}(U)$ and $\varepsilon > 0$. Since vf is continuous on $\overline{U}$, the set

$$N(z_0; \varepsilon) := \{z \in \overline{U} : |v(z)f(z) - v(z_0)f(z_0)| < \varepsilon\}$$

is an open neighbourhood of z_0 in $\overline{U}$ and $\mu\big(N(z_0;\varepsilon)\big) \subset N(z_0; f, \varepsilon)$.

As $\overline{U}$ is compact $\overline{U}/\mu$ is compact. Hence $\bar{\mu}$ is a uniform homeomorphism onto its image. On the other hand, by Lemma 9 $\mu|_U$ is injective. Let us show that $\mu|_U$ is an open map: given $A \subset U$ open there exists an open subset $B \subset \overline{U}$ such that $A = B \cap U$. Then $\mu(B)$ is open in $\mu(\overline{U})$. By Lemma 9 $\mu(A) = \mu(B) \cap \mu(U)$ and therefore $\mu(A)$ is open in $\mu(U)$. Thus $\mu|_U$ is a homeomorphism onto its image. □

Let U be a bounded open subset of $\mathbf{C}^n$, and let v be a continuous strictly positive weight which converges to 0 on the boundary of U. We know that the extreme points of the unit ball of $\mathscr{H}_{v_o}(U)'$ are contained in the set $\{\lambda v(z)\delta_z : \lambda \in \Gamma,\ z \in U\}$. Let us investigate the topological structure of this set.

Lemma 11. *Let U be an open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight which converges to 0 on the boundary of U. Then* $\overline{\mathrm{Ext}(B_{\mathscr{H}_{v_o}(U)'})}^{\sigma^*} = \big\{\lambda v(z)\delta_z : z \in \overline{\mathscr{B}_v(U)},\ \lambda \in \Gamma\big\}$.

Proof. As $B_{\mathscr{H}_{v_o}(U)}$ is separable it follows from [14, Proposition 2.5.12] that $\big(\mathscr{H}_{v_o}(U)', \sigma\big(\mathscr{H}_{v_o}(U)', \mathscr{H}_{v_o}(U)\big)\big)$ is metrizable. Let $\phi \in \overline{\mathrm{Ext}(B_{\mathscr{H}_{v_o}(U)'})}^{\sigma^*}$. Then there exist $(\lambda_n)_n \subset \Gamma$ and $(z_n)_n \subset \mathscr{B}_v(U)$ such that

$$\phi = w^* \lim_n \lambda_n v(z_n)\delta_{z_n} = w^* \lim_n \lambda_n \mu(z_n).$$

Since $\lambda_n \in \Gamma$, there exists a subsequence $(\lambda_{n_k})_k$ of $(\lambda_n)_n$ converging to some λ_o of modulus 1. Then $\big(\mu(z_{n_k})\big)_k$ w^*-converges to $1/\lambda_o\phi$. By Lemma 10 and using that $\bar{\mu}$ is a uniform homeomorphism onto its image, $z_{n_k} = \mu^{-1}\mu(z_{n_k})$ converges to some $z_o \in \overline{\mathscr{B}_v(U)}$. Hence $\phi = w^* \lim_{k\to\infty} \lambda_{n_k}\mu(z_{n_k}) = \lambda_o v(z_o)\delta_{z_o}$.

Conversely, if $z \in \overline{\mathscr{B}_v(U)}$ we have $z = \lim_n z_n$ where z_n belongs to $\mathscr{B}_v(U)$. Then $v(z)\delta_z = w^* \lim_n v(z_n)\delta_{z_n} \in \overline{\mathrm{Ext}(B_{\mathscr{H}_v(U)'})}^{\sigma^*}$. □

Note that if we start in the proof with $\phi \in \overline{\mathrm{Ext}(B_{\mathscr{H}_{v_o}(U)'})}^{\sigma^*} \setminus \mathrm{Ext}(B_{\mathscr{H}_{v_o}(U)'})$ then $z_0 = \lim_k z_{n_k} \in \overline{\mathscr{B}_v(U)} \setminus \mathscr{B}_v(U)$.

Proposition 12. *Let U be an open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight which converges to 0 on the boundary of U. Then $\mathscr{B}_v(U)$ is a G_δ subset of U.*

Proof. We use the fact that $\big(\mathscr{H}_{v_o}(U)', \sigma\big(\mathscr{H}_{v_o}(U)', \mathscr{H}_{v_o}(U)\big)\big)$ is metrizable (see Lemma 11). Applying [35, Proposition 1.3] it follows that $\mathrm{Ext}B_{\mathscr{H}_{v_o}(U)'}$ is a G_δ set in $\mathscr{H}_{v_o}(U)' \setminus \{0\}$ endowed with the weak*-topology. Let

$$\mathrm{Ext}B_{\mathscr{H}_{v_o}(U)'} = \bigcap_{n=1}^{\infty} G_n$$

with each G_n open in $\mathscr{H}_{v_o}(U)' \setminus \{0\}$ endowed with the weak*-topology. We can assume by the proof of [35, Proposition 1.3] that each G_n is radial. Define an equivalence relation $\equiv$ on $\mathscr{H}_{v_o}(U)' \setminus \{0\}$ by $\phi_1 \equiv \phi_2$ if $\phi_1 = \lambda\phi_2$ for some $\lambda \in \Gamma$ and let π be the quotient mapping from $\mathscr{H}_{v_o}(U)' \setminus \{0\}$ onto $\mathscr{H}_{v_o}(U)' \setminus \{0\}/\equiv$. We shall consider $\mathscr{H}_{v_o}(U)' \setminus \{0\}/\equiv$ endowed with the quotient topology of the weak* topology on $\mathscr{H}_{v_o}(U)' \setminus \{0\}$. Since $\pi^{-1}\big(\pi(A)\big) = \bigcup_{\lambda\in\Gamma} \lambda A$, π is an open mapping. As each G_n is radial it is easily checked that $\pi\big(\bigcap_{n=1}^{\infty} G_n\big) = \bigcap_{n=1}^{\infty} \pi(G_n)$ and then $\mathrm{Ext}B_{\mathscr{H}_{v_o}(U)'}/\equiv$ is a G_δ set in $\mathscr{H}_{v_o}(U)' \setminus \{0\}/\equiv$. Further, as the image of each z in U under μ lies in at most one $\equiv$-equivalence class of $\mathscr{H}_{v_o}(U)' \setminus \{0\}$, it is readily checked that $\pi \circ \mu$ is a homeomorphism onto its image. Under this mapping $\mathscr{B}_v(U)$ is mapped onto $\mathrm{Ext}B_{\mathscr{H}_{v_o}(U)'}/\equiv$ and result follows. □

We shall see in [11] that U is equal to the intersection of U with the $\mathscr{H}^\infty$-convex hull of $\mathscr{B}_v(U)$.

4. Convergence of weighted holomorphic functions

The study of the v-boundary allows us to show in [11, Theorem 24 and 25] that $\mathscr{H}_{v_o}(U)$ and $\mathscr{H}_v(U)$ are never smooth and (under moderate conditions on the weight) that both spaces are not rotund. In [12] the v-boundary is central in the isometric classification of $\mathscr{H}_{v_o}(U)$ and $\mathscr{H}_v(U)$.

We present some applications of the v-boundary to the geometry of $\mathscr{H}_{v_o}(U)$ and $\mathscr{H}_v(U)$. The following theorem characterises weak convergence and weak compactness in $\mathscr{H}_{v_o}(U)$.

Theorem 13. *Let U be a bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight on U which converges to 0 on the boundary of U. Then*

(a) *A bounded sequence $(f_k)_k$ in $\mathscr{H}_{v_o}(U)$ converges weakly to f in $\mathscr{H}_{v_o}(U)$ if and only if $(f_k)_k$ converges pointwise to f.*

(b) *A bounded subset of $\mathscr{H}_{v_o}(U)$ is weakly relatively compact if and only if it is relatively countably compact for the topology of pointwise convergence on $\mathscr{H}_{v_o}(U)$.*

Proof. We showed in Proposition 1 that the extreme points of the unit ball of $\mathscr{H}_{v_o}(U)'$ are contained in the set $\{\lambda v(z)\delta_z : z \in U,\ \lambda \in \Gamma\}$. Part (a) now follows from Rainwater's theorem, [36], while part (b) is a consequence of a theorem of Bourgain and Talagrand, [10]. □

Proposition 14. *Let U be a balanced bounded open subset of $\mathbf{C}^n$ and let v be a continuous strictly positive radial weight which converges to 0 on the boundary of U. Let $(z_\alpha)_\alpha$ be a net in U and z be a point of U. If $v(z_\alpha)f(z_\alpha) \to v(z)f(z)$ for all $f \in \mathscr{H}_{v_o}(U)$ then $f(z_\alpha)v(z_\alpha) \to f(z)v(z)$ for all $f \in \mathscr{H}_v(U)$.*

Proof. By [24, Examples III.1.4] $\mathscr{H}_{v_o}(U)$ is an M-ideal in $\mathscr{H}_v(U)$. (The proof in [24] is for the open unit disc Δ but is easily extended to arbitrary balanced domains in $\mathbf{C}^n$.) The result now follows by [24, Corollary III.2.15] (see also [22]). □

Proposition 15. *Let U be a balanced bounded open subset of $\mathbf{C}^n$ and let v be a continuous strictly positive radial weight which converges to 0 on the boundary of U. Then given $(z_n)_n$ in U and $(f_m)_m$ in the unit ball of $\mathscr{H}_v(U)$ we have that*

$$\lim_{m\to\infty}\lim_{n\to\infty} v(z_n)f_m(z_n) = \lim_{n\to\infty}\lim_{m\to\infty} v(z_n)f_m(z_n).$$

Proof. It follows from Lemma 10 that the set $\{v(z)\delta_z : z \in U\}$ is relatively compact for the $\sigma\big(\mathscr{H}_{v_o}(U)', \mathscr{H}_{v_o}(U)\big)$-topology. Applying [24, Corollary III.2.15] we now see that it is therefore weakly relatively compact. It now follows from [25, Lemma 19.A.1] that for all $(z_n)_n$ in U and $(f_m)_m$ in the unit ball of $\mathscr{H}_v(U)$ we have that

$$\lim_{m\to\infty}\lim_{n\to\infty} v(z_n)f_m(z_n) = \lim_{n\to\infty}\lim_{m\to\infty} v(z_n)f_m(z_n). \ \square$$

5. $C(K)$-spaces

Let X be a locally compact Hausdorff space and A be a closed subspace of $C_o(X)$. According to Araujo and Font [2] A is strongly separating if for each pair of points x_1 and x_2 in X there is $f \in A$ such that $|f(x_1)| \neq |f(x_2)|$. They define the Choquet boundary of A as $\{x \in X : e_x \text{ is an extreme point of } B_{A'}\}$, e_x is evaluation at x. It follows from [2, Corollary 4.2] that if $\mathscr{H}_{v_o}(U)$ is isometrically isomorphic to a strongly separating subspace A of $C_o(X)$ then $\mathscr{B}_v(U)$ is homeomorphic to the Choquet boundary of A.

A compact set K is said to be *perfect* if it has no isolated points. A compact set K is said to be *scattered* (dispersed) if it contains no perfect subsets.

Theorem 16. *Let U be a bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight on U which converges to 0 on the boundary of U. If $\mathscr{H}_{v_o}(U)$ is isometrically isomorphic to a subspace of (complex) $C(K)$ with K scattered then $\mathscr{B}_v(U)$ is a countable (and therefore discrete) subset of U.*

Proof. Define an equivalence relation $\equiv$ on $\mathrm{Ext}B_{\mathscr{H}_{v_o}(U)'}$ by $\lambda_1 v(z)\delta_z \equiv \lambda_2 v(w)\delta_w$ if $z = w$. Suppose that T is an isometry from $\mathscr{H}_{v_o}(U)$ onto a subspace M of $C(K)$. Then T^* maps $\mathrm{Ext}B_{C(K)'}$ with the weak*-topology, which is homeomorphic to $\Gamma \times K$, onto a set which contains $\{\lambda v(z)\delta_z : \lambda \in \Gamma,\ z \in \mathscr{B}_v(U)\}$ endowed with the $\sigma\big(\mathscr{H}_{v_o}(U)', \mathscr{H}_{v_o}(U)\big)$-topology. As T^* is linear it induces a map T_1^* from K onto a set containing $\{\lambda v(z)\delta_z : \lambda \in \Gamma,\ z \in \mathscr{B}_v(U)\}/\equiv$ or equivalently a set containing $\mathscr{B}_v(U)$. As T^* is continuous and surjective, it is $\sigma(M', M) - \sigma\big(\mathscr{H}_{v_o}(U)', \mathscr{H}_{v_o}(U)\big)$ open, see [26, Proposition 3.17.17]. Hence $\mu^{-1} \circ T_1^*$ is a continuous, open mapping and so $\mu^{-1} \circ T_1^*(K)$ is scattered. It follows that $\mathscr{B}_v(U)$ is also scattered. As U is metrizable the proof of [25, Lemma 25.D] gives us that $\mathscr{B}_v(U)$ is countable and hence must be discrete. ▫

Corollary 17. *Let U be a bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight on U which converges to 0 on the boundary of U. If v is complete or if U is balanced and v is radial then $\mathscr{H}_{v_o}(U)$ cannot be isometrically isomorphic to a subspace of $C(K)$ with K scattered. In particular, $\mathscr{H}_{v_o}(U)$ cannot be isometrically isomorphic to a subspace of c_o.*

Proof. In either of the above cases $\mathscr{B}_v(U)$ contains a non-trivial connected component and so cannot be scattered. ▫

In [28, Corollary 2.4(i)] Lusky proved that if v is a continuous strictly positive radial weight on Δ which converges to 0 on the boundary of Δ then $\mathscr{H}_{v_o}(\Delta)$ is isomorphic to a subspace of c_o. The weight $v(z) = \big(1 - \log(1 - |z|)\big)^{\beta}$, $\beta < 0$, is an example of a weight on Δ which gives a Banach space $\mathscr{H}_{v_o}(\Delta)$ which is isomorphic to a subspace of c_o yet not isomorphic to c_o. See [29]. Bonet and Wolf [9] have recently extended this result by showing that if U is an open subset of $\mathbf{C}^n$ and v is a continuous strictly positive weight on U then $\mathscr{H}_{v_o}(U)$ is almost isometrically isomorphic to a subspace of c_o. The previous corollary shows that this isomorphism can never be an isometric embedding whenever U is bounded and v is either a complete or radial weight which converges to 0 on the boundary of U.

6. A Choquet theorem

In this section we present an 'analytic' representation of the v-boundary obtained from an application of Choquet's theorem.

Theorem 18 (A Choquet type theorem). *Let U be a bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive weight on U which converges to 0 on the boundary of U. Then for each $z \in U$ there is a $\mathbf{C}$-valued measure, ν_z, of bounded variation with support contained in $\overline{\mathscr{B}_v(U)}^U$ so that*

$$(*) \qquad f(z) = \int_{\overline{\mathscr{B}_v(U)}^U} f(w)\, d\nu_z(w)$$

for all f in $\mathscr{H}_{v_o}(U)$. Moreover we have that $\nu_z\big(\overline{\mathscr{B}_v(U)}^U\big) = 1$.

Proof. We consider $B_{\mathscr{H}_{v_o}(U)'}$ with the weak*-topology. By definition the set of extreme points of the unit ball of $\mathscr{H}_{v_o}(U)'$ is equal to $\{\lambda v(z)\delta_z : \lambda \in \Gamma,\ z \in \mathscr{B}_v(U)\}$. Hence by the Choquet–Bishop–de Leeuw theorem, [35, Chapter 4, Theorem] and Lemma 11, for each z in U there is a probability measure, μ_z, with support contained in $\Gamma \times \overline{\mathscr{B}_v(U)}^U$ so that

$$(**) \qquad f(z)v(z) = \int_{\Gamma\times\overline{\mathscr{B}_v(U)}^U} \lambda f(w)v(w)\, d\mu_z(\lambda, w)$$

for all f in $\mathscr{H}_{v_o}(U)$. Define a measure ν_z on $\overline{\mathscr{B}_v(U)}^U$ by

$$\nu_z(E) = \frac{1}{v(z)} \int_{\Gamma\times E} \lambda v(w)\, d\mu_z(\lambda, w).$$

By the Radon–Nikodým theorem we have that

$$f(z) = \int_{\overline{\mathscr{B}_v(U)}^U} f(w)\, d\nu_z(w)$$

and the first part of the result is proven.

Let $M = \sup_{w\in\mathscr{B}_v(U)} v(w)$ and E be a measurable subset of $\mathscr{B}_v(U)$. Then $|\nu_z(E)| \le M/v(z)$ and thus ν_z has total variation at most $4M/v(z)$. Taking $f \equiv 1$ we get that $\nu_z\big(\overline{\mathscr{B}_v(U)}^U\big) = 1$. □

If z belongs to $\mathscr{B}_v(U)$ then a result of Bauer, [3], (see [35, Proposition 1.4]) shows that the point mass $\varepsilon_{v(z)\delta_z}$ is the unique probability measure with support contained in $B_{\mathscr{H}_{v_o}(U)'}$ which represents z. However, it is not possible to 'lift' this result and conclude δ_z is the unique probability measure on U which satisfies $(*)$. To see this consider any strictly positive weight v on Δ which converges to 0 on the boundary of Δ. Let $z \in \mathscr{B}_v(\Delta)$ and let r be such that $|z| < r < 1$. By Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_{|\lambda|=r} \frac{f(\lambda)}{\lambda - z}\, d\lambda$$

for all $f \in \mathscr{H}_{v_o}(\Delta)$ and thus

$$\rho_z(E) = \frac{1}{2\pi i}\int_{E\cap\{\lambda:|\lambda|=r\}} \frac{d\lambda}{\lambda - z}$$

is another probability measure representing z.

Theorem 18 shows that the v-boundary is a determining set for $\mathscr{H}_{v_o}(U)$ in the sense that if $f, g \in \mathscr{H}_{v_o}(U)$ and $f = g$ on $\mathscr{B}_v(U)$ then $f = g$ on U.

The above result should be compared with [6, Proposition 3.2]. Theorem 18 and [27, Chapter 9] explain our use of the term "v-boundary".

7. The centralizer of weighted spaces of holomorphic functions

We introduce some notation of Behrends [4].

Definition 19. Let E be a Banach space and $T\colon E \to E$ be a continuous linear operator. Then T is a multiplier if every extreme point of the unit ball of E' is an eigenvalue of T^*. That is

$$T^*(e) = a_T(e)e$$

for some real or complex number $a_T(e)$ and every $e \in \mathrm{Ext}B_{E'}$. We let $\mathrm{Mult}(E)$ denote the set of all multipliers on E.

Definition 20. Let E be a Banach space. The centralizer of E, $Z(E)$, is the set of all $T \in \mathrm{Mult}(E)$ for which there is $\overline{T}$ in $\mathrm{Mult}(E)$ with $(\overline{T})^*(e) = \overline{a_T(e)}e$ for all $e \in \mathrm{Ext}B_{E'}$.

We say that $Z(E)$ is trivial if $Z(E) = \mathbf{K}.\mathrm{Id}$, ($\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ depending on whether E is a real or complex Banach space).

Let U be a bounded open subset of $\mathbf{C}^n$, v be a continuous strictly positive weight on U which converges to 0 on the boundary of U. Given $g \in \mathscr{H}^\infty(U)$ we let $M_g\colon \mathscr{H}_{v_o}(U) \to \mathscr{H}_{v_o}(U)$ be the pointwise multiplication operator

$$\big(M_g(f)\big)(z) = g(z)f(z).$$

Proposition 21. *Let U be a bounded connected open subset of $\mathbf{C}^n$, v be a continuous strictly positive weight on U which converges to 0 on the boundary of U. Suppose that the $\overline{\overset{\circ}{\mathscr{B}_v(U)}}$ is non-empty. Then $\mathrm{Mult}(\mathscr{H}_{v_o}(U)) = \{M_g : g \in \mathscr{H}^\infty(U)\}$ and $Z\big(\mathscr{H}_{v_o}(U)\big)$ is trivial.*

Proof. Suppose that $T \in \operatorname{Mult}\big(\mathscr{H}_{v_o}(U)\big)$. Then for $z \in \mathscr{B}_v(U)$ $v(z)\delta_z$ is an extreme point of the unit ball of $\mathscr{H}_{v_o}(U)'$. Therefore we have

$$T^* v(z)\delta_z = a(z)v(z)\delta_z$$

for some $a(z)$ in $\mathbf{C}$. Hence for each f in $\mathscr{H}_{v_o}(U)$ and z in $\mathscr{B}_v(U)$ we have that

$$(Tf)(z) = a(z)f(z).$$

Taking $f \equiv 1$ we see that a extends to a holomorphic function in $\mathscr{H}_{v_o}(U)$ which we also denote by a. By continuity we get that $(Tf)(z) = a(z)f(z)$ for all z in $\overline{\mathscr{B}_v(U)}$ and all $f \in \mathscr{H}_{v_o}(U)$ for some $a(z) \in \mathbf{C}$. Since $\overset{\circ}{\overline{\mathscr{B}_v(U)}}$ is non-empty the principle of analytic continuation implies that

$$(Tf)(z) = a(z)f(z)$$

for all $z \in U$. As

$$|a(z)| = \frac{\|T^*(\delta_z)\|}{\|\delta_z\|} \le \|T^*\|$$

a is bounded on U and this proves the first part of the proposition.

Suppose that $T = M_g$ is in $Z\big(\mathscr{H}_{v_o}(U)\big)$. Then $\overline{M}_g$ is also a multiplier and so

$$\big(\overline{M}_g(f)\big)(z) = \bar{g}(z)f(z) = M_h(f)(z) = h(z)f(z)$$

for all z in $\overset{\circ}{\overline{\mathscr{B}_v(U)}}\mathscr{B}(U)$ all $f \in \mathscr{H}_{v_o}(U)$ and some $h \in \mathscr{H}^\infty(U)$. Thus g is both analytic and conjugate analytic on $\overset{\circ}{\overline{\mathscr{B}_v(U)}}\mathscr{B}(U)$ and therefore must be constant. ▫

Corollary 22. *Let U be a bounded open subset of $\mathbf{C}^n$, v be a continuous strictly positive complete weight on U which converges to 0 on the boundary of U. Then $\operatorname{Mult}\big(\mathscr{H}_{v_o}(U)\big) = \{M_g : g \in \mathscr{H}^\infty(U)\}$ and $Z\big(\mathscr{H}_{v_o}(U)\big)$ is trivial.*

A different assumption also gives us trivial centralisers.

Proposition 23. *Let U be a bounded open subset of $\mathbf{C}^n$ and let v be a continuous strictly positive weight on U which converges to 0 on the boundary of U. Suppose that $\mathscr{H}^\infty(U)$ separates $\mathscr{H}_{v_o}(U)'$. Then $\operatorname{Mult}\big(\mathscr{H}_{v_o}(U)\big) = \{M_g : g \in \mathscr{H}^\infty(U)\}$.*

Proof. Arguing as in Proposition 21 we get a holomorphic function a in $\mathscr{H}_{v_o}(U)$ so that

$$(Tf)(z) = a(z)f(z)$$

for all z in $\overline{\mathscr{B}_v(U)}$ and all f in $\mathscr{H}_{v_o}(U)$. For each f in $\mathscr{H}^\infty(U)$ we have that af belongs to $\mathscr{H}_{v_o}(U)$. As

$$(Tf)(z) = a(z)f(z)$$

for all z in $\overline{\mathscr{B}_v(U)}$ applying Theorem 18 we see that

$$T(f)(z) = \int_{\overline{\mathscr{B}_v(U)}^U} T(f)(w)\, d\nu_z(w) = \int_{\overline{\mathscr{B}_v(U)}^U} a(w)f(w)\, d\nu_z(w) = a(z)f(z)$$

for all z in U. Hence, we have that

$$\langle T^*\delta_z, f\rangle = \langle a(z)\delta_z, f\rangle$$

for all z in U and all f in $\mathscr{H}^\infty(U)$. Since $\mathscr{H}^\infty(U)$ separates $\mathscr{H}_{v_o}(U)'$ we have that $T^*\delta_z = a(z)\delta_z$ for all z in U.

The remainder of the proposition follows as in Proposition 21. □

In particular we get:

Proposition 24. *Let U be a balanced bounded open subset of $\mathbf{C}^n$ and v be a continuous strictly positive radial weight which converges to 0 on the boundary of U. Then* $\mathrm{Mult}\big(\mathscr{H}_{v_o}(U)\big) = \{M_g : g \in \mathscr{H}^\infty(U)\}$. *Furthermore, when $n = 1$ $Z\big(\mathscr{H}_{v_o}(U)\big)$ is trivial.*

Proof. It is shown in [5, Proposition 1.2] that polynomials are dense in $\mathscr{H}_{v_o}(U)$. They therefore will separate $\mathscr{H}_{v_o}(U)'$. The first part of the result now follows from Proposition 23. Let $n = 1$ and suppose that $T = M_g$ is in the centraliser of $\mathscr{H}_{v_o}(U)$. As v is radial, $\mathscr{B}_v(U)$ contains a circle and so we have h in $\mathscr{H}^\infty(U)$ so that

$$\big(\overline{M}_g(f)\big)(z) = \overline{g}(z)f(z) = M_h(f)(z) = h(z)f(z)$$

for all z in $\overline{\mathscr{B}_v(U)}^{\circ}$ and all $f \in \mathscr{H}_{v_o}(U)$. Consider z in $\overline{\mathscr{B}_v(U)}^{\circ}$ with $|z| = r$. As v is radial we have $\overline{g}(z) = h(z)$ for $|z| = r$. Taking a Poisson integral we get $\overline{g}(z) = h(z)$ for $|z| \le r$ which proves that all g for which M_g is in the centraliser of $\mathscr{H}_{v_o}(U)$ are constant. □

From [24, Theorem III.2.3] we get:

Proposition 25. *Let U be a balanced bounded open subset of $\mathbf{C}$, v be a continuous strictly positive radial weight on U which converges to 0 on the boundary of U. Then $Z\big(\mathscr{H}_v(U)\big)$ is trivial.*

Definition 26. A Banach space E is a C_σ-space if there is a compact Hausdorff set K and an involutory homeomorphism $\sigma\colon K \to K$ ($\sigma^2 = \mathrm{Id}$) such that E is isometrically isomorphic to

$$\big\{f \in C(K) : f(x) = -f\big(\sigma(x)\big) \text{ for all } x \in K\big\}.$$

From Proposition 21 and [24, Theorem II.5.9] we get:

Proposition 27. *Let U be a bounded open subset of $\mathbf{C}^n$, v be a continuous strictly positive complete weight on U which converges to 0 on the boundary of U. Then $\mathscr{H}_{v_o}(U)$ is not a C_σ-space.*

The above proposition is also valid for radial weights on balanced bounded open subsets of $\mathbf{C}$.

Given a Banach space E we shall use Z_E to denote $\overline{\mathrm{Ext} B_{E'}}^{\sigma^*} \setminus \{0\}$. For a Banach space E there is a canonical isometric embedding of E into $C_o(Z_E)$ (see [24, Examples I.3.4]). We have seen that $\overline{\mathrm{Ext} B_{\mathscr{H}_v(U)'}}^{\sigma^*} \setminus\{0\}$ may be identified with $\overline{\mathscr{B}_v(U)}$. Thus in this case we are identifying $\mathscr{H}_{v_o}(U)$ with a subspace of $C_o(U)$.

Definition 28. A Banach space E has the strong Banach–Stone property if given locally compact Hausdorff spaces X and Y and an isometric isomorphism $T\colon C_o(X;E) \to C_o(Y;E)$ there is a homeomorphism $\phi\colon Y \to X$ and a continuous function h from Y into the isometries of E endowed with the strong operator topology such that

$$\big(T(f)\big)(y) = h(x) f\big(\phi(y)\big)$$

for all $f \in C_o(X,E)$ and all $y \in Y$.

We have:

Proposition 29. *Let U be a bounded open subset of $\mathbf{C}^n$, v be a continuous strictly positive weight on U which converges to 0 on the boundary of U. Suppose that $\overline{\overset{\circ}{\mathscr{B}_v}(U)}$ is non-empty. Then $\mathscr{H}_{v_o}(U)$ has the strong Banach-Stone property. Alternatively, if U is a balanced bounded open subset of $\mathbf{C}$ and v is a continuous strictly positive radial weight then $\mathscr{H}_{v_o}(U)$ and $\mathscr{H}_v(U)$ have the strong Banach–Stone property.*

Proof. Apply Proposition 21, Proposition 24, [4, Theorem 8.11] and Proposition 25. □

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