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SPHERICAL HARMONICS AND MAXIMAL ESTIMATES FOR THE SCHRÖDINGER EQUATION

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Abstract. Maximal estimates are considered for solutions to an initial value problem for the Schrödinger equation. The initial value function is assumed to be a linear combination of products of radial functions and spherical harmonics. This generalizes the case of radial functions. We also replace the solutions to the Schrödinger equation by more general oscillatory integrals.

1. Introduction

Let f belong to the Schwartz space $\mathscr{S}(\mathbf{R}^n)$ and set

$$S_t f(x) = u(x,t) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\cdot\xi}\, e^{it|\xi|^a} \hat{f}(\xi)\, d\xi, \quad x \in \mathbf{R}^n,\ t \in \mathbf{R},$$

where $a > 1$. Here $\hat{f}$ denotes the Fourier transform of f, defined by

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} e^{-i\xi\cdot x} f(x)\, dx.$$

It then follows that $u(x,0) = f(x)$ and in the case $a = 2$ the function u is a solution to the Schrödinger equation $i\,\partial u/\partial t = \Delta u$. We shall here consider the maximal functions

$$S^* f(x) = \sup_{0<t<1} |S_t f(x)|, \quad x \in \mathbf{R}^n,$$

and

$$S^{**} f(x) = \sup_{t>0} |S_t f(x)|, \quad x \in \mathbf{R}^n.$$

We shall also introduce Sobolev spaces H_s by setting

$$H_s = \bigl\{ f \in \mathscr{S}' : \|f\|_{H_s} < \infty \bigr\}, \quad s \in \mathbf{R},$$

where

$$\|f\|_{H_s} = \left(\int_{\mathbf{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2\, d\xi \right)^{1/2}.$$

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We shall also consider homogeneous Sobolev spaces $\dot{H}_s$ defined by

$$\dot{H}_s = \{f \in \mathscr{S}' : \|f\|_{\dot{H}_s} < \infty\}, \quad s \in \mathbf{R},$$

where

$$\|f\|_{\dot{H}_s} = \left(\int_{\mathbf{R}^n} |\xi|^{2s} \big|\hat{f}(\xi)\big|^2 d\xi \right)^{1/2}.$$

We shall here study the local and global estimates

$$\|S^* f\|_{L^q(B)} \le C_B \|f\|_{H_s}, \tag{1}$$

$$\|S^* f\|_{L^q(\mathbf{R}^n)} \le C \|f\|_{H_s}, \tag{2}$$

$$\|S^{**} f\|_{L^q(B)} \le C_B \|f\|_{H_s}, \tag{3}$$

and

$$\|S^{**} f\|_{L^q(\mathbf{R}^n)} \le C \|f\|_{\dot{H}_s}, \tag{4}$$

where B denotes an arbitrary ball in $\mathbf{R}^n$. We shall always assume $1 \le q \le \infty$ and $s \in \mathbf{R}$. Estimates of this type have been considered by P. Sjölin [4], [5], [6], [7], [8], [9], and F. Gülkan [3], and by several other authors. We do not give a complete list of references but refer to the references in the mentioned papers. The case when f is assumed to be radial is studied in some of the above papers. We shall here generalize the case of radial functions. We recall that $L^2(\mathbf{R}^n) = \sum_{k=0}^{\infty} \oplus \mathscr{D}_k$, where $\mathscr{D}_k$ is the space of all linear combinations of functions of the form fP, where f ranges over the radial functions and P over the solid spherical harmonics of degree k, so that fP belongs to $L^2(\mathbf{R}^n)$ (see Stein and Weiss [10, p. 151]).

Now fix $k \ge 0$ and let $P_1, P_2, \dots, P_{a_k}$ denote an orthonormal basis for the space of solid spherical harmonics of degree k (where we use the inner product in $L^2(S^{n-1})$). The elements in $\mathscr{D}_k$ can be written in the form

$$f(x) = \sum_{j=1}^{a_k} f_j(r) P_j(x), \quad \text{(here } r = |x|) \tag{5}$$

and

$$\int_{\mathbf{R}^n} |f(x)|^2 \, dx = \sum_1^{a_k} \int_0^\infty |f_j(r)|^2 r^{n+2k-1} \, dr.$$

From now on we shall assume $n \ge 2$ and use the convention that if g is a function on $[0,\infty)$ or $(0,\infty)$ we shall also use the notation g for the corresponding radial function in $\mathbf{R}^n$.

We shall now define spaces $\mathscr{H}_k = \mathscr{H}_k(\mathbf{R}^n)$ for $k = 0, 1, 2, \dots$. We let $\mathscr{H}_0$ denote the class of all radial functions in $\mathscr{S}(\mathbf{R}^n)$. For $k \ge 1$ we define $\mathscr{H}_k$ as the space of functions f given by (5) with $f_j \in \mathscr{S}(\mathbf{R}^n)$ for $j = 1, 2, \dots, a_k$.

We shall here study the inequalities

$$\|S^*f\|_{L^q(B)} \le C_B\|f\|_{H_s}, \quad f \in \mathscr{H}_k, \tag{6}$$

$$\|S^*f\|_{L^q(\mathbf{R}^n)} \le C\|f\|_{H_s}, \quad f \in \mathscr{H}_k, \tag{7}$$

$$\|S^{**}f\|_{L^q(B)} \le C_B\|f\|_{H_s}, \quad f \in \mathscr{H}_k, \tag{8}$$

and

$$\|S^{**}f\|_{L^q(\mathbf{R}^n)} \le C\|f\|_{\dot{H}_s}, \quad f \in \mathscr{H}_k, \tag{9}$$

where the constants may depend on k.

To formulate our results we introduce a set $E = E_k$ of pairs (s,q) in the following way (where we only consider q with $1 \le q \le \infty$):

If $s < \frac{1}{4}$ then $(s,q) \in E$ for no q.

If $\frac{1}{4} \le s < \frac{1}{2}n$ then $(s,q) \in E$ if and only if $q \le 2n/(n-2s)$.

If $s = \frac{1}{2}n$ and $k = 0$ then $(s,q) \in E$ if and only if $q < \infty$.

If $s = \frac{1}{2}n$ and $k \ge 1$ then $(s,q) \in E$ for all q.

If $s > \frac{1}{2}n$ then $(s,q) \in E$ for all q.

We then have the following four theorems.

Theorem 1. *The local estimate* (6) *holds if and only if* $(s,q) \in E$.

Theorem 2. *The global estimate* (7) *holds if* $(s,q) \in E$, *and* $q = 4n/(2n-1)$ *for* $s = \frac{1}{4}$, *and* $q > 4(a-1)n/\bigl(4s + a(2n-1) - 2n\bigr)$ *for* $s > \frac{1}{4}$, *and also* $q \ge 2$. *If* $(s,q) \notin E$ *or if* $q < 4(a-1)n/\bigl(4s + a(2n-1) - 2n\bigr)$ *or if* $q < 2$, *the* (7) *does not hold.*

Theorem 3. *The local estimate* (8) *holds if and only if* $(s,q) \in E$.

Theorem 4. *If* $k = 0$ *the global estimate* (9) *holds if and only if* $\frac{1}{4} \le s < \frac{1}{2}n$ *and* $q = 2n/(n-2s)$. *If* $k \ge 1$ *the estimate* (9) *holds if and only if* $\frac{1}{4} \le s \le \frac{1}{2}n$ *and* $q = 2n/(n-2s)$.

Theorems 1, 3 and 4 imply that we have decided for which pairs (s,q) the estimates (6), (8) and (9) hold. Theorem 2 means that we have decided for which pairs (s,q) the global estimate (7) holds, except in the case

$$q = 4(a-1)n/\bigl(4s + a(2n-1) - 2n\bigr) \quad \text{and} \quad 1/4 < s \le a/4.$$

Remark. During the preparation of this paper we have learnt that some of the results in the paper have also been obtained by Y. Cho and Y. Shim.

2. Preliminaries

We let $\mathscr{F}_n$ denote the Fourier transformation in $\mathbf{R}^n$. Assume that $f \in \mathscr{H}_k$ so that

$$f(x) = \sum_1^{a_k} f_j(r) P_j(x), \quad r = |x|,$$

where $f_j \in \mathscr{S}(\mathbf{R}^n)$ and $(P_j)_1^{a_k}$ are as in the introduction. It then follows from [10, p. 158], that

$$\hat{f}(x) = \sum_1^{a_k} F_j(r) P_j(x), \quad r = |x|,$$

where

$$F_j(r) = c_k \, r^{1-n/2-k} \int_0^\infty f_j(s) J_{n/2+k-1}(rs) \, s^{n/2+k} \, ds, \quad r > 0,$$

and J_m denotes Bessel functions.

We then have

$$\begin{aligned} \int_{\mathbf{R}^n} |\hat{f}(\xi)|^2 (1+|\xi|^2)^s \, d\xi &= \int_0^\infty \left(\int_{S^{n-1}} |\hat{f}(r\xi')|^2 \, d\theta(\xi') \right) (1+r^2)^s r^{n-1} \, dr \\ &= \int_0^\infty r^{2k} \left(\sum_j |F_j(r)|^2 \right) (1+r^2)^s r^{n-1} \, dr \\ &= \sum_j \int_0^\infty |F_j(r)|^2 (1+r^2)^s r^{2k+n-1} \, dr, \end{aligned}$$

where θ denotes the area measure on S^{n-1}. It follows that

$$\|f\|_{H_s} = \left(\sum_j \int_0^\infty |F_j(r)|^2 \, (1+r^2)^s \, r^{2k+n-1} \, dr \right)^{1/2}$$

and in the same way one obtains

$$\|f\|_{\dot{H}_s} = \left(\sum_j \int_0^\infty |F_j(r)|^2 \, r^{2s+2k+n-1} \, dr \right)^{1/2}.$$

3. The three basic results

We first mention that it is proved in [3] that

$$\|S^{**}f\|_{L^q(\mathbf{R}^n)} \le C\|f\|_{H_s}, \quad q = 2n/(n-2s),\ n/4 \le s < n/2, \tag{10}$$

for arbitrary functions $f \in \mathscr{S}(\mathbf{R}^n)$.

We shall in this section prove the three basic results

$$\|S^{*}f\|_{L^2(\mathbf{R}^n)} \le C\|f\|_{H_s}, \qquad s > a/4,\ f \in \mathscr{H}_k, \tag{11}$$

$$\|S^{**}f\|_{L^q(\mathbf{R}^n)} \le C\|f\|_{H_{1/4}}, \quad q = 4n/(2n-1),\ f \in \mathscr{H}_k, \tag{12}$$

and

$$\|S^{**}f\|_{L^\infty(\mathbf{R}^n)} \le C\|f\|_{\dot H_{n/2}}, \quad f \in \mathscr{H}_k,\ k \ge 1. \tag{13}$$

The sufficiency part in our theorems will then follow from these results by use of interpolation. Before proving the basic results we observe that for $f \in \mathscr{H}_k$ we have (using the notation in Section 2)

$$\begin{aligned} S_t f(x) &= (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} e^{it|\xi|^a} \Big(\sum_j F_j(|\xi|)P_j(\xi)\Big)\,d\xi \\ &= \sum_j (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} \big(e^{it|\xi|^a} F_j(|\xi|)P_j(\xi)\big)\,d\xi \\ &= \sum_j c_k s^{1-n/2-k} \Big(\int_0^\infty J_{n/2+k-1}(rs) e^{itr^a} F_j(r) r^{n/2+k}\,dr\Big) P_j(x), \end{aligned} \tag{14}$$

where $s = |x| > 0$. Using the fact that $F_j = c_k\,\mathscr{F}_{n+2k} f_j$ we obtain

$$S^* f(x) \le C \sum_j \big(S^*_{n+2k} f_j(s)\big) s^k, \quad s > 0, \tag{15}$$

where f_j on the right-hand side is considered as a radial function in $\mathbf{R}^{n+2k}$, and we write S^*_{n+2k} to emphasize that the operator acts on functions in $\mathbf{R}^{n+2k}$.

The first basic result (11) has been proved in [7, pp. 59–61], in the case $k = 0$, and we shall use the inequality (15) to obtain (11) for $k \ge 1$. The idea is to use a result for functions in $\mathscr{H}_0(\mathbf{R}^{n+2k})$ to obtain a result for functions in $\mathscr{H}_k(\mathbf{R}^n)$.

For $k \geq 1$ and $f \in \mathscr{H}_k$ and invoking (15) one obtains

$$\begin{aligned}
\|S^*f\|^2_{L^2(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} |S^*f|^2\,dx \leq C\sum_j \int_{\mathbf{R}^n} |S^*_{n+2k}f_j(v)|^2 v^{2k}\,dx \\
&= C\sum_j \int_0^\infty |S^*_{n+2k}f_j(v)|^2 v^{2k+n-1}\,dv \\
&= C\sum_j \int_{\mathbf{R}^{n+2k}} |S^*_{n+2k}f_j(v)|^2\,dx \leq C\sum_j \|f_j\|^2_{H_s(\mathbf{R}^{n+2k})} \\
&= C\sum_j \int_{\mathbf{R}^{n+2k}} |F_j|^2(1+r^2)^s\,d\xi \\
&= C\sum_j \int_0^\infty |F_j|^2(1+r^2)^s r^{n+2k-1}\,dr = C\|f\|^2_{H_s},
\end{aligned}$$

if $s > \frac{1}{4}a$, where we have used the notation $v = |x|$ and $r = |\xi|$. Hence the first basic result (11) is proved for all k.

To prove the second basic result (12) we shall again use the idea that a result for $\mathscr{H}_0(\mathbf{R}^{n+2k})$ can be used to obtain a result for $\mathscr{H}_k(\mathbf{R}^n)$. However, the situation is now somewhat more complicated since we are no longer dealing with L^2 estimates and since the parameter q in (12) depends on the dimension n. We first observe that the argument that gave (15) also yields

$$S^{**}f(x) \leq C\sum_j \bigl(S^{**}_{n+2k}f_j(s)\bigr)s^k, \quad s > 0, \tag{16}$$

for $f \in \mathscr{H}_k$, $k \geq 1$, where again $s = |x|$.

It is proved in [6] that

$$\|S^*f\|_{L^q(B)} \leq C_B\|f\|_{H_{1/4}}, \quad q = 4n/(2n-1),$$

for radial functions. It is also observed in [3] and [9] that the proof in [6] can be modified to give the second basic result (12) for $k = 0$.

In [6] one also has the weighted estimate

$$\left(\int_B |S^*f(x)|^q|x|^\alpha\,dx\right)^{1/q} \leq C_B\|f\|_{H_{1/4}}, \quad \alpha = q(2n-1)/4 - n,$$

for $2 \leq q \leq 4$ and f radial. In [3] it is observed that the proof in [6] also gives the estimate

$$\left(\int_B |S^{**}f(x)|^q|x|^\alpha\,dx\right)^{1/q} \leq C_B\|f\|_{H_{1/4}} \tag{17}$$

for the same values of q and α and f radial. We shall prove that B can be replaced by $\mathbf{R}^n$ in this inequality. For $f \in \mathscr{H}_0(\mathbf{R}^n)$ we define f_N by

$$\hat{f}_N(\xi) = \hat{f}(\xi/N), \quad N \geq 1.$$

It is then easy to see that

$$S^{**} f_N(x) = N^n \, S^{**} f(Nx) \tag{18}$$

and replacing f by f_N in (17) we obtain

$$\left(\int_B |S^{**} f_N(x)|^q |x|^\alpha \, dx \right)^{1/q} \leq C_B \|f_N\|_{H_{1/4}}. \tag{19}$$

Choosing B as the unit ball we conclude that the left-hand side in (19) equals

$$N^n \left(\int_B |S^{**} f(Nx)|^q |x|^\alpha \, dx \right)^{1/q} = N^{n-\alpha/q-n/q} \left(\int_{|y|\leq N} |S^{**} f(y)|^q |y|^\alpha \, dy \right)^{1/q}.$$

On the other hand the right-hand side in (19) is equal to

$$C\left(\int_{\mathbf{R}^n} (1+|\xi|^2)^{1/4} |\hat{f}(\xi/N)|^2 d\xi \right)^{1/2}$$
$$= CN^{n/2+1/4} \left(\int_{\mathbf{R}^n} \left(\frac{1}{N^2} + |\eta|^2 \right)^{1/4} |\hat{f}(\eta)|^2 \, d\eta \right)^{1/2} \leq CN^{n/2+1/4} \|f\|_{H_{1/4}}.$$

Observing that $n - \alpha/q - n/q = n/2 + 1/4$ we then obtain

$$\left(\int_{|y|\leq N} |S^{**} f(y)|^q |y|^\alpha \, dy \right)^{1/q} \leq C\|f\|_{H_{1/4}}.$$

Letting $N \to \infty$ we conclude that

$$\left(\int_{\mathbf{R}^n} |S^{**} f(x)|^q |x|^\alpha \, dx \right)^{1/q} \leq C\|f\|_{H_{1/4}(\mathbf{R}^n)}, \quad \alpha = q(2n-1)/4 - n, \tag{20}$$

for $2 \leq q \leq 4$ and $f \in \mathscr{H}_0(\mathbf{R}^n)$. Replacing n by $n+2k$ we obtain

$$\left(\int_{\mathbf{R}^{n+2k}} |S^{**}_{n+2k} f_j(x)|^2 |x|^\alpha \, dx \right)^{1/q} \leq C\|f_j\|_{H_{1/4}(\mathbf{R}^{n+2k})},$$

for $2 \le q \le 4$ and $\alpha = q\big(2(n+2k)-1\big)/4-(n+2k)$, where f_j is considered as a radial function in $\mathbf{R}^{n+2k}$. Assuming $f=\sum_j f_jP_j$ as usual, we then also get

$$\begin{aligned}\left(\int_0^\infty |S^{**}_{n+2k}f_j(s)|^q s^\alpha s^{n+2k-1}\,ds\right)^{1/q} &\le C\left(\int_0^\infty |F_j|^2(1+r^2)^{1/4}r^{n+2k-1}\,dr\right)^{1/2}\\ &\le C\|f\|_{H_{1/4}(\mathbf{R}^n)} \end{aligned}\tag{21}$$

for the same values of q and α.

Now take $2\le q\le 4$, $\beta = q(2n-1)/4-n$, and $f\in \mathscr{H}_k(\mathbf{R}^n)$, $k\ge 1$. Invoking (16) one obtains

$$\begin{aligned}\left(\int_{\mathbf{R}^n} |S^{**}f(x)|^q|x|^\beta\,dx\right)^{1/q} &\le C\sum_j\left(\int_{\mathbf{R}^n} |S^{**}_{n+2k}f_j(s)|^q s^{kq}s^\beta\,dx\right)^{1/q}\\ &= C\sum_j\left(\int_0^\infty |S^{**}_{n+2k}f_j(s)|^q s^{kq+\beta+n-1}\,ds\right)^{1/q}. \end{aligned}\tag{22}$$

We now choose α so that $\alpha+n+2k-1 = kq+\beta+n-1$, which gives

$$\begin{aligned}\alpha &= kq+\beta-2k = kq+\frac{q}{4}(2n-1)-n-2k\\ &= \frac{q}{4}(2(n+2k)-1)-(n+2k).\end{aligned}$$

It follows that the right-hand side in (22) equals

$$C\sum_j\left(\int_0^\infty |S^{**}_{n+2k}f_j(s)|^q s^{\alpha+n+2k-1}\,ds\right)^{1/q}$$

and invoking (21) we conclude that this is dominated by $C\|f\|_{H_{1/4}(\mathbf{R}^n)}$.

Hence we have proved that (20) holds for $f\in\mathscr{H}_k(\mathbf{R}^n)$, $k\ge 1$ (cf. [11, p. 26]). Taking $q=4n/(2n-1)$ we obtain $\alpha=0$ in (20) and the second basic result (12) follows also for $k\ge 1$.

It remains to prove the third basic result (13). We first remark that the estimate

$$\|S^{**}f\|_{L^\infty(\mathbf{R}^n)} \le C\|f\|_{\dot H_{n/2}}\tag{23}$$

does not hold for $f\in\mathscr{H}_0$. To see this let χ_m, $m=5,6,7,\dots$, denote C^∞ functions on $[0,\infty)$ such that $0\le\chi_m(r)\le 1$ for all r and $\chi_m(r)=1$ for $3\le r\le m-1$, and $\chi_m(r)=0$ for $0\le r\le 2$ and for $r\ge m$. Then define $f_m\in\mathscr{H}_0(\mathbf{R}^n)$ by setting

$$\hat f_m(r) = \frac{1}{r^n\log r}\chi_m(r).$$

Then the Fourier inversion formula yields

$$f_m(0) \geq c \int_3^{m-1} \frac{1}{r^n \log r} r^{n-1}\, dr = c \int_3^{m-1} \frac{1}{r \log r}\, dr \to \infty$$

as $m \to \infty$. On the other hand we also have

$$\begin{aligned} \|f_m\|^2_{\dot{H}_{n/2}} &= \int_{\mathbf{R}^n} |\hat{f}_m(\xi)|^2 |\xi|^n \, d\xi \\ &\leq C \int_2^m \frac{1}{r^{2n}(\log r)^2} r^{2n-1}\, dr \\ &\leq C \int_2^\infty \frac{1}{r(\log r)^2}\, dr = C \end{aligned}$$

for all m, and it follows that (23) does not hold for radial functions. We shall then prove (23) for $f \in \mathscr{H}_k$, $k \geq 1$. Assuming $f = \sum_j f_j P_j \in \mathscr{H}_k$ we have

$$|S_t f(x)| \leq C \sum_j s^{1-n/2} \left| \int_0^\infty J_{n/2+k-1}(rs) e^{itr^a} F_j(r) r^{n/2+k}\, dr \right|, \quad s > 0,$$

according to (14). Hence

$$\begin{aligned} S^{**} f(x) &\leq C \sum_j s^{1-n/2} \int_0^\infty |J_{n/2+k-1}(rs)|\, |F_j(r)|\, r^{n/2+k}\, dr \\ &= C \sum_j A_j + C \sum_j B_j, \end{aligned}$$

where

$$A_j = s^{1-n/2} \int_0^{1/s} |J_{n/2+k-1}(rs)|\, |F_j(r)|\, r^{n/2+k}\, dr$$

and

$$B_j = s^{1-n/2} \int_{1/s}^\infty |J_{n/2+k-1}(rs)|\, |F_j(r)|\, r^{n/2+k}\, dr$$

for $s > 0$.

Invoking standard estimates for Bessel functions (see [10, p. 158]) one then obtains

$$\begin{aligned} A_j &\leq C s^{1-n/2} \int_0^{1/s} (rs)^{n/2+k-1} r^{n/2+k}\, |F_j(r)|\, dr \\ &= C s^k \int_0^{1/s} |F_j(r)| r^{n+2k-1}\, dr \end{aligned}$$

and

$$\begin{aligned}B_j &\le Cs^{1-n/2}\int_{1/s}^{\infty}(rs)^{-1/2}r^{n/2+k}\,|F_j(r)|\,dr\\ &= Cs^{1/2-n/2}\int_{1/s}^{\infty}|F_j(r)|r^{n/2+k-1/2}\,dr.\end{aligned}$$

Applying the Schwarz inequality we then get

$$\begin{aligned}A_j &\le Cs^k\int_0^{1/s}|F_j(r)|r^{n+k-1/2}r^{k-1/2}\,dr\\ &\le Cs^k\biggl(\int_0^{\infty}|F_j(r)|^2r^{2n+2k-1}dr\biggr)^{1/2}\biggl(\int_0^{1/s}r^{2k-1}\,dr\biggr)^{1/2}\\ &\le Cs^k\|f\|_{\dot H_{n/2}}s^{-k}=C\|f\|_{\dot H_{n/2}},\end{aligned}$$

where we have used the fact that $k\ge 1$ and also the fact that

$$\|f\|_{\dot H_{n/2}}=\biggl(\sum_j\int_0^{\infty}|F_j(r)|^2r^{2k+2n-1}\,dr\biggr)^{1/2}.$$

Invoking the Schwarz inequality again we also obtain

$$\begin{aligned}B_j &\le Cs^{1/2-n/2}\int_{1/s}^{\infty}|F_j(r)|r^{n+k-1/2}r^{-n/2}\,dr\\ &\le Cs^{1/2-n/2}\|f\|_{\dot H_{n/2}}\biggl(\int_{1/s}^{\infty}r^{-n}\,dr\biggr)^{1/2}\\ &\le Cs^{1/2-n/2}\|f\|_{\dot H_{n/2}}(s^{n-1})^{1/2}=C\|f\|_{\dot H_{n/2}}.\end{aligned}$$

We have proved that

$$S^{**}f(x)\le C\|f\|_{\dot H_{n/2}}$$

and the third basic result (13) follows.

4. Five counter-examples

To prove the necessity part in the theorems we shall use the following five statements.

Statement 1. If $s<\frac14$ then the local estimate (6) holds for no q.

Proof. We shall use the method in [7, pp. 55–58]. From the formula (14) for $S_tf(x)$ we conclude that it is sufficient to prove that there is no inequality

$$\int_0^1|TF(u)|u^{n-1}\,du\le C\biggl(\int_0^{\infty}|F(r)|^2(1+r^2)^sr^{2k+n-1}\,dr\biggr)^{1/2}\tag{24}$$

for $s < \frac{1}{4}$, where

$$TF(u) = u^{1-n/2} \int_0^\infty J_{n/2+k-1}(ru) e^{it(u)r^a} F(r) r^{n/2+k}\, dr, \quad 0 < u \le 1.$$

Here $t(u)$ is a measurable function on $(0,1]$ taking values in $(0,1)$.

We let $\varphi \in C_0^\infty(\mathbf{R})$ with $\operatorname{supp}\varphi \subset (-1,1)$ and choose F such that

$$F(r) r^k = N^{-1/2} \varphi(-N^{-1/2} r + N^{1/2})\, r^{1/2-n/2}, \quad r > 0.$$

It is proved in [7] that the right-hand side of (24) is less than $CN^{s-1/4}$ for N large. The proof in [7] also shows that the left-hand side of (24) is bounded from below for a suitable choice of the functions φ and $t(u)$. It follows that (24) cannot hold for $s < \frac{1}{4}$.

Statement 2. If $1/4 \le s < n/2$ then $q \le 2n/(n-2s)$ is a necessary condition for the local estimate (6).

Proof. The statement is proved for $k = 0$ in [7, pp. 58–59], and we shall prove that a modification of the method in [7] works also for $k \ge 1$. Assume that $f = f_1 P_1 \in \mathscr{H}_k$ so that $\hat{f} = F_1 P_1$ with $F_1 = c_k \mathscr{F}_{n+2k} f_1$. Then let $\varphi \in C_0^\infty(\mathbf{R}^n)$ be radial and non-negative and assume that $\operatorname{supp}\varphi \subset \{\xi : 1 < |\xi| < 2\}$ and that $\varphi(\xi) = 1$ for $\frac{5}{4} \le |\xi| \le \frac{7}{4}$. Then choose f_1 so that $F_1(\xi) = \varphi(\xi/N)$. It is then easy to see that

$$\|f\|_{H_s} \le CN^{n/2+s+k}$$

for large values of N.

One also has

$$S_0 f = f = f_1 P_1 = c_k \big(\mathscr{F}_{n+2k}(\varphi(\xi/N))\big) P_1$$

and

$$S_0 f(x) = c_k N^{n+2k} (\mathscr{F}_{n+2k}\varphi)(Nx) P_1(x).$$

Then choose $\delta > 0$ so small that

$$|(\mathscr{F}_{n+2k}\varphi)(x)| \ge c$$

for $|x| \le \delta$. It follows that

$$S^* f(x) \ge cN^{n+k}$$

for

$$\frac{\delta}{2N} \le |x| \le \frac{\delta}{N}$$

and $|x' - x_0'| \le c_1$, if x_0' is suitably chosen (here $x' = x/|x|$).

If (6) holds we obtain

$$\left(\int_{\delta/(2N)\le|x|\le\delta/N,\,|x'-x_0'|\le c_1} N^{(n+k)q}\,dx\right)^{1/q} \le C\,N^{n/2+s+k}$$

and

$$N^{n+k-n/q} \le CN^{n/2+s+k}.$$

Hence

$$N^{n-n/q} \le CN^{n/2+s}$$

and letting $N\to\infty$ we obtain

$$n-\frac{n}{q}\le\frac{n}{2}+s.$$

It follows that $q\le 2n/(n-2s)$.

Statement 3. Assume that $\frac14\le s\le\frac14 a$ and that the global estimate (7) holds. Then one has

$$q\ge\frac{4(a-1)n}{4s+a(2n-1)-2n}. \tag{25}$$

Proof. The statement is proved in [7, pp. 62–65], in the case $k=0$. For $k\ge1$ we generalize the method in [7]. First let $\varphi\in C_0^\infty(\mathbf{R})$ with $\operatorname{supp}\varphi\subset(-1,1)$ and choose $f\in\mathscr{H}_k$ such that

$$\hat f(\xi)=\varphi\big(-N^{a/2-1}r+N^{a/2}\big)r^{1/2-n/2}r^{-k}P_1(\xi),$$

where $r=|\xi|$. It is then easy to see that

$$\|f\|_{H_s}\le CN^{s+1/2-a/4}.$$

Using formula (14) for S_tf the above mentioned argument in [7] then gives the inequality (25).

Statement 4. A necessary condition for the global estimate (7) is $q\ge2$.

Proof. This is proved for $k=0$ in [7]. To extend this result to the case $k\ge1$ we construct a function f in the following way. Let $\psi\in C^\infty(\mathbf{R})$ and assume that $\psi(t)=0$, $t\le2$, and $\psi(t)=1$, $t\ge3$. Set $f(x)=0$, $|x|\le2$, and

$$f(x)=\frac{1}{r^{n/2}\log r}\psi(r)r^{-k}P_1(x),\quad |x|\ge2,$$

where $r=|x|$. Then $f\in H_s$ for every s but $f\notin L^q$ if $q<2$, and f can be used to prove the statement.

Statement 5. Assume that $s\ge\frac14$ and that the global estimate (9) holds. Then $s\le\frac12 n$ and $q=2n/(n-2s)$.

Proof. This is proved for $k=0$ in [9, pp. 135–136], and the same proof works in the general case.

5. Proofs of the theorems

In the proofs of the theorems we shall use interpolation. It follows from results in Bergh and Löfström [2, pp. 120–121], and Bennett and Sharpley [1, p. 213], that if the inequality (6) holds for two pairs (s_0, q_0) and (s_1, q_1), then it also holds for all pairs (s, q) with the property that the point $(s, 1/q)$ lies on the line segment between the points $(s_0, 1/q_0)$ and $(s_1, 1/q_1)$ in the plane. The same remark of course holds if the inequality (6) is replaced by one of the inequalities (7), (8) and (9).

Proof of Theorem 1. Interpolating between the inequality (10) and the second basic result (12) we conclude that (6) holds for all (s, q) with $\frac{1}{4} \le s < \frac{1}{2}n$ and $q = 2n/(n-2s)$.

A trivial estimate also shows that (6) holds for $s > \frac{1}{2}n$ and $q = \infty$.

We also use the observation that if (6) holds for a pair (s, q) then it holds for all pairs (s_1, q_1) with $s_1 \ge s$ and $q_1 \le q$.

Taking this into consideration and invoking the third basic result and the above counter-examples, one completes the proof of Theorem 1.

Proof of Theorem 2. Interpolating between the inequality (10) and the second basic result we first conclude that (7) holds for all (s, q) with $\frac{1}{4} \le s < \frac{1}{2}n$ and $q = 2n/(n-2s)$. We then interpolate between this result and the first basic result (11). The rest of the proof is easy if we use the counter-examples in Section 4. The condition $q > 4(a-1)n/\big(4s + a(2n-1) - 2n\big)$ comes from the fact that $q = 4(a-1)n/\big(4s + a(2n-1) - 2n\big)$ if the point $(s, 1/q)$ lies on the straight line which connects the pairs $(1/4, (2n-1)/4n)$ and $(a/4, 1/2)$.

Proof of Theorem 3. The proof is essentially the same as the proof of Theorem 1.

Proof of Theorem 4. We first observe that the proof of Statement 1 also shows that if $s < \frac{1}{4}$ then the global estimate (9) holds for no q. Statement 5 then implies that $\frac{1}{4} \le s \le \frac{1}{2}n$ and $q = 2n/(n-2s)$ is a necessary condition for (9). The necessity of the conditions in Theorem 4 then follows if we also invoke the counter-example given before the proof of the third basic result in Section 3. To prove the sufficiency we first interpolate between (10) and (12) to obtain the global estimate

$$\|S^{**}f\|_{L^q(\mathbf{R}^n)} \le C\|f\|_{H_s}, \quad f \in \mathscr{H}_k,$$

for $\frac{1}{4} \le s < \frac{1}{2}n$ and $q = 2n/(n-2s)$. Using the proof of Theorem 2.6 in [3] we can then conclude that we also have the homogeneous estimate (9) for the same values of s and q. To complete the proof of the theorem we then only have to invoke also the third basic result.

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