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AN INVERSE PROBLEM FOR TIME-HARMONIC ELECTROMAGNETIC CURRENTS IN A SMOOTHLY LAYERED MEDIUM

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Abstract. In this article we study the inverse source problem for Maxwell transmission problem; i.e. we study the uniqueness of constructing the electric and magnetic currents in a body with a smoothly layered medium from electromagnetic field measurements on the boundary of the body. We use Green's functions for Maxwell equations in this medium and the mapping properties of the operator corresponding to these Green's functions. Then the inverse problem is studied for a general class of currents belonging to the Sobolev spaces. These currents contain the currents on the surfaces of discontinuity in the medium. The uniqueness is achieved in many cases with the currents on a single surface.

1. Introduction

We consider in this work the following inverse problem: A bounded and nested body is given. The permittivity, conductivity and permeability are smooth functions in each domain between the intersurfaces and constant outside the nested body. The body is assumed to contain some electric and magnetic currents, harmonic with one frequency, that generate an electromagnetic field outside the body. The problem is to study the uniqueness of constructing these electric and magnetic currents from a certain set of electromagnetic field measurements on the boundary of the body. The currents are modelled by distributions belonging to the Sobolev spaces.

The main results in this paper are new at least to the author's knowledge. Especially, no paper in this field uses Sobolev space methods. Further, in this paper the currents may also be located on the surfaces of discontinuity in the medium. The question of the current on one surface causing zero measurement on the boundary is answered for a large amount of currents, also for the currents on the surface of discontinuity in the medium. The paper also contains some representation formulas that are interesting in themselves.

In the literature there are not many papers about inverse source problems for Maxwell equations. In [6] one studies the problem of determining the neural current from measurements of the magnetic field outside the head in the static case and shows the uniqueness with the assumption of energy minimization. In

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[10] one deals with the question of determining dipole sources in the static homogeneous case. The paper [9] gives a detailed discussion on the quasi-static model. The most recent paper [3] studies the problem of determining dipole point sources from measurements of the electric and magnetic field outside a bounded domain in the dynamic case. Recordings of the changes in electric and magnetic fields outside the skull by the electrical activity of the human brain are called the electroencephalogram (EEG) and magnetoencephalogram (MEG). Earlier, in [8] interdependence of the MEG and EEG was considered. One uses the quasistatic approximation of Maxwell equations. Several other authors, with some further prior knowledge, have studied this question, see e.g. [20]. In [14] the problem of the source detection was considered.

The closely related problem of identifying the crack in a conducting body at low frequencies is studied e.g. in [10] and [2]. In these papers the authors use the equivalent current distribution model in which the perturbed field is treated as the field given by a current distributed over the crack area. The question of determining a planar crack or equivalently the corresponding current on an open surface having the same normal unit vector on each of its points is studied in [2]. Some more specific questions of the same problem have been studied earlier in the literature. On the identification of cracks for the Laplace equation there are better mathematical results, see e.g. [1]. General inverse source problems are also studied in [22], [13] and [12].

Let us give a brief outline of the paper. In Chapter 4 we define potential operators and boundary integral operators with Maxwell Green's functions as kernels. We give extensions of these operators in Sobolev spaces.

In Chapter 5 we give a representation formula with electric and magnetic Green's functions as kernels of the operators. By using this we obtain a necessary and sufficient condition for the currents that give zero measurement on the boundary. Then we get an equivalent condition for this condition, which determines the currents as an ortogonal complement of a certain space. We also formulate the problem and give the corresponding results for a more general class of currents containing the currents on surfaces of discontinuity in the medium.

In Chapter 6 we study the currents, with one fixed frequency, that are located in smooth closed surfaces each of which is enclosing a simply connected domain. The uniqueness is proved, if the electric (or magnetic) current is on a single surface and the frequency is not an eigenfrequency of a certain Maxwell boundary value problem. If the frequency is an eigenfrequency, we obtain the currents that give zero measurement on the boundary. We also give some examples, where the location of the surface or the shape of the surface can be determined uniquely.

2. Definitions and preliminaries

We use the standard functional spaces, e.g. the L_2-based Sobolev-spaces $H^s(U)$, $H^s(\partial U)$ and the spaces $C_0^\infty(U)$ and $C^\infty(\overline{U})$, see e.g. [21]. Here, U is a bounded domain with a smooth boundary ∂U. For $u \in C^\infty(\overline{U})$ $\mathrm{Ex}(u)$ denotes the extension by zero into $\mathbf{R}^3 \setminus \overline{U}$ of u. For $s < 0$ we recall from [21] the definition of the Sobolev-spaces by duality: e.g. $H^s(U) := \big(H_0^{-s}(U)\big)'$, $H^s(\partial U) := \big(H^{-s}(\partial U)\big)'$. We also denote $H_{\overline{U}}^{-s}(\mathbf{R}^3) = \{f \in H^s(\mathbf{R}^3) \mid \operatorname{supp}(f) \subset \overline{U}\}$, for $s > 0$. We identify the spaces $\big(H^s(U)\big)'$ and $H_{\overline{U}}^{-s}(\mathbf{R}^3)$ with $C_0^\infty(U)$ a dense subspace, see [25, Chapter 12.6]. We use the prefix T or N to denote the corresponding tangential or normal space, e.g. $T\mathscr{D}'(\partial U) = \big\{u \in (\mathscr{D}'(\partial U))^3 \mid n\cdot u = 0\big\}$.

Let Div denote the surface divergence on ∂U. We define

$$TH^s_{\mathrm{Div}}(\partial U) = \big\{u \in TH^s(\partial U) \mid \mathrm{Div}(u) \in H^s(\partial U)\big\}, \quad s \in \mathbf{R},$$
$$H^s_{\mathrm{Div}}(U) = \big\{u \in (H^s(U))^3 \mid \mathrm{Div}(n \times u|_{\partial U}) \in H^{s-1/2}(\partial U)\big\}, \quad s > \tfrac{1}{2}.$$

For $s \le \frac{1}{2}$ we define the space $H^s_{\mathrm{Div}}(U)$ in the same way, now meaning those functions for which the trace is defined by Theorem 2.1. The Div-spaces are discussed to some extent in [7] and [19]. In this paper the notation $H^s(U)$ also denotes the space $\big(H^s(U)\big)^3$.

For an open set $U \subset \Omega$ we use the Hilbert spaces $\mathbf{H}^s_U := \prod_{i=0}^N H^s(\Omega_i \cap U)$ and $\mathbf{H}^s_{U,\mathrm{Div}} := \prod_{i=0}^N H^s_{\mathrm{Div}}(\Omega_i \cap U)$, with the norm $\|u\|^2_{\mathbf{H}^s_U} := \sum_{i=0}^N \|u\|^2_{H^s(\Omega_i \cap U)}$ and $\|u\|^2_{\mathbf{H}^s_{U,\mathrm{Div}}} := \sum_{i=0}^N \|u\|^2_{H^s_{\mathrm{Div}}(\Omega_i \cap U)}$, $s \in \mathbf{R}$. Then we define

$$MW^s(U) := \Bigg\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbf{H}^s_U \times \mathbf{H}^s_U \;\Bigg|\; \nabla \times w_1 = i\omega\mu w_2, \quad \nabla \times w_2 = -i\omega\gamma w_1; [n \times w_1]_{\Gamma_i} = 0, [n \times w_2]_{\Gamma_i} = 0 \Bigg\},$$

$$MW^s(\mathbf{R}^3 \setminus \overline{\Omega}) := \Bigg\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \overline{H}^s_{\mathrm{loc}}(\mathbf{R}^3 \setminus \overline{\Omega}) \times \overline{H}^s_{\mathrm{loc}}(\mathbf{R}^3 \setminus \overline{\Omega}) \;\Bigg|\; \nabla \times w_1 = i\omega\mu w_2, \quad \nabla \times w_2 = -i\omega\gamma w_1;\ (w_1, w_2) \text{ satisfy the Silver–Müller radiation condition} \Bigg\}.$$

Here $\overline{H}^s_{\mathrm{loc}}(\mathbf{R}^3 \setminus \overline{\Omega}) = \{u|_{\mathbf{R}^3 \setminus \overline{\Omega}}; u \in H^s_{\mathrm{loc}}(\mathbf{R}^3)\}$. Correspondingly, we define $MW^s_{\mathrm{Div}}(U)$ with the space $\mathbf{H}^s_{U,\mathrm{Div}}$ instead of $\mathbf{H}^s_U$.

The following trace theorem is proved in [16, Theorem 2.6.5 and Remark 2.6.5], for scalar operators and in [23, Theorem 7] for elliptic systems, see also [21, p. 213].

Theorem 2.1. *Let U be an open set with a smooth boundary ∂U and $s \in \mathbf{R}$. Furthermore, let P be an elliptic system of differential operators. The trace mapping $u \mapsto u|_{\partial U}$ of $C^\infty(\overline{U})$ into $C^\infty(\partial U)$ is defined in $\{u \in H^{s+1/2}(U) \mid Pu = 0 \text{ in } U\}$ so that $\|u|_{\partial U}\|_{H^s(\partial U)} \le c\|u\|_{H^{s+1/2}(U)}$.*

In this paper we also consider the tensor product $E\otimes F$ of the locally convex spaces E and F. Two different topologies for $E\otimes F$, which are identical if one of the spaces is nuclear, is given in [25]. If the spaces E, F are complete then $E\hat{\otimes}_\varepsilon F$ (the completition of the space $E\otimes F$ in ε-topology) can be identified with the closure of $E\otimes F$ in $\mathscr{B}_\varepsilon(E'_\sigma, F'_\sigma)$, the space of separately continuous bilinear forms on $E'_\sigma \times F'_\sigma$ provided with the ε-topology given in Chapter 43 in [25]. Here E'_σ is the weak dual of E. The following theorem is from [25, Proposition 50.5]. In this paper E' denotes the strong dual of E.

Theorem 2.2. *Let E and F be two locally convex Hausdorff spaces satisfying: E and F are complete, E is barreled and E' is nuclear and complete. Then $L(E;F)$ is complete, and we have $E'\hat{\otimes}F = L(E;F)$ (topological isomorfism). Here $L(E;F)$ is the vector space of continuous linear mappings from E into F with the topology of bounded convergence.*

In this paper the integral denotes also the operator given by Theorem 2.2; e.g. $\int_U A(x,y)u(y)\,dy$ with $A(x,y)\in \mathscr{D}'(U)\hat{\otimes}\mathscr{D}'(U)$ and $U\subset \mathbf{R}^3$ an open set.

3. Statement of the problem and results

We assume that the bounded body $\Omega\subset\mathbf{R}^3$ contains N domains Ω_i, $i=1,\dots,N$, such that $\overline{\Omega}=\cup\overline{\Omega}_i$, where Ω_1 is simply connected with a connected complement and Ω_1 is surrounded by Ω_2, etc. The boundaries $\Gamma_1,\dots,\Gamma_{N-1}$ between the domains $\Omega_1,\dots,\Omega_N$ are smooth closed surfaces. We also use the notation $\Omega_{N+1} := \mathbf{R}^3\setminus\overline{\Omega}$. The permittivity ε, conductivity σ and permeability μ, $\mu>0$, $\varepsilon>0$ and $\sigma\ge 0$, are smooth functions in each Ω_i and $\gamma=\varepsilon+i\sigma/\omega$ with ω a frequency. We have $\varepsilon=\varepsilon_0$, $\mu=\mu_0$ and $\sigma=0$ in $\mathbf{R}^3\setminus\overline{\Omega}$, where ε_0 and μ_0 are positive constants, and ε, σ and μ are smooth functions in a neighborhood of $\Gamma=\partial\Omega$.

Let ω, $\operatorname{Re}\omega$, $\operatorname{Im}\omega\ge 0$, be a fixed frequency. The direct problem is: Let the currents $J,M\in\mathbf{H}^s_\Omega$, be smooth in some interior and exterior neighborhood of each Γ_i and $J=M=0$ in $\mathbf{R}^3\setminus\overline{\Omega}$. Find the fields $E,H\in\mathbf{H}^s_\Omega\times\overline{H}^{\,s}_{\mathrm{loc}}(\mathbf{R}^3\setminus\overline{\Omega})$ that satisfy

$$\begin{aligned}&\nabla\times E = i\omega\mu H + M,\\ &\nabla\times H = -i\omega\gamma E + J \text{ in } \Omega_i,\quad i=1,\dots,N+1,\end{aligned}\tag{1}$$

with the transmission and the Silver–Müller radiation conditions

$$\begin{aligned}&[n\times E]_{\Gamma_i} = M_{i0},\quad [n\times H]_{\Gamma_i}=J_{i0}\quad \text{for } i=1,\dots,N,\\ &\mu_0\frac{x}{|x|}\times H+\sqrt{\varepsilon_0\mu_0}\,E = o(|x|^{-1}),\qquad E=O(|x|^{-1}),\ \text{as } |x|\to\infty,\end{aligned}\tag{2}$$

where $J_{i0},M_{i0}\in TH^{s+1/2}(\Gamma_i)$ and $[v]_{\Gamma_i} := v|^+_{\Gamma_i} - v|^-_{\Gamma_i}$ denotes the jump of v across Γ_i.

The transmission condition (2) ensues from the transition of the electromagnetic field through a surface of discontinuity, see e.g. [24, Section 1.13]. This problem is generalized in Definition 5.2. Theorem 5.2, (4) and (17) yield the existence of the solution of the problem (1)–(2).

The following inverse problem is considered: Let the traces $E|_\Gamma$ and $H|_\Gamma$ of the fields E and H in (1)–(2) be known. Does this determine the currents J, M, J_{i0} and M_{i0} uniquely? The currents that give zero measurement, i.e. $E|_\Gamma = H|_\Gamma = 0$, do not depend on the material outside the body, as can easily be seen. Thus the smoothness of the electromagnetic parameters in a neigborhood of Γ is not a restriction.

We give here one of the main results in this paper.

Theorem 3.1. *Let the currents J and M have the form in Theorem* 6.1 *with $m = 1$.*

(i) *Let $M \equiv 0$. Then the currents J_0 that satisfy* (18), *i.e. give zero measurement, are given by the condition $J_0 \in \mathscr{M}$, where $\mathscr{M} := \{n \times H|_{\Gamma_a} \mid (E,H) \in MW^s_{\mathrm{Div}}(\Omega_a),\ n \times E|_{\Gamma_a} = 0\}$ and Ω_a is the area surrounded by Γ_a; see Chapter* 6.

(ii) *Let $J \equiv 0$. Then the currents M_0 that satisfy* (18) *are given by the condition $M_0 \in \overline{sp}\{n \times E_4|_{\Gamma_a}\}$, where (E_4, H_4) is a solution of the problem*

$$(w_1, w_2) \in MW^s(\Omega_a) \quad \text{and} \quad n \times w_2|_{\Gamma_a} = c, \quad c \in TH^{s-1/2}_{\mathrm{Div}}(\Gamma_a), \tag{5}$$

with $c = 0$. Here $\overline{sp}$ denotes the space spanned by the vectors in the parenthesis.

This theorem is proved in Chapter 6. Theorem 5.2 determines the electromagnetic field given by the current located on surfaces of discontinuity in the medium. This result is unknown in the literature; Green's function (the kernel of the operator in Theorem 5.2) is used only by the author.

4. Green's functions and integral operators

We denote by $\mathscr{E}_E(x,y)$, $\mathscr{H}_E(x,y)$, $\mathscr{E}_H(x,y)$ and $\mathscr{H}_H(x,y)$, with the components $\mathscr{E}_{Ej}(x,y)$, $j = 1,2,3$, etc. electric and magnetic Green's functions considered as distributions in $\mathbf{R}^3$ with fixed $y \in \cup\Omega_i$. Here, $\big(\mathscr{E}_{Ej}(x,y), \mathscr{H}_{Ej}(x,y)\big)$ (correspondingly $(\mathscr{E}_{Hj}, \mathscr{H}_{Hj})$) satisfy the transmission problem (1)–(2) with $M_{i0} = J_{i0} = 0$ and $M = 0$, $J = \delta(x-y)e_j$, $J = 0$, $M = \delta(x-y)e_j$, with e_j, $j = 1,2,3$, the Cartesian coordinate vectors.

Green's functions are given in [17] and [18]. Let U be a bounded domain with the boundary Γ_a so that the electromagnetic parameters are smooth in U and in a neighborhood of Γ_a. In [18] we defined and used the operators

$$D_U u := \int_{\Gamma_a} \mathscr{E}_E(x,y)u(y)\,ds(y), \qquad K_U u := \int_{\Gamma_a} \mathscr{H}_E(x,y)u(y)\,ds(y),$$

$$D_{UH} u := \int_{\Gamma_a} \mathscr{H}_H(x,y)u(y)\,ds(y), \quad K_{UH} u := \int_{\Gamma_a} \mathscr{E}_H(x,y)u(y)\,ds(y),$$

$x \in U$. The boundary operators D, D_H, K, K_H are defined in (25)–(26) in [18]. We denote the kernel of these operators by the same symbols as above; e.g. $Du := \int_{\Gamma_a} n(x) \times \mathscr{E}_E(x,y)u(y)\,ds(y)$. Here n is the unit normal on Γ_a directed into $\mathbf{R}^3 \setminus \overline{U}$. The operators $D_U, D_{UH}, K_U, K_{UH}\colon TH^s_{\mathrm{Div}}(\Gamma_a) \to H^{s+1/2}_{\mathrm{Div}}(U)$ and the operators $D, D_H, K, K_H\colon TH^s_{\mathrm{Div}}(\Gamma_a) \to TH^s_{\mathrm{Div}}(\Gamma_a)$ are continuous for $s \in \mathbf{R}$; see [18, Theorems 4.4 and 4.5]). From Chapter 7 in [18] we also obtain that the operator $u \mapsto \int_{\Gamma_i} \mathscr{E}_E(x,y)u(y)\,ds(y)$ (and with other Green's functions) is continuous from $TH^s_{\mathrm{Div}}(\Gamma_i)$ into $H^{s+1/2}_{\mathrm{Div}}(\Omega_j)$, $i,j = 1,\dots,N$, and $s \in \mathbf{R}$. From [18] we have

$$
\begin{aligned}
n \times K_{\mathbf{R}^3\setminus\overline{U}}\, u|^+_{\Gamma_a} &= Ku + \tfrac{1}{2}u,\\
n \times K_U u|^-_{\Gamma_a} &= Ku - \tfrac{1}{2}u,\\
n \times K_{\mathbf{R}^3\setminus\overline{U}\,H} u|^+_{\Gamma_a} &= K_H u + \tfrac{1}{2}u,\\
n \times K_{UH} u|^-_{\Gamma_a} &= K_H u - \tfrac{1}{2}u,\\
n \times D_{\mathbf{R}^3\setminus\overline{U}}\, u|^+_{\Gamma_a} &= Du = n \times D_U u|^-_{\Gamma_a},\\
n \times D_{\mathbf{R}^3\setminus\overline{U}\,H} u|^+_{\Gamma_a} &= D_H u = n \times D_{UH} u|^-_{\Gamma_a}
\end{aligned}
\tag{4}
$$

for Γ_a a closed surface with electromagnetic parameters smooth in the neighborhood of Γ_a. If $\Gamma_a = \Gamma_i$ for some i, we have that

$$
\begin{aligned}
n \times K_{\mathbf{R}^3\setminus\overline{U}}\, u|^+_{\Gamma_a} - n \times K_U u|^-_{\Gamma_a} &= u,\\
n \times D_{\mathbf{R}^3\setminus\overline{U}}\, u|^+_{\Gamma_a} - n \times D_U u|^-_{\Gamma_a} &= 0,\\
n \times K_{\mathbf{R}^3\setminus\overline{U}\,H} u|^+_{\Gamma_a} - n \times K_{UH} u|^-_{\Gamma_a} &= u,\\
n \times D_{\mathbf{R}^3\setminus\overline{U}\,H} u|^+_{\Gamma_a} - n \times D_{UH} u|^-_{\Gamma_a} &= 0.
\end{aligned}
\tag{5}
$$

The following lemma is proved in [18].

Lemma 4.1. *We have* $\mathscr{E}_E(x,y) = K^y(x) + \mathscr{E}^0_E(x,y)$. *Here* $K^y = \sum_{j=1}^2 A_j(K^y_{Bj})$ *for every fixed* $y \in \mathbf{R}^3$, *where* $A_1, A_2 \in L\bigl(TH^s(\Gamma_1); H^{s-3/2}(\Omega_i)\bigr)$, $i = 1,2$, $s \in \mathbf{R}$, *and* $K^y_{B1} = -\mathscr{E}^{s,2}_E(x,y) + \mathscr{E}^{s,1}_E(x,y)$, $K^y_{B2} = -\mathscr{H}^{s,2}_E(x,y) + \mathscr{H}^{s,1}_E(x,y)$. *Here* $\mathscr{E}^{s,i}_E(x,y)$ *denotes Green's function for the electromagnetic parameters that are smoothly extended from* O_i *into* $\mathbf{R}^3$, *where* $O_i := U_1 \cap \Omega_i$ *and* U_1 *is an open set with* $\Omega_1 \cup \Gamma_1 \subset U_1$.

The transpose, B^t_j, *of the operator* B_j *with the kernel* K_{Bj}, *is continuous from* $TH^{-s}(\Gamma_1)$ *into* $H^{-s-1/2}_{\mathrm{loc}}(\mathbf{R}^3)$, $s > 0$. *The function* $\mathscr{E}^0_E(x,y)$ *is defined (for fixed* $y \in O_1 \cup O_2$*) as* $\bigl(\mathscr{E}^{s,1}_E(x,y)|_{O_1}, \mathscr{E}^{s,2}_E(x,y)|_{O_2}\bigr)$.

Further, the operator $T_{y_2}\colon H^s_{\mathrm{comp}}(\mathbf{R}^3) \to H^{s+2}_{\mathrm{loc}}(\mathbf{R}^3)$ *with the kernel* y_2 *(for smooth electromagnetic parameters) given in* (25)–(27), (29) *in* [17] *is continuous.*

We also have that

$$\mathscr{E}_E^s = i\nabla\Big(-\frac{i}{\gamma\omega}\nabla\cdot(\mu^{-1/2}y_2)\Big) + \omega\mu^{1/2}y_2 \quad \text{and} \quad \mathscr{H}_E^s = -\frac{i}{\mu}\nabla\times(\mu^{1/2}y_2),$$

where the index s denotes the smooth electromagnetic parameters in $\mathbf{R}^3$.

5. Harmonic currents in Ω

We define the matrices and operators that are needed in this paper and give a representation theorem, which is a base for our analysis.

Definition 5.1. We define the matrices

$$L_1 := \begin{pmatrix} 0 & \nabla\times \\ \nabla\times & 0 \end{pmatrix}, \quad G := \begin{pmatrix} \mathscr{E}_E & \mathscr{E}_H \\ \mathscr{H}_E & \mathscr{H}_H \end{pmatrix}, \quad K := i\omega\begin{pmatrix} \gamma & 0 \\ 0 & -\mu \end{pmatrix}$$

and $M := L_1 + K$. We denote by M_i (or K_i) the operator M (K) in Ω_i with the electromagnetic parameters in Ω_i smoothly extended outside Ω_i.

Theorem 5.1. *Let $E, H \in C^\infty(\overline{\Omega}_i)$, and $[n\times E]_{\Gamma_i} = 0$, $[n\times H]_{\Gamma_i} = 0$, for $i = 1,\dots,N$. Then we have*

$$\begin{aligned} &\int_\Omega \big(G(x,y)\big)^T M \begin{pmatrix} E(x) \\ H(x) \end{pmatrix} dx \\ &\qquad - \int_{\partial\Omega} \begin{pmatrix} \mathscr{H}_E^T & \mathscr{E}_E^T \\ \mathscr{H}_H^T & \mathscr{E}_H^T \end{pmatrix}(x,y) \begin{pmatrix} n\times E(x) \\ n\times H(x) \end{pmatrix} ds(x) = \begin{pmatrix} E(y) \\ H(y) \end{pmatrix}, \end{aligned} \tag{6}$$

for $y \in \cup\Omega_i$. Here the integrals are defined as a sum of terms, which are defined by the duality of the Sobolev spaces and as normal integrals.

Proof. Lemma 4.1 yields that $x \mapsto \mathscr{E}_E^T(x,y), x \mapsto \mathscr{H}_E^T(x,y) \in H^{-2}(\Omega_i)$. We have that $H = \mathrm{Ex}(H_1) + H_2$, where $H_1 \in C_0^\infty(U_0)$, $H_2 \in C^\infty(\overline{\Omega_i})$ and $H_2 = 0$ in U_1 for some open sets U_i, $i = 0,1$, $y \in U_1 \subset U_0$. By using duality and Green's formula, for $\mathscr{E}_E, \mathrm{Ex}(H_i)$, $i = 1,2$, (and other pairs in (7)) we get

$$\begin{aligned} &\int_{\Omega_i} \Big\{ \big(L_1 G(x,y)\big)^T \begin{pmatrix} E(x) \\ H(x) \end{pmatrix} - G^T(x,y)\Big(L_1\begin{pmatrix} E(x) \\ H(x) \end{pmatrix}\Big)\Big\} dx \\ &\qquad = \int_{\partial\Omega_i} \Big\{\Big[\begin{pmatrix} 0 & n\times \\ n\times & 0 \end{pmatrix} G(x,y)\Big]^T \begin{pmatrix} E(x) \\ H(x) \end{pmatrix}\Big\} ds(x), \end{aligned} \tag{7}$$

for $y \in \cup\Omega_i$ and $i = 1,\dots,N$ with n pointing into the exterior of Ω_i. Then we sum these formulas with the transmission conditions and use the definition of Green's functions. □

We note that Lemma 5.7 and the continuity properties of the boundary operators in Chapter 4 yield that (6) is also valid as $C^\infty(\overline{\Omega}_i)$ functions. The following reciprocity lemma in a more powerful form in Lemma 5.7 is needed e.g. in the proof of Theorem 5.2.

Lemma 5.1 *We have* $G^T(x,y) = G(y,x)$ *for* $x, y \in \mathbf{R}^3 \setminus \cup\Gamma_i$, $x \neq y$.

Proof. We proceed as in the proof of Theorem 5.1 with $\big(E(x), H(x)\big)$ replaced by $G(x,z)$. Let $B_R \supset \Omega$ be a ball with radius R. By summing the formulas corresponding to (7) for $i = 1, \dots, N$ and for the set $B_R \setminus \overline{\Omega}$ with the transmission conditions and then by taking the limit as $R \to \infty$ we get that

$$\int_{\mathbf{R}^3\setminus\cup\Gamma_i} \big\{ \big(L_1 G(x,y)\big)^T G(x,z) - G^T(x,y)\big(L_1 G(x,z)\big) \big\}\, dx = 0 \tag{8}$$

for fixed $y, z \in \mathbf{R}^3 \setminus \cup\Gamma_i$, $y \neq z$. Here we have noticed that if (E_i, H_i), $i = 1,2$, satisfy Maxwell equations in $\mathbf{R}^3 \setminus \overline{\Omega}$, then $\int_{\partial B_r} \{E_2 \cdot n \times H_1 + H_2 \cdot n \times E_1\}\, ds = I_1 - I_2$, where $I_i = \int_{\partial B_r} E_i \cdot \{n \times H_j + kE_j\}\, ds \to 0$, as $r \to \infty$; $i, j \in \{1,2\}$, $i \neq j$.

The formula $\big(L_1 + K(x)\big)G(x,t) = \delta(x-t)\mathbf{1}_6$ with (8) yields the claim. ▫

Next we will study the electromagnetic inverse problem. We will generally define the electromagnetic field given by the electromagnetic current $F \in H^{-s}_{\overline{\Omega}}(\mathbf{R}^3)$. We study the case, where we have one interior surface Γ_1 and the electromagnetic parameters are smooth in $\mathbf{R}^3 \setminus \overline{\Omega_1}$; i.e. from now on until (16) we have $\Omega = \overline{\Omega_1} \cup \Omega_2$ and $\Omega_3 = \mathbf{R}^3 \setminus \overline{\Omega}$. The most important result in this analysis is Theorem 5.2. The general case with a similar method is technically slightly more difficult; see [18]. We start with a natural representation formula.

Lemma 5.2. *Let* $s > 0$ *and* Ω, Ω_i, $i = 1,2,3$, *be defined as above. Further, let* $A_3 := \{w \in L^2_{\mathrm{loc}}(\mathbf{R}^3) \mid w|_{\Omega_3} \in C^\infty(\overline{\Omega_3}),\ \mathrm{supp}(w) \subset \overline{\Omega_3}, \}$. *Then every* $u \in H^{-s}_{\mathrm{loc}}(\mathbf{R}^3)$ *with* $u|_{\mathbf{R}^3\setminus\overline{\Omega}} \in C^\infty\big(\overline{\mathbf{R}^3 \setminus \overline{\Omega}}\big)$ *can be given in a form* $u = u_1 + u_2 + u_3$, *where* $u_i \in H^{-s}_{\overline{\Omega_i}}$, $i = 1,2$, *and* $u_3 \in A_3$.

Proof. We use the isomorphism $(H^s_{\overline{\Omega}}(\mathbf{R}^3)/\mathrm{Ker}_s(|_{\Omega_i}))' = H^{-s}_{\overline{\Omega_i}}(\mathbf{R}^3)$, see [16, Chapter 12.6]. Let $L \in (H^s_{\overline{\Omega}}(\mathbf{R}^3)/\mathrm{Ker}_s(|_{\Omega_i}))'$. We take $L_1 \in (H^s_{\overline{\Omega}}(\mathbf{R}^3)/\mathrm{Ker}_s(|_{\Omega_i}))'$ so that $L_1(w) = L(w)$ for $w \in H^s_{\overline{\Omega}}(\mathbf{R}^3) \cap \mathrm{Ker}_s(|_{\Omega_2})$ and $L_1(v)$ arbitrarily for $v|_{\Gamma_1}$ some base vector of $C^\infty(\Gamma_1)$. This yields the claim with $u_3 := \mathrm{Ex}(u|_{\mathbf{R}^3\setminus\overline{\Omega}})$. ▫

We give a definition for the electromagnetic field that is generated by the electromagnetic current. This is a natural generalization of the problem (1)–(2).

Definition 5.2. Let $s, k \in \mathbf{R}$. The electromagnetic field generated by the electromagnetic current $F \in H^{-s}_{\overline{\Omega_1}}(\mathbf{R}^3)$ is a distribution $u \in H^{-s-k}_{\mathrm{loc}}(\mathbf{R}^3)$ that satisfies the condition: Let (f_i) be a sequence, $f_i \in C_0^\infty(\Omega_1)$, such that $f_i \to F$ in $H^{-s}(\mathbf{R}^3)$. Then there exists a sequence (u_i), $u_i \in L_{2,\mathrm{loc}}(\mathbf{R}^3)$, $u_i|_{\Omega_i} \in C^\infty(\Omega_i)$ and $u_i \to u$ in $H^{-s-k}_{\mathrm{loc}}(\mathbf{R}^3)$ so that u_i is a solution of the problem (1)–(2) with $(J,M) = f_i$ and $J_{i0} = M_{i0} = 0$.

Further, if the current $F \in H^{-s}_{\overline{\Omega_2}}(\mathbf{R}^3)$ generates the field $v \in H^{-s-k}_{\mathrm{loc}}(\mathbf{R}^3)$, then we say that the field $u + v$ is generated by the current $\big(F + E, (F,E)\big)$, where the last notation denotes the division (F,E) of the current $F + E$ given in Lemma 5.2.

Remark 5.1. We can also define an equivalence relation r for (F,E) in Definition 5.2; e.g. by $(F,E)r(F_1,E_1)$ if $F=F_1+w$ and $E=E_1-w$ for some $w\in H^{-s}_{\Gamma_1}(\mathbf{R}^3)$. Then we define that this class generates the same field as the current (F_0,E_0), according to Definition 5.2 that satisfies

$$\|F_0\|+\|E_0\|=\inf_{w\in H^{-s}_{\Gamma_1}(\mathbf{R}^3)}(\|F_1+w\|+\|E_1-w\|)$$

with $H^{-s}(\mathbf{R}^3)$-norm. We see that $(F_0,E_0)r(F_1,E_1)$ and now the sequence (f_i) in Definition 5.2 can be taken as $f_i\in C_0^\infty(\mathbf{R}^3)$.

Let the electromagnetic field w satisfy $w_i:=w|_{\Omega_i}\in C^\infty(\overline{\Omega_i})$. By Theorem 5.1 we have $M_iw_i=g_i^0+(-1)^i(n\times w_i)\otimes\delta_\Gamma$, $i=1,2$, where $g_i^0=\big(\mathrm{Ex}(M_iw_i|_{\Omega_i})\big)$. This yields that Definition 5.2 is equivalent with the classical definition for these currents.

The uniqueness of the fields u and v in Definition 5.2 is easily proved by using the uniqueness of the fields u_i and v_i, see Theorem 5.3 in [18]. Theorem 5.2 yields the existence of the fields. We denote by $\hat{\otimes}$ either the completion of the tensor product space or the extension of the tensor product of two operators, see Chapters 2 and 43 in [25]. We use the following lemma, which is a consequence of the (sequential) density of the smooth kernels in the tensor product spaces and the definition of these spaces.

Lemma 5.3. *Let U_1 be an open set so that $\Omega_1\cup\Gamma_1\subset U_1$. Further, let $B\colon C_0^\infty(U_1)\to T\mathscr{D}'(\Gamma_1)$ and $A\colon T\mathscr{D}'(\Gamma_1)\to\mathscr{D}'(\Omega_i)$, $i=1,2$, be continuous operators. Then the distributional kernel of the operator $A\circ B$ is $(A\hat{\otimes}\mathbf{1})K_B$, i.e. $\langle(A\hat{\otimes}\mathbf{1})K_B,v\otimes u\rangle=\langle v,(A\circ B)u\rangle$ for $v\in C_0^\infty(\Omega_i)$ and $u\in C_0^\infty(U_1)$. Here, K_B is the distributional kernel of B and $\mathbf{1}$ is the identity operator in $\mathscr{D}'(U_1)$.*

By the definitions of the spaces and the isomorphism $\mathscr{E}'(U)=\bigcup_{s\in\mathbf{R}}H^s_c(\mathbf{R}^3)$ we have the isomorphism $T\mathscr{D}'(\Gamma_1)=\bigcup_{s\in\mathbf{R}}TH^s(\Gamma_1)$ as topological vector spaces; see e.g. Chapter 31 in [25]. We use the isomorphism $\mathbf{1}_{0,E}\colon\prod_{i=1}^2L_2(\overline{O_i};E)\to L_2(U_1)\hat{\otimes}E$, embeddings $\mathbf{1}_{1,E}\colon\prod_{i=1}^2C^0(\overline{O_i};E)\hookrightarrow\prod_{i=1}^2L_2(\overline{O_i};E)$, $\mathbf{1}_{2,E}\colon L_2(U_1)\hat{\otimes}E\hookrightarrow\mathscr{D}'(U_1)\hat{\otimes}E$ and $i_E:=\mathbf{1}_{2,E}\mathbf{1}_{0,E}\mathbf{1}_{1,E}\colon\prod_{i=1}^2C^0(\overline{O_i};E)\hookrightarrow\mathscr{D}'(U_1)\hat{\otimes}E$.

Lemma 5.4. (i) *Let $G_s(x,y)$ be the matrix G given in Definition 5.1 for the smooth electromagnetic parameters in $\mathbf{R}^3$. We have $G_s(x,y)=G_s^T(y,x)$ in $\mathscr{D}'(\mathbf{R}^3)\hat{\otimes}\mathscr{D}'(\mathbf{R}^3)$.*

(ii) *Let* Gr *be any of the Green's functions $\mathscr{E}^s_E$, $\mathscr{H}^s_E$, $\mathscr{E}^s_H$, $\mathscr{H}^s_E$. Then $\mathrm{Gr}_i(x,y):=\mathrm{Gr}(x,y)\in C^0_y(\overline{O}_i)\hat{\otimes}T\mathscr{D}'_x(\Gamma_1)$, $i=1,2$.*

(iii) *Let $\mathrm{Gr}^0(x,y)=i_{T\mathscr{D}'(\Gamma_1)}\big(\mathrm{Gr}_1(x,y),\mathrm{Gr}_2(x,y)\big)$. Then $\mathrm{Gr}(x,y)=\mathrm{Gr}^0(x,y)$.*

Proof. (i) (Sketch) Lemma 4.1 and the kernels theorem yield with the transpose of the operator that $G_s^T(x,y)\in\mathscr{D}'_x(B_R)\hat{\otimes}C^0_y(B_R)$.

Thus also $G_s(x,y)\in\mathscr{D}'_x(B_R)\hat{\otimes}C^0_y(B_R)$ and these kernels define distributions in $\mathbf{R}^3$ for a fixed $y\in\mathbf{R}^3$. Let $w(x)=\int_{\mathbf{R}^3}G_s(x,z)u(z)\,dz$, $u\in C_0^\infty(\mathbf{R}^3)$ and

$y \in \mathbf{R}^3$ be fixed. By the uniqueness of Green's functions we can proceed as in Lemma 5.1 with $G(x,z)$ replaced by $w(x)$. We get, for fixed $y \in \mathbf{R}^3$, that $\int_{\mathbf{R}^3} G_s(y,x)u(x)\,dx - \int_{\mathbf{R}^3} G_s^T(x,y)u(x)\,dx = 0$, for $u \in C_0^\infty(\mathbf{R}^3)$ and $y \in \mathbf{R}^3$. This implies the claim, with the continuity results in the beginning of the proof and with Theorem 2.2.

(ii) By using the mapping properties in Chapter 4 for the operators D_U, K_U, etc. and the embedding $H^s(O_i) \hookrightarrow C^0(\overline{O}_i)$ we have $\mathrm{Gr}(x,y) \in C_x^0(\overline{O}_i)\hat{\otimes} T\mathscr{D}_y'(\Gamma_1)$. This yields that Gr defines an operator from $(C^0(\overline{O}_i))'$ into $T\mathscr{D}'(\Gamma_1)$ (see the proof of Lemma 5.6(ii)) and then by the kernels theorem we get $\mathrm{Gr}^T(x,y) \in C_x^0(\overline{O}_i)\hat{\otimes} T\mathscr{D}_y'(\Gamma_1)$. By using Lemma 5.1 we obtain the claim.

(iii) In this proof we denote by T_A the operator in $\mathbf{R}^3$ with the kernel A. By using Lemma 4.1 and Theorem 2.2 we get a continuous operator $T^{\Gamma_1}_{\mathscr{H}_E^s} := T_{\mathscr{H}_E^s} \circ (|_{\Gamma_1})^t$ from $H^{-s+1/2}(\Gamma_1)$ into $H^{-s+1}_{\mathrm{loc}}(\mathbf{R}^3)$, $s > \frac{1}{2}$. From Chapter 4 with the kernels theorem we have $(\overline{\mathrm{Gr}}_i(x,y) :=)\mathscr{H}_E^s(x,y) \in C_x^0(\overline{O}_i)\hat{\otimes} T\mathscr{D}_y'(\Gamma_1)$. Let $\overline{\mathrm{Gr}}^0(x,y) := i_{T\mathscr{D}'(\Gamma_1)}\big(\overline{\mathrm{Gr}}_1(x,y), \overline{\mathrm{Gr}}_2(x,y)\big)$. Then $\overline{\mathrm{Gr}}^0(x,y)$ defines a continuous mapping $T^{\Gamma_1,0}_{\mathscr{H}_E^s}\colon TC^\infty(\Gamma_1) \to L_2(U_1)$. As we have $T^{\Gamma_1}_{\mathscr{H}_E^s}u|_{O_i} = T^{\Gamma_1,0}_{\mathscr{H}_E^s}u|_{O_i}$ for $u \in TC^\infty(\Gamma_1)$ and $T^{\Gamma_1}_{\mathscr{H}_E^s}u \in L_2(U_1)$, case (i), with the fact that $\mathscr{H}_E^s(x,y)$ defines a continuous operator from $(C_x^0(\overline{O}_i))'$ into $T\mathscr{D}_y'(\Gamma_1)$, implies the claim for Green's function $\mathscr{E}_H^s$.

Next we study Green's function $\mathscr{E}_E^s$. By using Lemma 4.1 we get, as above, the continuous operator $T^{\Gamma_1}_{\mathscr{E}_E^s}\colon H^{-s+1/2}(\Gamma_1) \to H^{-s}_{\mathrm{loc}}(\mathbf{R}^3)$, $s > \frac{1}{2}$, and the operator $T^{\Gamma_1,0}_{\mathscr{E}_E^s}\colon TC^\infty(\Gamma_1) \to L_2(U_1)$. Let $u \in TC^\infty(\Gamma_1)$. Theorem 2.2 yields that $T_{\nabla\times\mathscr{H}_E^s}u = \nabla \times (T_{\mathscr{H}_E^s}u)$. Let $a := T^{\Gamma_1,0}_{\mathscr{H}_E^s}u$ and $c := 1/i\omega\gamma$. As $a \in C^\infty(\overline{O}_i) \cap L_2(U_1)$, we obtain by using Green's theorem and the jump relations in Chapter 4 that

$$\begin{aligned} -(cT^{\Gamma_1}_{\nabla\times\mathscr{H}_E^s}u, b) &= -\big(a, \nabla \times (cb)\big) \\ &= -\sum_{i=1}^{2}\left[\int_{O_i} c(\nabla \times a)b\,dx\right] - \int_{\Gamma_1} [n \times a]_{\Gamma_1} cb\,ds(x) \\ &= (T^{\Gamma_1,0}_{\mathscr{E}_E^s}u, b) - \int_{\Gamma_1} ucb\,ds(x), \end{aligned} \tag{9}$$

where $b \in C_0^\infty(U_1)$ and n is the unit normal on Γ_1 directed into the exterior of O_1. As $\delta(x-y) \in C^0\big(\Gamma_1; \mathscr{D}_\sigma'(\mathbf{R}^3)\big)$, we can calculate, by using the kernels theorem and the isomorphism $L(TC^\infty(\Gamma_1); \mathscr{D}_\sigma'(U_1)) = T\mathscr{D}'(\Gamma_1)\hat{\otimes}\mathscr{D}_\sigma'(U_1)$, that $T_\delta^{\Gamma_1}u = \delta * (u \otimes \delta_{\Gamma_1})$. This yields that $\langle cT_\delta^{\Gamma_1}u, b\rangle = \int_{\Gamma_1} ucb\,ds(x)$. Then (9) and the definition of Green's functions in Chapter 4 imply that the kernels of the operators $T^{\Gamma_1}_{\mathscr{E}_E^s}$ and $T^{\Gamma_1,0}_{\mathscr{E}_E^s}$ are identical in $L_{2,x}(U_1)\hat{\otimes} T\mathscr{D}_y'(\Gamma_1)$. As $\mathscr{E}_E^s(x,y)$ defines a continuous operator from $\big(L_{2,x}(U_1)\big)'$ into $T\mathscr{D}_y'(\Gamma_1)$ (see the proof of

Lemma 5.6(ii)), we get that the transposes of the kernels are also identical in $L_{2,x}(U_1)\hat{\otimes}T\mathscr{D}'_y(\Gamma_1)$. Then with (i) we obtain the claim for $\mathscr{E}^s_E$. The claim for $\mathscr{H}^s_E$ and $\mathscr{H}^s_H$ is proved correspondingly. □

Lemma 5.5. *The pointwise constructed function K^y in Lemma* 4.1 *is the distributional kernel of the operator $\sum_{j=1}^2 A_j \circ B_j$.*

Proof. Lemma 4.1 with the known continuous embeddings and the isomorphism $T\mathscr{D}'(\Gamma_1) = \bigcup_{s\in\mathbf{R}} TH^s(\Gamma_1)$ gives us with Lemma 5.3 the kernel $Kk_j := (A\hat{\otimes}\mathbf{1}_{\mathscr{D}'(U_1)})K_{Bj}$ of the operator $A_j \circ B_j$. Here by Lemma 5.4 case (iii) $K_{Bj} = i_{T\mathscr{D}'(\Gamma_1)}(K_{Bj,1}, K_{Bj,2})$ and $K_{Bj,i}$ denotes K_{Bj} for $y \in \overline{O_i}$ given by Lemma 5.4 case (ii). Let

$$K^j_\Pi := \begin{pmatrix} (A\hat{\otimes}\mathbf{1}_{C^0(\overline{O_1})})K_{Bj,1} \\ (A\hat{\otimes}\mathbf{1}_{C^0(\overline{O_2})})K_{Bj,2} \end{pmatrix}.$$

We get that $i_{\mathscr{D}'(\Omega_i)}K^j_\Pi = Kk_j$.

Let $AK^y_{Bj,i}$, for $y \in \overline{O}_i$, be the pointwise constructed kernel in Lemma 4.1. By using the (sequential) density of $C^\infty_0(\overline{O}_i) \otimes TC^\infty(\Gamma_1)$ in $C^0(\overline{O}_i)\hat{\otimes}T\mathscr{D}'(\Gamma_1)$ and the continuity of the operators $(A\hat{\otimes}\mathbf{1}_{C^0(\overline{O_i})})$ and A we obtain that $(K^j_\Pi)^y = AK^y_{Bj,i}$ for $y \in \overline{O}_i$. □

We need the following two lemmas to get Theorem 5.2. The second lemma improves the result in Lemma 5.1.

Lemma 5.6. (i) *Let $A_2 \subset \Omega_i$, $i \in \{1, 2\}$, and $\partial A_2 \cap \partial\Omega_i = \emptyset$. The operator $u \to \int_{\Omega_i} G(x, y)u(y)\,dy$ is continuous from $C^\infty_0(A_2)$ into $C^0(\overline{O_j})$, $j = 1, 2$ (or into $L_2(U_1)$).*

(ii) *Further, the operator $u \to \int_{\Omega_i} G^T(x, y)u(x)\,dx$ is continuous from $C^\infty_0(\Omega_i)$ into $C^0(\overline{O_j})$, $j = 1, 2$ (or into $L_2(U_1)$).*

Proof. (i) By using the smoothness of the solution of the homogeneous elliptic equation we get with Lemma 5.6 and the embedding ii: $H^s_{\mathrm{loc}}(O_j) \hookrightarrow C^0(\overline{O_j})$ for $s > \frac{3}{2}$ that $K_j \in C^0_x(\overline{O_j})\hat{\otimes}C^\infty_y(A_2)$. Here K_j is part of the kernel K in Lemma 4.1.

The definition of Green's functions in [18] (Theorem 5.1 and Remark 8.2) gives us $K^y = iii(K^y_1, K^y_2)$ for fixed $y \in O_1 \cup O_2$, where iii is the embedding $\prod_{j=1}^2 C^0(\overline{O_j}) \hookrightarrow L_2(U_1)$. Correspondingly, as in the the proof of Lemma 5.5, we have that $K = (K_1, K_2)$ in $L_{2,x}(U_1)\hat{\otimes}C^\infty_y(A_2)$. Then the continuous embedding $C^\infty_0(A_2) \hookrightarrow \mathscr{E}'(A_2)$ yields the claimed continuity for K.

The mapping properties of $\mathscr{E}^{s,j}_E(x, y)$ and the definition in Lemma 4.1 yield that $\mathscr{E}^0_E(x, y) \in L_{2,x}(U_1)\hat{\otimes}\mathscr{D}'_y(A_2)$. These results imply the claim for $\mathscr{E}_E(x, y)$ and then correspondingly also for $G(x, y)$.

(ii) The continuity of $\mathscr{E}^{s,i}_E(x, y)$ from $L_2(U_1)$ into $\mathscr{D}'(\Omega_i)$ together with the definition of $\mathscr{E}^0_E(x, y)$, the transpose of the operator and Lemma 5.5 yield that $G(x,y)\in\mathscr{D}'_x(\Omega_i)\hat{\otimes}L_{2,y}(U_1)$. Thus $G(x,y)\in\mathscr{B}\big((\mathscr{D}'_x(\Omega_i))'_\sigma,(L_{2,y}(U_1))'_\sigma\big)\subset\mathscr{B}\big((\mathscr{D}'_x(\Omega_i))',(L_{2,y}(U_1))'\big)$. For a separately continuous bilinear mapping B in $E\times F$, where E, F are Frechet spaces there are continuous seminorms P_E in E and P_F in F so that $|B(u,v)|\le P_E(u)P_F(v)$. As $C_0^\infty(\Omega_i)$ is a countable strict inductive limit of Frechet spaces, this yields that $G(x,y)$ defines a continuous operator from $\big(L_2(U_1)\big)'$ into $\mathscr{D}'(\Omega_i)$. This proves the claim with the obvious changes for the first part of the claim. ◻

Lemma 5.7. *We have* $G(x,y)=G^T(y,x)$ *in* $\mathscr{D}'_x(U_1)\hat{\otimes}\mathscr{D}'_y(\Omega_i)$, $i=1,2$.

Proof. (Sketch) By Lemma 5.6 we have $G^T(x,y)\in\mathscr{D}'_x(\Omega_i)\hat{\otimes}C^0_y(\overline{O}_j)$. Thus also $G(x,y)\in\mathscr{D}'_x(\Omega_i)\hat{\otimes}C^0_y(\overline{O}_j)$. The (sequential) density of $C_0^\infty(\Omega_i)\otimes C^0_y(\overline{O}_j)$ in $\mathscr{D}'_x(\Omega_i)\hat{\otimes}C^0_y(\overline{O}_j)=L\big(C_0^\infty(\Omega_i);C^0_y(\overline{O}_j)\big)$ yields that

$$\big(G^T(x,y),v(x)\big)=\int_{\Omega_i}G^T(x,y)v(x)\,dx$$

for $v\in C_0^\infty(\Omega_i)$ and fixed y. Here in the first term we have a distribution in Ω_i for a fixed y.

Let $w(x)=\int_{\Omega_i}G(x,z)u(z)\,dz$, $u\in C_0^\infty(\Omega_i)$. Again we can proceed as in Lemma 5.1 with $G(x,z)$ replaced by $w(x)$. We obtain for a fixed $y\in\mathbf{R}^3\setminus\cup\Gamma_i$ that

$$\int_{\mathbf{R}^3\setminus\cup\Gamma_i}\big\{\big(L_1G(x,y)\big)^Tw(x)-G^T(x,y)\big(L_1w(x)\big)\big\}\,dx=0. \tag{10}$$

This yields that $\int_{\Omega_i}G(y,x)u(x)\,dx-\int_{\Omega_i}G^T(x,y)u(x)\,dx=0$, for $u\in C_0^\infty(\Omega_i)$ and $y\in O_i$ (by Lemma 5.6 for $y\in\overline{O_i}$). This implies the claim with Lemma 5.6 and Theorem 2.2 and with x and y interchanged. ◻

Finally, we obtain one of our main results in this paper. The existence of the fields u and v in Definition 5.2 is a consequence of this theorem. This continuity result is unknown in the literature. The kernel Green's function was obtained and is used only by the author.

Theorem 5.2. *Let* $s>0$. (i) *The mapping* $J\to\int_{\Omega_i}\mathscr{E}_E(x,y)J(y)\,dy$ *defined for* $J\in C_0^\infty(\Omega_i)$ *has a continuous extension as an operator from* $H^{-s+3/2}_{\overline{\Omega}_i}(\mathbf{R}^3)$ *into* $H^{-s-1/2}_{\mathrm{loc}}(U_1)$, $i=1,2$, *where* $U_1\supset\Omega$ *is an open set in* $\mathbf{R}^3$.

(ii) *Furthermore, the operator* $J\to\int_{\Omega_i}\mathscr{E}_E(x,y)J(y)\,dy|^+_{\Gamma_i}$ *has a continuous extension as an operator from* $H^{-s+3/2}_{\overline{\Omega}_i}(\mathbf{R}^3)$ *into* $H^{-s-1}(\Gamma_i)$, $i=1,2$.

Proof. (i) By using Theorem 2.2 and Lemma 5.5 we have that the transpose of the operator $\sum_{j=1}^2(A_j\circ B_j)$ and the operator given by the kernel $K^T(x,y)$

are the same as operators from $C^\infty_{0,x}(\Omega_i)$ into $\mathscr{D}'_y(U_1)$. Lemma 4.1 implies that B_j, $j = 1,2$, are continuous from $H^{s+1/2}_{\mathrm{comp}}(U_1)$ into $TH^s(\Gamma_1)$, $s > 0$. Then by using Lemma 4.1 we have that $\left[\sum_{j=1}^2 (A_j \circ B_j)\right]^t$, and thus also the operator given by the kernel $K^T(x,y)$, has a continuous extension from $H^{-s+3/2}_{\overline{\Omega}_i,x}(U_1)$ into $H^{-s-1/2}_{loc,y}(U_1)$ for $s > 0$. On the other hand, by using Lemma 4.1 we have that the operator $T_{\mathscr{E}^{s,i}_E}$ given by the kernel $\mathscr{E}^{s,i}_E(x,y)$ is continuous from $H^{s-3/2}_{\mathrm{comp},y}(U_1)$ into $H^{s-3/2}_x(\Omega_i)$. Then by recalling the definition in Lemma 4.1 for $\mathscr{E}^0_E$ we get that $(\mathscr{E}^0_E)^T(x,y)$ defines a continuous operator from $H^{-s+3/2}_{\overline{\Omega}_i,x}(U_1)$ into $H^{-s+3/2}_{\mathrm{loc},y}(U_1)$. As $\mathscr{E}^0_E(x,y)$ defines a continuous mapping from $H^{-s}_{\overline{O_j}}(\mathbf{R}^3)$ into $\mathscr{D}'(\Omega_i)$, we get that $(\mathscr{E}^0_E)^T(x,y) \in \mathscr{D}'_x(\Omega_i)\hat{\otimes} C^0_y(\overline{O}_j)$, $j = 1,2$. Thus also $\mathscr{E}^0_E(x,y) \in \mathscr{D}'_x(\Omega_i)\hat{\otimes} C^0_y(\overline{O}_j)$. This together with the same property for $K(x,y)$ in Lemma 5.5 yields that $\mathscr{E}^T_E(x,y) = K^T(x,y) + (\mathscr{E}^0_E)^T(x,y)$ in $\mathscr{D}'_x(\Omega_i)\hat{\otimes}\mathscr{D}'_y(U_1)$. By Lemma 5.7 $\mathscr{E}^T_E(x,y) = \mathscr{E}_E(y,x)$ in $\mathscr{D}'_x(\Omega_i)\hat{\otimes}\mathscr{D}'_y(U_1)$, which now implies the claim.

(ii) As $\mathscr{E}_E$ satisfies an elliptic homogeneous equation in $\mathbf{R}^3 \setminus \overline{\Omega_i}$, Theorem 2.1 and (i) yield the claim. □

The same results are also obtained for the corresponding operators with $\mathscr{E}_H, \mathscr{H}_E$ and $\mathscr{H}_H$ as kernels.

Remark 5.2. The operator given by the kernel G, $u \to Gu$, has no continuous extension $G\colon \mathscr{E}'(\mathbf{R}^3) \to \mathscr{D}'(\mathbf{R}^3)$ or $G\colon H^{-s}_{\mathrm{comp}}(\mathbf{R}^3) \to H^{-s-k}_{\mathrm{loc}}(\mathbf{R}^3)$ with some $k \in \mathbf{R}$ and $s > \frac{3}{2}$. Namely, let such an extension exist and let $u \in H^{-s}_{\Gamma_1}(\mathbf{R}^3)$. Let (v_{ji}), $j = 1,2$, be sequencies so that $v_{ji} \in C^\infty_0(\Omega_j)$, and $v_{ji} \to u$, $i \to \infty$ in $H^{-s}(\mathbf{R}^3)$. Definition 5.1 and Theorem 5.2 yield that $G(M_1 u) = u$ and $G(M_2 u) = u$. Thus $G(M_1 u - M_2 u) = 0$. On the other hand, we can choose u so that $M_1 w = (K_2 - K_1)u$, with some $w \neq 0$, $\mathrm{supp}(w) \subset \Gamma_1$ and K_i, $i = 1,2$, are from Definition 5.1. This yields by Theorem 5.2 that $G\big((M_2 - M_1)u\big) = w \neq 0$. Thus the contradiction is false and the claim is true.

If the electromagnetic parameters are smooth we can generalize Theorem 5.1 for the currents $J, M \in H^{-s}_{\mathrm{comp}}(\mathbf{R}^3)$. Also in the layered case we obtain a representation formula that is given in Theorem 5.3. Theorem 5.3 also proves that electromagnetic Green's function is the left-sided inverse of the Maxwell-operator.

Theorem 5.3. *Let Ω and Ω_i, $i = 1,2,3$, and A_3 be as in Lemma 5.2 and $G\colon \prod_{i=1}^2 H^{-s+3/2}_{\overline{\Omega}_i}(\mathbf{R}^3) \times H^{-s+3/2}_{\overline{B_R \setminus \overline{\Omega}}}(\mathbf{R}^3) \to H^{-s-1/2}_{\mathrm{loc}}(\mathbf{R}^3)$ be the continuous operator obtained in Theorem 5.2.*

(i) *Let $u = (u_1, u_2, u_3) \in \prod_{i=1}^2 H^{-s-1}_{\overline{\Omega}_i}(\mathbf{R}^3) \times A^0_3$, where $A^0_3 = \{w \in A_3 \mid \mathrm{supp}(M_3 w)$ is compact and w satisfies the Silver–Müller radiation condition$\}$.*

Then we have

$$(u_1, u_2, u_3) = (Gf_1, Gf_2, Gf_3), \quad \text{where } f_i = M_i u_i\,.$$

(ii) *Let* $B_3 := \{w \in A_3 \mid \operatorname{supp}(w) \subset B_r$ *for some* $r\}$ *and* $\overline{G}$, $\overline{M}$ *be the operators* $\overline{M}(u) := \sum_{i=1}^{3} M_i u_i$, $\overline{G}(u) := \sum_{i=1}^{3} G(u_i)$. *Here* $u = u_1 + u_2 + u_3$, $u_i \in H^{-s}_{\overline{\Omega}_i}(\mathbf{R}^3)$ *and* $u_3 \in A_3^0$ *or* $u_3 \in B_3$.

Then $\overline{G}\colon H^{-s}_{\overline{\Omega}}(\mathbf{R}^3)+B_3 \to H^{-s-2}_{\mathrm{loc}}(\mathbf{R}^3)$ *and* $\overline{M}\colon H^{-s}_{\overline{\Omega}}(\mathbf{R}^3)+A_3^0 \to H^{-s-1}_{\mathrm{loc}}(\mathbf{R}^3)$ *continuously and we have* $\overline{G}(\overline{M}u) = u$ *for* $u \in H^{-s}_{\overline{\Omega}}(\mathbf{R}^3) + A_3^0$.

Proof. (i) We use the density of $C_0^\infty(\Omega_i)$ in $H^{-s}_{\overline{\Omega}_i}(\mathbf{R}^3)$ and Theorem 5.2. For u_3 in $B_r \setminus \overline{\Omega}$ we use the method in Theorem 5.1. Then we let $r \to \infty$ and get $u_3 = G(P_3 u_3)$.

(ii) This follows by using Lemma 5.2 and by summing the results in case (i). □

Remark 5.3. Definition 5.2 and Theorem 5.2 yield that we have $u|_{\mathbf{R}^3\setminus\overline{\Omega}} = \big(G(F_1)+G(F_2)\big)|_{\mathbf{R}^3\setminus\overline{\Omega}}$, where the field u is given by the current $\big(F_1+F_2, (F_1,F_2)\big)$. Thus the condition

$$\big(G(F_1) + G(F_2)\big)|_{\Gamma}^{+} = 0, \tag{11}$$

is a necessary condition for the currents that give zero measurement.

The condition (11) is also sufficient. Namely, if the current $F = (F_1, F_2)$ satisfies the condition (11), we define $(E,H) := G_1(F_1)+G_2(F_2)$. Then $(E,H)|_{\mathbf{R}^3\setminus\overline{\Omega}} = 0$ and (E,H) satisfies the condition in Definition 5.2.

By using Lemma 5.1 (or Lemma 5.7) and the transmission conditions for Green's functions we get the formula

$$\int_\Gamma u(x) \int_{\Omega_i} \mathscr{H}_E(x^+, y) J(y)\, dy\, ds(x) = \int_{\Omega_i} J(y) \int_\Gamma \mathscr{E}_H(y, x^-) u(x)\, ds(x)\, dy \tag{12}$$

for $u \in TC^\infty(\Gamma)$ and $J \in C_0^\infty(\Omega_i)$, $i = 1,2$. The indexes $+$ and $-$ denote the outer and inner limit value on Γ. This is also valid for $u \in H^{s+1}(\Gamma)$ and $J \in H^{-s+3/2}_{\overline{\Omega}_i}(\mathbf{R}^3)$ by using Theorem 5.2 and the duality of $H^{-s}_{\overline{\Omega}_i}(\mathbf{R}^3)$ and $H^s(\Omega_i)$ or by using the mapping property of the operator D_{Ω_i} given in Chapter 4. By treating analoguously the other Green's functions this yields, with the continuous extensions for $D_\Omega, K_{\Omega H}$ etc. in Div-spaces, an equivalent condition with (11):

$$\begin{pmatrix} J \\ M \end{pmatrix} \perp \Big\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in H^s(\Omega_i) \times H^s(\Omega_i),\ i = 1,2 \mid w_1 = D_\Omega u + K_{\Omega H} v; \\ w_2 = K_\Omega u + D_{\Omega H} v; u \in TH^{s-1/2}_{\mathrm{Div}}(\Gamma),\ v \in TH^{s-1/2}_{\mathrm{Div}}(\Gamma) \Big\} =: U \tag{13}$$

for $J, M \in H^{-s}_{\overline{\Omega}_1}(\mathbf{R}^3)$.

Remark 5.4. Here we give a mathematical definition, where the field on the surfaces Γ_i is left partly open. We show that also in this model we can use the formula (13).

We define: The electromagnetic field generated by the electromagnetic current $F \in H^{-s}_{\overline{\Omega}}(\mathbf{R}^3)$ with the division (F_1, F_2) is a distribution u given by an injective mapping M_D, $M_D(F_1, F_2) = u$. We assume that $M_D(0) = 0$ and $M_i u|_{\Omega_i} = F_i|_{\Omega_i}$, $i = 1, 2$, where M_i is given in Definition 5.1.

For brevity, we have one interior surface. Let F give zero measurement; i.e. $r_{\mathbf{R}^3\setminus\overline{\Omega}} u = 0$, where $r_{\mathbf{R}^3\setminus\overline{\Omega}}\colon H^{-s}_{\mathrm{loc}}(\mathbf{R}^3) \to H^{-s}_{\mathrm{loc}}(\mathbf{R}^3\setminus\overline{\Omega})$ is the continuous restriction mapping. Let (u_1, u_2) be a division of u and $f_i := M_i u_i$, $i = 1, 2$. Then $u_2, f_2 \in H^{-s}_{\Gamma}(\mathbf{R}^3)$ and the division $(w_1, w_2) := (u_1 + u_2, 0)$ satisfies $M_2 w_2 = 0$. Furthermore, Theorem 5.3 yields that

$$G_1(f_0)|_{\mathbf{R}^3\setminus\overline{\Omega}} = 0 \tag{14}$$

with $f_0 := f_1 + M_1 u_2$ and

$$G_1(f_1)|_{\mathbf{R}^3\setminus\overline{\Omega}} = 0 \quad \text{and} \quad G_2(f_2)|_{\mathbf{R}^3\setminus\overline{\Omega}} = 0. \tag{15}$$

We see by using Theorem 5.3 that the conditions (14) and (15) are equivalent for the currents (f_1, f_2) in Theorem 5.3(i). Thus (13) gives an equivalent condition with (14) for the current f_0 that denotes the current in Theorem 5.3 corresponding to the division (w_1, w_2), where $M_2 w_2 = 0$.

In the rest of this paper we use Definition 5.2, but we will get the same kind of results also with Remark 5.4 with appropriate changes.

Next we give a representation formula for the closure of U in $\mathbf{H}^s_\Omega \times \mathbf{H}^s_\Omega$.

By carefully using Theorems 5.1 and 5.2, the density of $C^\infty(\overline{U})$ in $H^s(U)$ for $U \subset \mathbf{R}^3$ open, and the continuity of the trivial extension mapping $u \to \mathrm{Ex}(u)$ from $H^s(U)$ into $H^s(\mathbf{R}^3)$, $-\frac{1}{2} < s < \frac{1}{2}$, we get the following result: Let $s > \frac{1}{2}$. Furthermore, let $(w_1, H_1) \in MW^s(\Omega)$ and $(w_2, H_2) \in MW^s(\mathbf{R}^3 \setminus \overline{\Omega})$. Then we have

$$\begin{pmatrix} K_{UH} & D_U \\ D_{UH} & K_U \end{pmatrix} \begin{pmatrix} n \times w_1|_{\Gamma_-} \\ n \times H_1|_{\Gamma_-} \end{pmatrix} = \begin{cases} -\begin{pmatrix} w_1(x) \\ H_1(x) \end{pmatrix}, & x \in \Omega, \\ 0, & x \in \mathbf{R}^3 \setminus \overline{\Omega}, \end{cases} \tag{16}$$

and the corresponding formula for $(n \times w_2|_{\Gamma_+}, n \times H_2|_{\Gamma_+})$ with Ω and $\mathbf{R}^3 \setminus \overline{\Omega}$ interchanged.

Theorem 5.4. *Let $s > \frac{1}{2}$. Then we have $\overline{U}^{\mathbf{H}^s_\Omega \times \mathbf{H}^s_\Omega} = MW^s(\Omega)$.*

Proof. We have that $MW^s(\Omega) \subset \mathbf{H}^s_{\Omega,\mathrm{Div}} \times \mathbf{H}^s_{\Omega,\mathrm{Div}}$. Then with (16) we get that $MW^s(\Omega) \subset U$. Thus $U = MW^s(\Omega)$. Theorem 2.1 and the injective embeddings of the Sobolev spaces imply that $MW^s(\Omega)$ is closed in $\mathbf{H}^s_\Omega \times \mathbf{H}^s_\Omega$. □

If the Maxwell exterior boundary value problem with the radiation condition in (2) has a unique solution in $\mathbf{R}^3 \setminus \overline{\Omega}$, then the measurements from the outside of the boundary are equivalent with the measurement $(E|_\Gamma,\ H|_\Gamma)$. These are e.g. the measurements $(n \times E|_\Gamma)$, $(n \times H|_\Gamma)$ and $(n \cdot E|_\Gamma,\ n \cdot H|_\Gamma)$ (see [26]).

6. Currents on surfaces

In this chapter we apply the results in Chapter 5 for currents that are on some closed surfaces. Let F be a current on the surface $T = \bigcup_{i=1}^{m} \Gamma_{ai}$, $F = \sum_{i=1}^{m} F_i$, where $F_i \in H^{-s}_{\Gamma_{ai}}(\mathbf{R}^3)$. Each surface Γ_{ai} in this chapter is a smooth closed surface in Ω which encloses a simply connected region, either $\Gamma_{ai} \subset \Omega_k$ or $\Gamma_{ai} = \Gamma_j$, for some k and j. We assume that the current $F_i = (J_i, M_i)$ has the form $J_i = J_{0i} \otimes \delta_{\Gamma_{ai}}$, $M_i = M_{0i} \otimes \delta_{\Gamma_{ai}}$ with $J_{0i}, M_{0i} \in H^{-s+1/2}(\Gamma_{ai})$.

By using the formula (17) for smooth L_0 and u and the duality and the density results in Chapter 2 we get the following result: Let $r > 0$ and $\Gamma_a \subset \Omega$ be a closed surface and Ω_a the area surrounded by Γ_a. The linear mapping $L_0 \mapsto L_0 \otimes \delta_{\Gamma_a}$ defined on $C^\infty(\Gamma_a)$ has a continuous extension from $H^{-r}(\Gamma_a)$ into $H^{-r-1/2}_{\overline{\Omega}_a}(\mathbf{R}^3)$. Further, the formula

$$(L, u)_{(H^{-r-1/2}_{\overline{\Omega}_a}(\mathbf{R}^3), H^{r+1/2}(\Omega_a))} = (L_0, g)_{(H^{-r}(\Gamma_a), H^{r}(\Gamma_a))} \tag{17}$$

is valid for $L_0 \in H^{-r}(\Gamma_a)$ and $u \in H^{r+1/2}(\Omega_a)$ with $u|_{\Gamma_a} = g$ and $L = L_0 \otimes \delta_{\Gamma_a}$.

The tangential currents (17) yield, by using Theorem 5.4 and the condition (13), the following theorem.

Theorem 6.1. *Let $s > \frac{1}{2}$ and the current $(J, M) := F$ have the form above with $J_{0i}, M_{0i} \in TH^{-s+1/2}(\Gamma_{ai})$. Then the currents that give zero measurement are determined by the condition*

$$\sum_{i=1}^{m} \big\{ (n \times J_{0i}, n \times E)_{(-s+1/2, s-1/2)} + (n \times M_{0i}, n \times H)_{(-s+1/2, s-1/2)} \big\} = 0, \tag{18}$$

where $(E, H) \in MW^s(\Omega)$ and the duality $\big(TH^{-s+1/2}(\Gamma_{ai}), TH^{s-1/2}(\Gamma_{ai})\big)$ is denoted by $(\,\cdot\,,\cdot\,)_{(-s+1/2, s-1/2)}$.

Next we study the case, where we know the location of the current on one surface Γ_a. We see that the currents that give zero measurement are independent of the electromagnetic parameters on the exterior domain. That is why we can assume that the electromagnetic parameters are smoothly extended outside the domain surrouded by Γ_a. As ω can be an eigenfrequency of the Maxwell transmission problem, we proceed in the following way. Our presentation partly uses the ideas given in Chapter 4.4 in [4], Theorem 4.23 and Theorem 4.26, and generalizes them.

We use the Fredholm theory in dual systems presented in Chapter 1.3 in [4], see also [15]. Definition 1.22 in [4] shows that $\langle TH^{s-1/2}_{\mathrm{Div}}(\Gamma_a), TH^{s-1/2}_{\mathrm{Div}}(\Gamma_a)\rangle$ is a dual system (for $s \ge \frac{1}{2}$) with the bilinear form $\langle a, b\rangle := \int_{\Gamma_a} a(x) b(x)\, ds(x)$. In [18] we proved that K is a compact operator from $TH^{s-1/2}_{\mathrm{Div}}(\Gamma_a)$ into $TH^{s-1/2}_{\mathrm{Div}}(\Gamma_a)$. Let K' be the adjoint of the operator K in $TH^{s-1/2}_{\mathrm{Div}}(\Gamma_a)$. To get a formula for K' we need the following lemma, see Chapter 4 for the definition of these Green's functions on Γ_a.

Lemma 6.1. *Let Γ_a be a closed surface which encloses a simply connected region so that the electromagnetic parameters are smooth in the neighborhood of Γ_a. Then $\mathscr{H}_E^T(x,y) = \mathscr{E}_H(y,x)$ in $T\mathscr{D}'(\Gamma_a)\hat{\otimes}T\mathscr{D}'(\Gamma_a)$.*

Proof. By using the formulas (29), (98) and (99) in [17] we have that

$$\begin{aligned}\int_{\Gamma_a} \mathscr{H}_E^T(x,y)u(x)\,ds(x) &= -\int_{\Gamma_a} (\nabla_x \times \phi(x-y)\mathbf{1}_3)^T u(x)\,ds(x)\\ &\quad + \int_{\Gamma_a} K_2^T(x,y)u(x)\,ds(x)\\ &:= T_{U1}u + T_{U2}u \int_{\Gamma_a} \mathscr{E}_H(y,x)u(x)\,ds(x)\\ &= -\int_{\Gamma_a} (\nabla_y \times \phi(y-x)\mathbf{1}_3)u(x)\,ds(x)\\ &\quad + \int_{\Gamma_a} K_3(y,x)u(x)\,ds(x)\\ &:= K_{UH1}u + K_{UH2}u,\end{aligned}$$

where ϕ is the radiating fundamental solution of the Helmholtz operator $\Delta + k^2$. By Lemma 5.1 the operators on the left-hand side are identical as operators from $TC^\infty(\Gamma_a)$ into $C^\infty(\overline{U_{\Gamma_a}})$ with U_{Γ_a} a half neighborhood of Γ_a. The continuity of the tangential trace of the operator K_{UH2} is proved in [18], or see (4). Then by using the trace theorem and the fact that $\nabla_x\phi(x-y) = -\nabla_y\phi(x-y)$ in $\mathscr{D}'(\mathbf{R}^3\times\mathbf{R}^3)$ (see Lemma 4.3 in [18]), we obtain that $n\times T_{U2}u|_{\Gamma_a} = n\times K_{UH2}u|_{\Gamma_a}$.

The boundary operator K_H with the kernel $n(x) \times \mathscr{E}_H(x,y)$ is defined as $K_Hu = K_{H1}u + n\times K_{UH2}u|_{\Gamma_a}$, where K_{H1} has the kernel $n(y)\times(\nabla_y\times\phi(y-x)\mathbf{1}_3)$. The operator K is defined correspondingly. Then the transpose of this operator gives the operator T with the kernel $n(x)\times\mathscr{H}_E^T(x,y)$, $Tu = T_1u + n\times T_{U2}u|_{\Gamma_a}$. Here T_1 has the kernel $n(y)\times(\nabla_x\times\phi(x-y)\mathbf{1}_3)^T$. The result for the constant electromagnetic parameters (see e.g. [5, p. 164]) yields that $T_1u = K_{H1}u$. Then $Tu = K_Hu$, which implies the claim. □

Using Lemma 6.1 with some vector calculus we obtain

$$K'u = n\times K_H(n\times u), \quad u \in TH^{s-1/2}_{\mathrm{Div}}(\Gamma_a). \tag{19}$$

For Theorem 3.1 it would be enough to prove (20) and Theorem 6.2 for $s > M$ with $M > 0$. However, the method is almost the same for $s \geq \frac{1}{2}$. These results correspond to Theorems 4.23 and 4.26 in [4].

Correspondingly, as in [4], we get by defining (E,H) as $(E,H) := (D_Uu, K_Uu)$ in U for $U = \Omega_a$ and $U = \mathbf{R}^3\setminus\overline{\Omega_a}$ and by using (4) and (16) the following result:

$$\operatorname{Ker}\left(\tfrac{1}{2}\mathbf{1}_3 + K\right) = \mathscr{M}. \tag{20}$$

In Theorem 6.2 we study the boundary value problem: Let $c \in TH^{s-1/2}_{\mathrm{Div}}(\Gamma_a)$. Find (E_1, H_1) that satisfies

$$(E_1, H_1) \in MW^s(\Omega_a) \quad \text{and} \quad n \times E_1|_{\Gamma_a} = c. \tag{21}$$

Theorem 6.2. *Let* $s \geq \frac{1}{2}$. *The problem* (21) *is solvable, if and only if,* $\int_{\Gamma_a} c(x) \cdot H_2(x)\, ds(x) = 0$ *for all the solutions* (E_2, H_2) *of the problem* (21) *with* $c = 0$.

Proof. Sufficiency: We have with (19) the equivalency of the equations $\bigl(K_H - \frac{1}{2}\mathbf{1}_3\bigr)v = c$ and $\bigl(K' + \frac{1}{2}\mathbf{1}_3\bigr)w = n \times c$ with $w := -n \times v$. By using (20) and the assumption we get that $\int_{\Gamma_a} (n \times c) \cdot u\, ds = 0$ for $u \in \ker\bigl(K + \frac{1}{2}\mathbf{1}_3\bigr)$. These results imply with the Fredholm alternative, Theorem 1.30 in [4], that the equation $\bigl(K_H - \frac{1}{2}\mathbf{1}_3\bigr)v = c$ is solvable. Then

$$(E, H) := (K_{\Omega_a H} v, D_{\Omega_a H} v) \quad \text{in } \Omega_a \tag{22}$$

is a solution of the problem (21).

Necessity: For $s \geq 1$ this is obtained by using Green's formula with (21) in each $U_i := \Omega_i \cap \Omega_a$. For $\frac{1}{2} \leq s \leq 1$ this can be proved by using the continuity of the boundary operators given in Chapter 4, Theorem 2.1 and the case $s \geq 1$. □

Now we can give the proof of Theorem 3.1, which is the main result for our inverse problem.

Proof of Theorem 3.1. (i) For the currents $J_0 \in TH^0(\Gamma_a)$ the claim follows by Theorem 6.2. But for $J_0 \in TH^{-s+1/2}(\Gamma_a)$, $s > \frac{1}{2}$, we proceed in the following way. Let $C := \overline{\{n \times E|_{\Gamma_a} \in TH^0_{\mathrm{Div}}(\Gamma_a) \mid (E, H) \in MW^{1/2}(\Omega_a)\}}^{L_2(\Gamma_a)}$, $C_r := \bigl(C \cap TH^r_{\mathrm{Div}}(\Gamma_a)\bigr)$ and $\widetilde{C}_r := \{n \times E|_{\Gamma_a} \in TH^r(\Gamma_a) \mid (E, H) \in MW^{r+1/2}(\Omega_a)\}$, $r \geq 0$. By the smoothness of the elements of $\mathscr{M}$ and as the field in Theorem 6.2 is given by (22), we get that $C_r \subset \widetilde{C}_r$. This clearly yields $C_r = \widetilde{C}_r$.

Let $M \equiv 0$, $J_0 \in TH^{-s+1/2}(\Gamma_a)$ satisfy the condition (18) and $r := s - \frac{1}{2}$. We prove that $J_0 \in \mathscr{M}$. Let $J_i \in C^\infty(\Gamma_a)$ and $J_i \to n \times J_0$ in $H^{-r}(\Gamma_a)$ as $i \to \infty$, and $J_i = J_{1i} + J_{2i}$ with $J_{1i} \in n \times \mathscr{M}$, $J_{2i} \in C_r$ for $i \in \mathbf{N}$. As $C_r = \widetilde{C}_r$, the condition (18) with the result $(J_{1i}, g)_{(-r,r)} = (J_{1i}, g)_{(L_2,L_2)} = 0$, for $g \in C_r$, yields that

$$(J_{2i}, g)_{(-r,r)} \to 0 \quad \text{for } g \in C_r. \tag{23}$$

The sequencies (J_{1i}) and (J_{2i}) are Cauchy-sequencies in $H^{-r}(\Gamma_a)$. Namely, we have that $(J_i, v)_{(-r,r)} = (J_{1i}, v)_{(-r,r)}$ for $v \in n \times \mathscr{M}$. This yields, by choosing $v = J_{1i}/\|J_{1i}\|_0$ and by using duality and the equivalence of the norms in $n \times \mathscr{M}$, that there exists $a, c > 0$ such that

$$c\|J_{1i}\|_{-r} \leq \|J_{1i}\|_0 = |(J_i, v)_{(-r,r)}| \leq \|J_i\|_{-r} \frac{\|J_{1i}\|_r}{\|J_{1i}\|_0} \leq \frac{1}{a}\|J_i\|_{-r}.$$

Thus (J_{1i}) (and (J_{2i})) is a Cauchy-sequence in $H^{-r}(\Gamma_a)$. Let $J_{2i} \to J_2$ in $H^{-r}(\Gamma_a)$. Now we have by the formula (23) that

$$(J_2, g)_{(-r,r)} = 0 \quad \text{for } g \in C_r . \tag{24}$$

Finally, let $w \in TH^r_{\mathrm{Div}}(\Gamma_a)$ with $w = u+v$, where $u \in n\times\mathscr{M}$ and $v \in C_r$. As $(J_{2i}, u)_{(L_2,L_2)} = 0$, we have with (24) and the density of $TH^r_{\mathrm{Div}}(\Gamma_a)$ in $TH^r(\Gamma_a)$ that $J_2 = 0$. Thus $n \times J_0 \in n \times \mathscr{M}$.

(ii) This is proved correspondingly with the corresponding result of Theorem 6.2 for the problem (3). □

Note that the unique continuation principle is not useful in proving Theorem 3.1, because the current is not assumed to vanish in some part of the surface. In Theorem 6.3 we have this assumption and can use the unique continuation principle.

Remark 6.1. If neither $J \equiv 0$ nor $M \equiv 0$, the pair (J, M) cannot be determined uniquely by the measurement $(E|_\Gamma, H|_\Gamma)$ for any frequency ω. This follows from the formula (18) with (16).

Theorem 3.1 (or Theorem 6.3) also gives the uniqueness of dipole currents with the form $u \otimes \delta_a$ and currents on one-dimensional sets with the form $u \otimes \delta_L$ located either on some Γ_i, $i = 1, \ldots, N$, or in some Ω_j, $j = 1, \ldots, N$. Here $n \cdot u = 0$, if the set L or the point a is on Γ_i with n the normal on Γ_i.

We can prove that $\overline{sp}(\{u \in H^{s-1/2}(\Gamma_1 \cup \Gamma_2) | u = n \times E|_{\Gamma_1\cup\Gamma_2}$, with $(E, H) \in MW^s(\Omega)\})$ is not dense in $TH^{s-1/2}(\Gamma_1 \cup \Gamma_2)$. Thus by Theorem 6.1 the currents on two or more surfaces are not uniquely determined for any frequency. But if we assume that the currents are not spread on the whole surface, and we know the location of the current, we obtain the value of the current.

Theorem 6.3. *Let $M \equiv 0$ and J have the form of Theorem 6.1. Further, we assume that on each surface there is a point x_{ai} such that $J = 0$ on some neighborhood $U_{x_{ai}}$ of x_{ai}. We also assume that $\Gamma_{ai} \cap \Gamma_{ai} = \emptyset$ for $i \neq j$. Then the functions J_{0i} can be determined uniquely.*

Proof. Let the currents $J_k = \sum_{i=1}^m u_{ki} \otimes \delta_{\Gamma_{ai}}$, $k = 1, 2$, give the same measurement on the boundary Γ. By using the uniqueness of the exterior boundary value problem for Maxwell equations, the unique continuation principle and the assumption we obtain with the same method as in Theorem 3.1 in [17] (extending the fields over the interior surfaces) that the electromagnetic field given by the current $J_1 - J_2$ vanishes in $\mathbf{R}^3 \setminus \left(\bigcup_{i=1}^m \Gamma_{ai}\right)$.

Then by using the representation formula for the electromagnetic fields (Theorem 5.3), (17) and the jump relations in (49 and (5) we obtain that $u_{ki} = 0$ for every i and $k = 1, 2$. □

The proof of the next theorem shows that the assumption $\Gamma_{ai} \cap \Gamma_{ai} = \emptyset$ for $i \neq j$, can be replaced by the weaker assumption: no subset of $\bigcup_{i=1}^m \Gamma_{ai} \setminus \bigl(\bigcup_{i=1}^m \{U_{x_{ai}}\}\bigr)$ is a piecewise smooth closed surface which encloses a simply-connected region.

To determine the shape of the surface or the location of the surface is more complicated and usually we do not get uniqueness. In [2] the current was assumed to have the form $Q\delta_S u/|S|$, where the surface S is an open surface and admits the same normal vector u on each of its points. The uniqueness of the shape of the surface S and the constants Q and u was proved. These results are also achieved here in Theorem 6.5. We give here a result about the uniqueness of the shape or the location of the surface.

Theorem 6.4. (i) *Let $M \equiv 0$ and J have the form of Theorem* 6.1, *where each surface Γ_{ai} is a ball. We assume that on every surface $\Gamma_{ai} J = 0$ on some neighborhood of the points on the line from the center of this ball into the center of an other ball. We also assume that the location of the center of the balls Γ_{ai} are known. Then the radii of the balls can be determined uniquely.*

(ii) *Let the current J, $J = u \otimes \delta_{\Gamma_a}$, $u \in TH^{-s+1/2}(\Gamma_a)$, be on a half-ball Γ_a. Further, let the radius and the direction of the normal on the top of the half-ball be known. Then we can determine u and the location of the surface Γ_a.*

Proof. (ii) Let the currents $J_i = J_{i0} \otimes \delta_{B_i^{1/2}}$, $i = 1,2$, on the half-balls $B_i^{1/2}$ with the electromagnetic fields (E_i, H_i) give the same measurent. By using Theorem 5.3 or Remark 5.3 and the relations (5) we have that $[n \times H_i]_{B_i} = J_{i0}$, where B_i is a smooth closed surface containing $B_i^{1/2}$. The set $L := B_1^{1/2} \cap B_2^{1/2}$ is a one-dimensional set. This yields with the method in Theorem 6.3 and the unique continuation principle that the fields $H_1 - H_2$ and $E_1 - E_2$ vanish outside the surfaces $B_i^{1/2}$, $i = 1,2$.

Let $S_1 \subset B_1^{1/2}$ be a smooth surface piece so that $L \cap \overline{S_1} = \emptyset$. Further, let $U_1 \subset \mathbf{R}^3$ be an open set containing S_1, $U_1 \cap L = \emptyset$, and $u \in C_0^\infty(U_1)$ so that $u = 1$ in a neighborhood of S_1. As $uH_2 \in C^\infty(U_1)$ we get with Theorem 2.1 and the relations (5) that $0 = [n \times (u(H_1 - H_2))]_{B_1} = u|_{B_1} J_{10}$. Then the definition of S_1 and u yields that $\operatorname{supp}(J_{10}) = L$. With the same result for J_{20} we obtain with Theorem 6.3 the claim.

(i) Let the currents $J_k = \sum_{i=1}^m J_{i0} \otimes \delta_{B_{ki}}$, $k = 1,2$, give the same measurent. The assumption yields that there is no subset of $\bigcup_{k=1}^2 \bigl(\bigcup_{i=1}^m B_{ki}\bigr)$, where $J_1 - J_2 \neq 0$ and which encloses a simply connected region. The method in (ii) yields the claim. □

It is clear that in case (ii) we can, instead of a half-ball, choose any surface that satisfies the condition that the intersection of two surfaces of this kind does not enclose a simply connected region. Thus both the magnitude and the location of the finitely many currents on curves or the dipole currents can be determined uniquely. The last theorem treats the currents with the normal component on the surface.

Theorem 6.5. *Let* $\gamma, \mu \in C^\infty(\mathbf{R}^3)$. *Further, let the current* J, $J = u \otimes \delta_{\Gamma_a}$, $u \in NH^{-s+1/2}(\Gamma_a)$, $s > \frac{1}{2}$, *be on some surface* Γ_a. *We assume that the shape of the surface is known. We also assume that the intersection of two different surfaces devides each surface so that the part outside the intersection of each surface is connected. Then we can determine the distribution* u *and the location of the surface* Γ_a.

Proof. Let the currents J_1 and J_2 give the same measurement. By the assumption there exists a smooth surface Γ_{00}, which encloses a simply connected region and $\Gamma_a \cup \Gamma_b \subset \Gamma_{00}$. Let $J_1 - J_2 = (u_0 n) \otimes \delta_{\Gamma_{00}}$ with n the unit normal on Γ_{00}. As

$$n \cdot E = \frac{i}{\omega\gamma} \mathrm{Div}(n \times H),$$

formulas (13) and (17) yield that

$$\begin{aligned}(25)\qquad &\left(\mathrm{Grad}\left(\frac{i}{\omega\gamma}u_0\right), (n \times H)\right)_{(H^{-s-1/2}(\Gamma_{00}), H^{s+1/2}(\Gamma_{00}))} \\ &= -\left((u_0), \frac{i}{\omega\gamma}\mathrm{Div}(n \times H)\right)_{(H^{-s+1/2}(\Gamma_{00}), H^{s-1/2}(\Gamma_{00}))} = 0\end{aligned}$$

for $(E,H) \in MW^s(\Omega)$. Here the first equality is valid by the continuity of the surface gradient operator $w \mapsto \mathrm{Grad}(w)$ from $H^s(\Gamma_{00})$ into $TH^{s-1}(\Gamma_{00})$, $s \in \mathbf{R}$.

By the assumption we have that $\Gamma_{00} \setminus (\Gamma_a \cup \Gamma_b) \neq \emptyset$ with an area. Then the corresponding condition of Theorem 6.2 for the problem (3) yields that in (25) $n \times H|_{\Gamma_a \cup \Gamma_b} \in TH^{s+1/2}(\Gamma_a \cup \Gamma_b)$ is arbitrary. Thus (25) yields that $cu_0/\gamma =$ constant on Γ_{00}. As $u_0 = 0$ on $\Gamma_{00} \setminus (\Gamma_a \cup \Gamma_b)$, we get the claim. □

The condition in Theorem 6.5 means that the surface is either a plane or its location (and direction) is known in such a way that the condition is satisfied.

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