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ON THE DIMENSION OF p-HARMONIC MEASURE

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Abstract. In this paper we study the dimension of a measure associated with a positive p-harmonic function which vanishes on the boundary of a certain domain.

Introduction

Denote points in Euclidean 2-space $\mathbf{R}^2$ by $x = (x_1, x_2)$. Let $\langle \cdot , \cdot \rangle$ be the standard inner product on $\mathbf{R}^2$ and let $|x| = \langle x, x\rangle^{1/2}$ be the Euclidean norm of x. Set $B(x,r) = \{y \in \mathbf{R}^2 : |x-y| < r\}$ whenever $x \in \mathbf{R}^2$ and $r > 0$. Let dx denote Lebesgue measure on $\mathbf{R}^2$ and define k dimensional Hausdorff measure, in $\mathbf{R}^2$, $0 < k \leq 2$, as follows: For fixed $\delta > 0$ and $E \subseteq \mathbf{R}^2$, let $L(\delta) = \{B(x_i, r_i)\}$ be such that $E \subseteq \bigcup B(x_i, r_i)$ and $0 < r_i < \delta$, $i = 1, 2, \dots$. Set

$$\phi_\delta^k(E) = \inf_{L(\delta)} \Bigl(\sum \alpha(k)\, r_i^k\Bigr),$$

where $\alpha(k)$ denotes the volume of the unit ball in $\mathbf{R}^k$. Then

$$H^k(E) = \lim_{\delta \to 0} \phi_\delta^k(E), \quad 0 < k \leq 2.$$

If O is open and $1 \leq q \leq \infty$, let $W^{1,q}(O)$ be the space of equivalence classes of functions u with distributional gradient $\nabla u = (u_{x_1}, u_{x_2})$, both of which are qth power integrable on O. Let

$$\|u\|_{1,q} = \|u\|_q + \|\nabla u\|_q$$

be the norm in $W^{1,q}(O)$ where $\|\cdot\|_q$ denotes the usual Lebesgue q-norm in O. Let $C_0^\infty(O)$ be infinitely differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$. Let Ω be a domain (i.e. an open connected set) and suppose that the boundary of Ω (denoted $\partial\Omega$) is bounded and non-empty. Let N be a neighborhood of $\partial\Omega$, p fixed, $1 < p < \infty$,

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and u a positive weak solution to the p Laplace partial differential equation in $\Omega \cap N$. That is, $u \in W^{1,p}(\Omega \cap N)$ and

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, dx = 0 \tag{1.1}$$

whenever $\theta \in W_0^{1,p}(\Omega \cap N)$. Observe that if u is smooth and $\nabla u \neq 0$ in $\Omega \cap N$, then $\nabla \cdot (|\nabla u|^{p-2} \nabla u) \equiv 0$, in the classical sense, where $\nabla \cdot$ denotes divergence. We assume that u has zero boundary values on $\partial\Omega$ in the Sobolev sense. More specifically if $\zeta \in C_0^\infty(N)$, then $u\,\zeta \in W_0^{1,p}(\Omega \cap N)$. Extend u to $N \setminus \Omega$ by putting $u \equiv 0$ on $N \setminus \Omega$. Then $u \in W^{1,p}(N)$ and it follows from (1.1), as in [HKM], that there exists a positive finite Borel measure μ on $\mathbf{R}^2$ with support contained in $\partial\Omega$ and the property that

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx = -\int \phi \, d\mu \tag{1.2}$$

whenever $\phi \in C_0^\infty(N)$. For the reader's convenience we outline another proof of existence for μ under the assumption that u is continuous in N. We claim that it suffices to show

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx \leq 0$$

whenever $\phi \geq 0$ and $\phi \in C_0^\infty(N)$. Indeed once this claim is established, we get existence of μ as in (1.2) from basic Caccioppoli estimates (see Lemma 2.6) and the same argument as in the proof of the Riesz representation theorem for positive linear functionals on the space of continuous functions. To prove our claim we note that $\theta = \big[(\eta + \max[u - \varepsilon, 0])^\varepsilon - \eta^\varepsilon\big]\phi$ can be shown to be an admissible test function in (1.1) for small $\eta > 0$. From (1.1) we see that

$$\int_{\{u \geq \varepsilon\} \cap N} \big[(\eta + \max[u - \varepsilon, 0])^\varepsilon - \eta^\varepsilon\big] |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx \leq 0.$$

Using dominated convergence, letting first η and then $\varepsilon \to 0$ we get our claim.

We note that if $\partial\Omega$ is smooth enough, then

$$d\mu = |\nabla u|^{p-1} \, dH^1|_{\partial\Omega}. \tag{1.3}$$

In this paper we study for given p, $1 < p < \infty$, the Hausdorff dimension of μ (denoted H-dim μ) defined as follows:

$$\text{H-dim}\,\mu = \inf\{k : \text{there exists } E \text{ Borel} \subset \partial\Omega \text{ with } H^k(E) = 0 \text{ and } \mu(E) = \mu(\partial\Omega)\}.$$

To outline previous work, let $p = 2$ and u be the Green's function for Ω with pole at some point $x_0 \in \Omega$. Then μ is called harmonic measure for Ω relative to x_0. Carleson [C] showed $\text{H-dim}\,\mu = 1$ when $\partial\Omega$ is a snowflake and that $\text{H-dim}\,\mu \le 1$ for any scale invariant Cantor set. He was also the first to recognize the importance of $\int_{\partial\Omega_n} |\nabla g_n| \log |\nabla g_n|\, dH^1$ where g_n is the Green's function for Ω_n with pole at $x_0 \in \Omega_0$ and (Ω_n) is an increasing sequence of smooth domains whose union is Ω. Later Makarov [M] proved for any simply connected domain $\Omega \subset \mathbf{R}^2$, that $\text{H-dim}\,\mu = 1$. Jones and Wolff [JW] proved that $\text{H-dim}\,\mu \le 1$ when $\Omega \subset \mathbf{R}^2$ and μ exists. Wolff [W1] strengthened [JW] by showing that harmonic measure is concentrated on a set of σ-finite H^1 measure. We also mention results of Batakis [Ba], Kaufmann–Wu [KW], and Volberg [V] who showed for certain fractal domains and domains whose complements are Cantor sets that

$$\text{Hausdorff dimension of } \partial\Omega = \inf\{k : H^k(\partial\Omega) = 0\} > \text{H-dim}\,\mu. \tag{1.4}$$

Finally we note that higher-dimensional results for the dimension of harmonic measure can be found in [B], [W], and [LVV]. In this paper we prove the following theorems.

Theorem 1. *Let u, μ be as in* (1.1), (1.2). *If $\partial\Omega$ is a certain snowflake and $1 < p < 2$, then* $\text{H-dim}\,\mu > 1$ *while if* $2 < p < \infty$, *then* $\text{H-dim}\,\mu < 1$.

Theorem 2. *Let u, μ be as in* (1.1), (1.2). *If $\partial\Omega$ is a certain self-similar Cantor set and $2 < p < \infty$, then* $\text{H-dim}\,\mu < 1$.

Theorem 3. *Let u, μ be as in* (1.1), (1.2). *If $\partial\Omega$ is a quasicircle, then* $\text{H-dim}\,\mu \le 1$ *for* $2 \le p < \infty$, *while* $\text{H-dim}\,\mu \ge 1$ *for* $1 < p \le 2$.

The snowflakes and Cantor sets we consider are defined in Sections 3 and 5, respectively. The definition of a k-quasicircle, $0 < k < 1$, is given in Section 2. As motivation for our theorems we note that Wolff in [W] was able to estimate the dimension of certain snowflakes in $\mathbf{R}^3$. The first step in his construction was to determine the sign of

$$\int_{\partial\widetilde{D}(\varepsilon)} |\nabla\tilde{g}|(\,\cdot\,,\varepsilon) \ln |\nabla\tilde{g}(\,\cdot\,,\varepsilon)|\, dH^2 \tag{1.5}$$

where $\widetilde{D}(\varepsilon)$ is a certain domain with smooth boundary and $\tilde{g}(\,\cdot\,,\varepsilon)$ is Green's function for the Laplacian in $\widetilde{D}(\varepsilon)$ with pole at ∞. In analogy with Wolff and for p fixed, $1 < p < \infty$, put $D(\varepsilon) = \{x \in \mathbf{R}^2 : x_2 > \varepsilon\,\psi(x_1)\}$ where $\psi \in C_0^\infty(\mathbf{R})$ with $\psi \equiv 0$ in $\mathbf{R} \setminus (-1,1)$. Let $g(\,\cdot\,,\varepsilon)$ be a positive weak solution to the p-Laplace equation in $D(\varepsilon)$ with $g(\,\cdot\,,\varepsilon) = 0$ continuously on $\partial D(\varepsilon)$ and $g(x,\varepsilon) = x_2 + \omega(x,\varepsilon)$ where $\omega(\,\cdot\,,\cdot\,)$ is infinitely differentiable in $\overline{D}(\varepsilon) \times (-\varepsilon_0, \varepsilon_0)$ and

$$|\nabla\omega| \le k(1 + |x|)^{-2}, \quad x \in D(\varepsilon),$$

for some constant k independent of $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Existence of $g(\,\cdot\,,\varepsilon)$ and $\omega(\,\cdot\,,\varepsilon)$ can be deduced by Picard iteration and Schauder type estimates for $\varepsilon_0 > 0$, sufficiently small. In fact proceeding operationally, assume $g(x,\varepsilon) = x_2 + \omega(x,\varepsilon)$ exists. Assuming that $\nabla g \neq 0$ in $D(\varepsilon)$ and writing out the p-Laplace equation for g, one deduces that if $w(x_1,x_2) = \omega\big(x_1, \sqrt{p-1}\,x_2\big)$, then w satisfies

$$\Delta w(x) = F\left(w_{x_1}, \frac{w_{x_2}}{\sqrt{p-1}}, w_{x_1x_1}, \frac{w_{x_1x_2}}{\sqrt{p-1}}, \frac{w_{x_2x_1}}{\sqrt{p-1}}, \frac{w_{x_2x_2}}{p-1}\right) = H(w,x),$$

$x \in D(\varepsilon)$, where

$$\begin{aligned} F(q_1,q_2,q_{11},q_{12},q_{21},q_{22}) = -(p-2)&\sum_{i,j=1}^{2} q_i q_j q_{ij} \\ &- \left(2q_2 + \sum_{i,j=1}^{2} q_i^2\right)\left(\sum_{i=1}^{2} q_{ii}\right) - 2(p-2)\sum_{i,j=1}^{2} q_i q_{ij}. \end{aligned}$$

Also, $w = -\sqrt{p-1}\,x_2$ on $\partial D(\varepsilon)$. Let w_0 be the bounded solution to Laplace's equation in $D(\varepsilon)$ with $w_0 = -\sqrt{p-1}\,x_2$ on $\partial D(\varepsilon)$. Proceeding by induction, let w_{n+1} for $n = 1,2,\dots,$ be the bounded solution to

$$\Delta w_{n+1}(x) = H(w_n,x) \text{ in } D(\varepsilon) \text{ with } w_{n+1} = -\sqrt{p-1}\,x_2 \text{ continuously on } \partial D(\varepsilon).$$

Using Schauder type estimates one can show for $\varepsilon_0 > 0$, sufficiently small that $\lim_{n\to\infty} w_n = w$ exists with the desired smoothness properties. Thus $g(\,\cdot\,,\varepsilon)$ exists.

From (1.3) we get that the analogue of (1.5) for p fixed, $1 < p < \infty$, is

$$I(\varepsilon) = \int_{\partial D(\varepsilon)} |\nabla g(\,\cdot\,,\varepsilon)|^{p-1} \ln |\nabla g(\,\cdot\,,\varepsilon)|\, dH^1. \tag{1.6}$$

Following Wolff one calculates that $I(0) = I'(0) = 0$ and

$$I''(0) = \frac{p-2}{p-1}\int_{\mathbf{R}} \left(\frac{d\psi}{dx_1}\right)^2 dx_1.$$

Now if ε_0 is small enough, then I has three continuous derivatives on $(-\varepsilon_0,\varepsilon_0)$ and so by Taylor's theorem,

(1.7)

$I(\varepsilon) > 0$ when $p > 2$ and $I(\varepsilon) < 0$ for $1 < p < 2$, when ε is sufficiently small.

Initially we found it quite surprising that (1.7) held for fixed p, independently of ψ, especially in view of the examples in [W], [LVV]. Our first attempt at explaining (1.7) was to observe that if $\nabla g(\,\cdot\,,\varepsilon) \neq 0$ in $D(\varepsilon)$ and $v = \log |\nabla g|$ satisfies

$$\nabla \cdot (|\nabla g|^{p-2}\nabla v) < 0 \ (> 0) \text{ when } 1 < p < 2 \ (p > 2) \tag{1.8}$$

then (1.7) follows from the divergence theorem applied to $|\nabla g|^{p-2}[g\nabla v - v\nabla g]$. However a direct calculation shows no reason for (1.8) to hold. Next we tried to imitate Wolff's construction in order to produce examples of snowflakes where the conclusion of Theorem 1 held. To indicate the difficulties involved we note that Wolff shows Carleson's integral over $\partial\Omega_n$ can be estimated at the nth step in the construction of certain snowflakes $\subset \mathbf{R}^3$, provided the integral in (1.5) has a sign for small $\varepsilon > 0$. His calculations make key use of a boundary Harnack inequality for positive harmonic functions vanishing on a portion of the boundary in a non-tangentially accessible domain (see [JK]). Thus we prove a 'rate theorem' for the ratio of two positive p harmonic functions u_1, u_2 which are defined in $B(z,r)\cap\Omega$ and vanish continuously on $B(z,r)\cap\partial\Omega$, whenever $z\in\partial\Omega$ and $\partial\Omega$ is a quasicircle. We show that u_1/u_2 is bounded in $B(z,r/2)\cap\Omega$, by a constant that depends only on p, Ω, when $1<p<\infty$ (see Lemma 2.16), but are not able to show u_1/u_2 is Hölder continuous in $\Omega\cap B(z,r/2)$ when $1<p<\infty$, $p\neq 2$, as is the case when $p=2$ (see [JK]). The $p=2$ argument for Hölder continuity uses linearity of the Laplacian which is clearly not available for the p-Laplacian. Using just boundedness in the boundary Harnack inequality, we are still able to deduce that μ has a certain weak mixing property. An argument of Carleson–Wolff can then be applied to obtain an invariant ergodic measure ν on $\partial\Omega$ (with respect to a certain shift) satisfying (see Section 3),

$$\mu,\ \nu \text{ are mutually absolutely continuous.} \tag{1.9}$$

From ergodicity and (1.9) it follows that the ergodic theorem of Birkhoff and entropy theorem of Shannon–McMillan–Breiman can be used to get that

$$\lim_{r\to 0}\frac{\log\mu[B(x,r)]}{\log r} = \text{H-dim}\,\mu \text{ for } \mu \text{ almost every } x\in\partial\Omega. \tag{1.10}$$

In [W], Wolff uses Hölder continuity of the ratio in order to make effective use of (1.10) in his estimates of H-dim μ. We first tried to avoid the use of Hölder continuity in our estimates by a finess type argument which was supposed to take advantage of the constant sign in (1.7) when $p\neq 2$ is fixed. However, later this argument was shown to be incorrect because of a calculus type mistake. Finally in desperation we returned to our original idea of using the divergence theorem and finding a partial differential equation for which u is a solution and $v=\log|\nabla u|$ is a subsolution (super solution) when $p>2$ ($1<p<2$). To describe our efforts we note for u as in Theorem 1, that if $\eta\in\mathbf{R}^2$ with $|\eta|=1$, while ∇u is nonzero and sufficiently smooth in $\Omega\cap N$, then $\zeta=\langle\nabla u,\eta\rangle$, is a strong solution in $\Omega\cap N$ to

$$L\zeta = \nabla\cdot\big[(p-2)|\nabla u|^{p-4}\langle\nabla u,\nabla\zeta\rangle\nabla u + |\nabla u|^{p-2}\nabla\zeta\big] = 0. \tag{1.11}$$

Clearly,

$$Lu = (p-1)\nabla\cdot\big[|\nabla u|^{p-2}\nabla u\big] = 0 \quad \text{in } \Omega\cap N. \tag{1.12}$$

(1.11) can be rewritten in the form

$$L\zeta = \sum_{i,k=1}^{2} \frac{\partial}{\partial x_i}[b_{ik}(x)\zeta_{x_k}(x)] = 0, \tag{1.13}$$

where at $x \in \Omega \cap N$,

$$b_{ik}(x) = |\nabla u|^{p-4}\big[(p-2)u_{x_i}u_{x_k} + \delta_{ik}|\nabla u|^2\big](x), \quad 1 \leq i,k \leq 2, \tag{1.14}$$

and δ_{ij} is the Kronecker δ.

Next we assume at x that

$$\nabla u(x) = (1,0)|\nabla u(x)| \tag{1.15}$$

which is permissible since (1.11) is rotationally invariant. Then

$$v_{x_k} = |\nabla u|^{-2}\sum_{l=1}^{2} u_{x_l}u_{x_l x_k}$$

and so

$$Lv = \sum_{i,k=1}^{2} \frac{\partial(b_{ik}v_{x_k})}{\partial x_i} = \sum_{i=1}^{2}\frac{\partial}{\partial x_i}\bigg(|\nabla u|^{-2}\sum_{k,l=1}^{2} b_{ik}u_{x_l}\,u_{x_l x_k}\bigg).$$

Using (1.13) on the right-hand side of this display with $\eta = (1,0),(0,1)$, we get

$$\begin{aligned} Lv &= -2|\nabla u|^{-4}\sum_{i,k,l,m=1}^{2} b_{ik}(u_{x_l}u_{x_l x_k}u_{x_m}u_{x_m x_i}) \\ &\quad + |\nabla u|^{-2}\sum_{i,k,l=1}^{2} b_{ik}u_{x_l x_i}u_{x_l x_k} = T_1 + T_2. \end{aligned} \tag{1.16}$$

From (1.14), (1.15) we see at x that

$$b_{11} = (p-1)|\nabla u|^{p-2}, \quad b_{22} = |\nabla u|^{p-2}, \quad \text{and} \quad b_{12} = b_{21} = 0. \tag{1.17}$$

Also from (1.12), (1.15) we find that

$$(p-1)u_{x_1x_1} + u_{x_2x_2} = 0. \tag{1.18}$$

Using (1.17), (1.18) in the definitions of T_1, T_2 we obtain at x,

$$T_1 = -2|\nabla u|^{p-4}\big[(p-1)(u_{x_1x_1})^2 + (u_{x_1x_2})^2\big]$$

and

$$T_2 = p|\nabla u|^{p-4}\big[(p-1)(u_{x_1x_1})^2 + (u_{x_1x_2})^2\big].$$

Putting these equalities for T_1, T_2 in (1.16) we deduce

$$Lv = (p-2)|\nabla u|^{p-4}\big[(p-1)(u_{x_1x_1})^2 + (u_{x_1x_2})^2\big]. \tag{1.19}$$

From (1.19) we conclude that

$$Lv \geq 0 \text{ when } p > 2 \quad \text{and} \quad Lv \leq 0 \text{ when } p < 2. \tag{1.20}$$

The sign in (1.7) can now be explained by using (1.20) and applying the divergence theorem to the vector field whose ith component ($i = 1, 2$) is

$$u \sum_{k=1}^{2} b_{ik} v_{x_k} - v \sum_{k=1}^{2} b_{ik} u_{x_k}.$$

Hence we use (1.19), (1.20), and the divergence theorem in place of Hölder continuity of the ratio to make estimates and finally get Theorem 1. Our examples are more general than in [W] as our estimates do not require a smallness assumption on the Lipschitz norm of the piecewise linear function defining a snowflake. Theorem 2 is proved similarly. That is, we follow the general scheme of Carleson–Wolff and use (1.19)–(1.20) to make final estimates. To prove Theorem 3 we follow [M] and first prove an integral inequality involving $\log|\nabla u|$ (see Lemma 6.1). Theorem 3 follows easily from this inequality and the estimates in Section 2 for u, μ. The theorems are proved in the order they were conceived.

As for the plan of this paper, in Section 2 we list smoothness results for the p-Laplacian. We also list some results for quasiregular mappings and discuss their relationship to $\nabla\tilde{u}$, where $\tilde{u}$ is a certain p-harmonic function satisfying the hypotheses of Theorem 1 and/or Theorem 2. In Section 2 we also prove the 'rate theorem' discussed above and then use it in Section 3 to set up the ergodic apparatus necessary to prove Theorem 1. Finally in Section 4, we obtain Theorem 1. In Section 5, we prove Theorem 2. In Section 6 we prove Theorem 3 and make concluding remarks.

Finally we remark that the term p-harmonic measure was first used by Martio in [Mar]. His definition and our definition agree for $p = 2$ provided u is chosen to be the Green's function corresponding to the Laplacian with pole at some point in D. However, for $p \neq 2$, the two definitions are quite different due to the nonlinearity of the p-Laplace equation. In fact Martio's measure need not even be subadditive (see [LMW] for references and examples).

2. Basic estimates

We begin with some definitions. A Jordan curve J is said to be a k-quasicircle, $0<k<1$, if $J=f\big(\partial B(0,1)\big)$ where $f\in W^{1,2}(\mathbf{R}^2)$ is a homeomorphism of $\mathbf{R}^2$ and

$$(2.1)\qquad |f_{\bar z}|\le k|f_z|,\quad H^2 \text{ almost everywhere in } \mathbf{R}^2.$$

Here we are using complex notation, $i=\sqrt{-1}$, $z=x_1+ix_2$, $2f_{\bar z}=f_{x_1}+if_{x_2}$, $2f_z=f_{x_1}-if_{x_2}$. We say that J is a quasicircle if J is a k-quasicircle for some $0<k<1$. Let w_1, w_2 be distinct points on the Jordan curve J and let J_1, J_2 be the arcs on J with endpoints w_1, w_2. Then J is said to satisfy the Ahlfors three-point condition provided there exists $1\le M<\infty$ such that whenever $w_1,w_2\in J$, we have

$$(2.2)\qquad \min\{\operatorname{diam} J_1,\operatorname{diam} J_2\}\le M|w_1-w_2|.$$

Ω is said to be a uniform domain provided there exists $\widehat M$, $1\le \widehat M<\infty$, such that if $w_1,w_2\in\Omega$, then there is a rectifiable curve $\gamma\colon[0,1]\to\Omega$ with $\gamma(0)=w_1$, $\gamma(1)=w_2$, and

$$(2.3)\qquad \begin{aligned}&\text{(a)}\quad H^1(\gamma)\le \widehat M|w_1-w_2|,\\ &\text{(b)}\quad \min\big\{H^1\big(\gamma([0,t])\big),H^1\big(\gamma([t,1])\big)\big\}\le \widehat M d\big(\gamma(t),\partial\Omega\big).\end{aligned}$$

Here as in the sequel, $d(E,F)$ denotes the distance between the non-empty sets E and F. If $1\le\widetilde M<\infty$ and Ω is a domain, then a ball $B(w,r)\subset\Omega$ is said to be $\widetilde M$ non-tangential provided

$$\widetilde M r> d\big(B(w,r),\partial\Omega\big)>\widetilde M^{-1}r.$$

If $w_1,w_2\in\Omega$, then a Harnack-chain from w_1 to w_2 in Ω is a sequence of $\widetilde M$-non-tangential balls such that the first ball contains w_1, the last ball contains w_2, and consecutive balls intersect. A domain Ω is called non-tangentially accessible (NTA) if there exist $\widetilde M$ (as above) such that:

(2.4) (α) *Corkscrew condition.* For any $w\in\partial\Omega$, $0<r\le\operatorname{diam}\Omega$, there exists $a=a_r(w)\in\Omega$ such that $\widetilde M^{-1}r<|a-w|<r$ and $d(a,\partial\Omega)>\widetilde M^{-1}r$,

(2.4) (β) $\mathbf{R}^2\setminus\Omega$ satisfies the corkscrew condition,

(2.4) (γ) *Harnack chain condition.* Given $\varepsilon>0$, $w_1,w_2\in\Omega$, $d(w_j,\partial\Omega)>\varepsilon$, and $|w_1-w_2|<C\varepsilon$, there is a Harnack chain from w_1 to w_2 whose length depends on C but not on ε.

In (2.4) (α), $\operatorname{diam}\Omega$ denotes the diameter of Ω. For use in the sequel we list the following equivalences.

Lemma 2.5. *Ω is a uniform domain if and only if* (2.4) (α), (2.4) (γ) *hold. If $\partial\Omega = J$ is a Jordan curve, then the conditions*:

J is a quasicircle,
J satisfies the Ahlfors three point condition,
Ω is a uniform domain,
Ω is non-tangentially accessible,
all imply each other and constants in one definition can be determined from the constants in another definition.

Proof. See [G]. □

Note that Ω in Lemma 2.5 can be either of the two components of $\mathbf{R}^2 \setminus J$. In the sequel c will denote a positive constant ≥ 1 (not necessarily the same at each occurrence), which may depend only on p, unless otherwise stated. In general, $c(a_1, \dots, a_n)$ denotes a positive constant ≥ 1, which may depend only on $p, a_1, \dots, a_n$, not necessarily the same at each occurrence. We also assume that Ω is a domain and $0 < r < \operatorname{diam}\partial\Omega < \infty$. Next we state some interior and boundary estimates for $\tilde{u}$ a positive weak solution to the p-Laplacian in $B(w,4r) \cap \Omega$ with $\tilde{u} \equiv 0$ in the Sobolev sense on $\partial\Omega \cap B(w,4r)$ when this set is non-empty. More specifically, $\tilde{u} \in W^{1,p}\big(B(w,4r)\cap\Omega\big)$ and (1.1) holds whenever $\theta \in W_0^{1,p}\big(B(w,4r)\cap\Omega\big)$. Also $\zeta\tilde{u} \in W_0^{1,p}\big(B(w,4r)\cap\Omega\big)$ whenever $\zeta \in C_0^\infty\big(B(w,4r)\big)$. Extend $\tilde{u}$ to $B(w,4r)$ by putting $\tilde{u} \equiv 0$ on $B(w,4r)\setminus\Omega$. Then there exists a locally finite positive Borel measure $\tilde{\mu}$ with support $\subset B(w,4r)\cap\partial\Omega$ and for which (1.2) holds with u replaced by $\tilde{u}$ and $\phi \in C_0^\infty\big(B(w,4r)\big)$. Let $\max_{B(z,s)}\tilde{u}, \min_{B(z,s)}\tilde{u}$ be the essential supremum and infimum of $\tilde{u}$ on $B(z,s)$ whenever $B(z,s) \subset B(w,4r)$.

Lemma 2.6. *Let $\tilde{u}$ be as above. Then*

$$c^{-1} r^{p-2} \int_{B(w,r/2)} |\nabla\tilde{u}|^p \, dx \leq \max_{B(w,r)} \tilde{u}^p \leq c r^{-2} \int_{B(w,2r)} \tilde{u}^p \, dx.$$

If $B(w,2r) \subset \Omega$, then

$$\max_{B(w,r)} \tilde{u} \leq c \min_{B(w,r)} \tilde{u}.$$

Proof. The first display in Lemma 2.6 is a standard subsolution estimate while the second display is a standard weak Harnack estimate for positive weak solutions to nonlinear partial differential equations of p-Laplacian type (see [S]). □

Lemma 2.7. *Let $\tilde{u}$ be as in Lemma* 2.6. *Then $\tilde{u}$ has a representative in $W^{1,p}\big(B(w,4r)\cap\Omega\big)$ (also denoted $\tilde{u}$) with Hölder-continuous partial derivatives* *in $B(w,4r)\cap\Omega$. That is, for some $\sigma=\sigma(p)\in(0,1)$ we have*

$$c^{-1}|\nabla\tilde u(w_1)-\nabla\tilde u(w_2)|\le(|w_1-w_2|/s)^\sigma\max_{B(z,s)}|\nabla\tilde u|\le cs^{-1}(|w_1-w_2|/s)^\sigma\max_{B(z,2s)}\tilde u$$

whenever $w_1,w_2\in B(z,s)$ and $B(z,4s)\subset B(w,4r)\cap\Omega$.

Proof. The proof of Lemma 2.7 can be found in [D], [L1] or [T] and in fact is true when $B(w,4r)\cap\Omega\subset\mathbf{R}^n$. In $\mathbf{R}^2$ the best Hölder exponent in Lemma 2.7 is known when $p>2$ while for $1<p\le 2$ a solution has continuous second partials (see [IM]). □

In order to describe further ($\mathbf{R}^2$) results for solutions to the p-Laplacian we note that h: $B(w,4r)\cap\Omega\to\mathbf{R}^2$ is said to be quasiregular in $B(w,4r)\cap\Omega$ provided $h\in W^{1,2}\big(B(w,4r)\cap\Omega\big)$ and (2.1) holds with f replaced by h in $B(w,4r)\cap\Omega$. From a factorization theorem for quasiregular mappings, it then follows that $h=\tau\circ f$ where f is quasiconformal in $\mathbf{R}^2$ and τ is an analytic function on $f\big(B(w,4r)\cap\Omega\big)$. We have

Lemma 2.8. *If $\tilde u$ is as in Lemma 2.6 and $z=x_1+ix_2$, then $\tilde u_z$ is quasiregular in $B(w,4r)\cap\Omega$ for some $0<k<1$ (depending only on p) and consequently $\nabla\tilde u$ has only isolated zeros in $B(w,4r)\cap\Omega$.*

Proof. For a proof of quasiregularity (see [ALR], [L]). The fact that the zeros of $\nabla\tilde u$ are isolated follow from the above factorization theorem and the corresponding theorem for analytic functions. □

From Lemma 2.8 we deduce

Lemma 2.9. *If $\tilde u$ is as in Lemma 2.6, then $\tilde u$ is real analytic in $B(w,4r)\cap\big[\Omega\setminus\{x:\nabla\tilde u(x)\ne 0\}\big]$. Moreover if $B(w,4r)\subset\Omega$, $\nabla\tilde u\ne 0$ in $B(w,4r)$, and $\max_{B(w,2r)}|\nabla\tilde u|\le\lambda\max_{B(w,r)}|\nabla\tilde u|$ then*

$$(+)\qquad \max_{B(w,2r)}|\nabla\tilde u|\le c(\lambda)\min_{B(w,r)}|\nabla\tilde u|,$$

$$(++)\qquad \max_{x\in B(w,r)}\sum_{i,j=1}^{2}|\tilde u_{x_ix_j}|(x)\le c(\lambda)r^{-1}\max_{x\in B(w,2r)}|\nabla\tilde u|(x),$$

$$(+++)\qquad \max_{x,y\in B(w,r/2)}\sum_{i,j=1}^{2}|\tilde u_{x_ix_j}(x)-\tilde u_{y_iy_j}(y)|$$
$$\le c(\lambda)(|x-y|/r)\max_{x\in B(w,r)}\sum_{i,j=1}^{2}|\tilde u_{x_ix_j}|(x).$$

Proof. To prove Lemma 2.9, we first observe from (1.1) with u replaced by $\tilde u$ and Lemmas 2.7, 2.8 that

$$\tilde u\in W^{2,2}\big(B(w,4r)\cap\Omega\big)\quad\text{and}\quad\sum_{i,k=1}^{2}a_{ik}(x)\tilde u_{x_ix_k}(x)=0\tag{2.10}$$

for H^2 almost every $x \in B(w,4r) \cap \Omega$. Here $a_{ik} = |\nabla u|^{2-p} b_{ik}$, where b_{ik}, $1 \le i,k \le 2$, are as in (1.14). It is easily checked that (a_{ik}) are measurable, L^∞ bounded, and uniformly elliptic in $B(w,4r)\cap\Omega$ with L^∞ norm and ellipticity constant depending only on p. Second, using this observation, Lemma 2.7, Schauder theory, and a bootstrap argument (see [GT, Section 9.6]), we get that $\tilde u \in C^\infty\bigl(\Omega \cap B(w,4r)\bigr)$. Real analyticity of $\tilde u$ (i.e. a power series expansion in x_1, x_2) at points of $B(w,4r)\cap\Omega$ follows from a theorem of Hopf (see [H], [F], [F1]). To prove $(+)$ we note from Lemma 2.8 that $v = \log|\nabla \tilde u|$ is a weak solution in $B(w,4r)$ to the divergence form partial differential equation (see [Re, Chapter III]),

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\bigl(A_{ij}(x) v_{x_j}\bigr) = 0$$

where if $D\tilde u_z$ denotes the Jacobian matrix of $\tilde u_z$, $\det D\tilde u_z$ the Jacobian of $\tilde u_z$ and $D^t \tilde u_z$ the transpose of $D\tilde u_z$, then $A = (A_{ij})$ is defined by

$$A(x) = \begin{cases} [\det D\tilde u_z](D^t\tilde u_z \cdot D\tilde u_z)^{-1} & \text{if } D\tilde u_z \text{ is invertible,} \\ \text{Identity matrix, otherwise.} \end{cases}$$

From quasiregularity of $\tilde u_z$ and/or (2.10) we find that (A_{ij}) are L^∞-bounded and uniformly elliptic (again with constants depending only on p). Using Harnack's inequality for positive solutions to partial differential equations of this type (see [S]), applied to $\max_{B(w,2r)} v - v$ in $B(w,r)$ we get $(+)$. $(++)$ and $(+++)$ follow from $(+)$, (2.10), and once again Schauder estimates. □

Next we consider the behaviour of $\tilde u$ near $B(w,4r)\cap\partial\Omega$ and the relation between $\tilde u$, $\tilde\mu$. By a simply connected domain Ω we shall always mean that $\mathbf{R}^2 \setminus \Omega$ is a connected set of more than one point.

Lemma 2.11. *Let $\tilde u$ be as in Lemma 2.6 and $w \in \partial\Omega$. If $p > 2$ and $\partial\Omega$ is bounded, then there exists $\alpha = \alpha(p) \in (0,1)$ such that $\tilde u$ has a Hölder α continuous representative in $B(w,r)$ (also denoted $\tilde u$). Moreover if $x, y \in B(w,r)$ then*

$$|\tilde u(x) - \tilde u(y)| \le c(|x-y|/r)^\alpha \max_{B(w,2r)} \tilde u.$$

If $1 < p \le 2$, and Ω is simply connected, then this inequality is also valid when $1 < p \le 2$, with $\alpha = \alpha(p)$.

Proof. For $p > 2$, Lemma 2.11 is a consequence of Lemma 2.6 and Morrey's theorem (see [E, Chapter 5]). If $1 < p \le 2$ and Ω is simply connected we deduce from the interior estimates in Lemma 2.7 that it suffices to consider only the case when $y \in B(w,r)\cap\partial\Omega$. We then show for some $\theta = \theta(p,k)$, $0 < \theta < 1$, that

$$\max_{B(z,\varrho/4)} \tilde u \le \theta \max_{B(z,\varrho/2)} \tilde u \text{ whenever } 0 < \varrho < r \text{ and } z \in \partial\Omega \cap B(w,r). \tag{2.12}$$

(2.12) can then be iterated to get Lemma 2.11 for x,y as above. To prove (2.12) we use the fact that $B(z,\varrho/4)\cap\partial\Omega$ and $B(z,\varrho/4)$ have comparable p-capacities (see [HKM]) and estimates for subsolutions to elliptic partial differential equations of p-Laplacian type (see [GZ], [L]). □

Lemma 2.13. *Let $\tilde u$, Ω, w be as in Lemma 2.11. Assume also that Ω is a uniform domain. Then there exists $c=c(\widehat M)$ with*

$$\max_{B(w,2r)} \tilde u \le c\tilde u\big(a_r(w)\big)$$

where $\widehat M$ is as in (2.3) and $a_r(w)$ is as in (2.4)(α). Hence,

$$|\tilde u(x)-\tilde u(y)| \le c(|x-y|/r)^{\alpha}\tilde u\big(a_r(w)\big) \quad \text{for } x,y\in B(w,r).$$

Proof. The first display in Lemma 2.13 follows from Harnack's principle in Lemma 2.6, Hölder continuity of $\tilde u$ in Lemma 2.11, the fact that Ω is a uniform domain and a general argument which can be found in [CFMS]. The second display in Lemma 2.13 is a consequence of the first display and Lemma 2.11. □

Lemma 2.14. *Let $\tilde u$, Ω, w, r be as in Lemma 2.11 and $\tilde\mu$ as in (1.2) relative to $\tilde u$. Then there exists c such that*

$$c^{-1}r^{p-2}\tilde\mu[B(w,r/2)] \le \max_{B(w,r)} \tilde u^{p-1} \le cr^{p-2}\tilde\mu[B(w,2r)].$$

Proof. The proof of Lemma 2.14 when $\tilde u$ is a subsolution to a class of partial differential equations that includes the p-Laplacian can be found in [KZ, Lemma 3.1]. Their proof of the above inequality essentially just uses Harnack's inequality—Lemma 2.11 and is modeled on a previous proof for solutions to partial differential equations of p-Laplacian type in [EL, Lemma 1]. □

From Lemmas 2.13, 2.14 and Harnack's principle, we note that if Ω is a uniform domain, $x\in\partial\Omega\cap B(w,4r)$, and $B(x,4\varrho)\subset B(w,4r)$, then

$$\varrho^{p-2}\tilde\mu\big(B(x,\varrho)\big) \approx \tilde u\big(a_\varrho(x)\big)^{p-1} \approx \Big(\max_{B(x,\varrho)}\tilde u\Big)^{p-1}, \tag{2.15}$$

where $e\approx f$ means e/f is bounded above and below by positive constants. In this case the constants depend only on p, $\widehat M$.

Next we prove the rate theorem referred to in Section 1.

Lemma 2.16. *Let $\partial\Omega$ be a quasicircle and $\tilde u$, $\tilde\mu$, w as in Lemma 2.11. Let $v>0$, ν be as defined above Lemma 2.6, (1.2), respectively, with $\tilde u$, $\tilde\mu$ replaced by*

v, ν. If $\tilde u\big(a_r(w)\big) = v\big(a_r(w)\big)$, then there exists $c_+ = c_+(\widehat M)$, $M_+ = M_+(\widehat M) \ge 1$, such that

$$c_+^{-1} < \frac{\tilde u(x)}{v(x)} < c_+ \text{ for all } x \in B(w, M_+^{-1} r) \cap \Omega .$$

Proof. Let $\gamma\colon]-1,1[\to \mathbf{R}^2$ be a parametrization of $\partial\Omega \cap B(w,2r)$ such that $\gamma(0) = w$. Let $r_1 = r/c_1$ where $c_1 = c_1(\widehat M) \ge 1$ will be chosen later. Let $t_1 = \sup\{t < 0 : |\gamma(t) - w| = r_1\}$, $t_2 = \inf\{t > 0 : |\gamma(t) - w| = r_1\}$, $z_1 = \gamma(t_1)$, and $z_2 = \gamma(t_2)$. Then $|w - z_1| = |w - z_2| = r_1$ and the part of $\partial\Omega$ between z_1 and z_2 is contained in $B(w, r_1)$. If $r_2 = r_1/c_1$, then from (2.2) we see for c_1 large enough that $B(z_1, r_2) \cap B(z_2, r_2) = \emptyset$. For any point $\zeta_1 \in B(z_1, r_2) \cap \partial\Omega$ and $\zeta_2 \in B(z_2, r_2) \cap \partial\Omega$ we can use (2.3) to construct a curve with endpoints ζ_1, ζ_2, in the following way: Take ϱ such that $B(\zeta_i, \varrho) \subset B(z_i, r_2)$ for $i = 1, 2$. Draw the curve from $a_\varrho(\zeta_1)$ to $a_\varrho(\zeta_2)$ guaranteed by (2.3). Similarly, connect $a_\varrho(\zeta_1)$ to $a_{\varrho/2}(\zeta_1)$ and then $a_{\varrho/2}(\zeta_1)$ to $a_{\varrho/4}(\zeta_1)$, etc. Since $a_{\varrho/2^n}(\zeta_1) \to \zeta_1$ as $n \to \infty$ this curve ends up at ζ_1. We go from $a_\varrho(\zeta_2)$ to ζ_2 in the same way. The total curve from ζ_1 to ζ_2 is denoted by Γ.

From our construction and (2.3) we note for c_1 large enough that

$$\begin{aligned} &\text{(a)}\quad \Gamma \setminus \{\zeta_1, \zeta_2\} \subset B(w, r) \cap \Omega, \\ &\text{(b)}\quad H^1(\Gamma) \le c_1 r, \qquad\qquad (2.17)\\ &\text{(c)}\quad \min\big\{H^1\big(\Gamma([0,t])\big), H^1\big(\Gamma([t,1])\big)\big\} \le c_1 d\big(\Gamma(t), \partial\Omega\big). \end{aligned}$$

Fix c_1 satisfying the above requirements. Now suppose that $u/v \ge \lambda$ at some point in $B(z, M_+^{-1} r) \cap \Omega$ where $M_+(\widehat M)$ is chosen so large that $\Gamma \cap B(w, M_+^{-1} r_1) = \emptyset$ independently of $\zeta_1 \in B(z_1, r_2) \cap \partial\Omega$, $\zeta_2 \in B(z_2, r_2) \cap \partial\Omega$. Existence of M_+ follows from (2.17). Note from Lemma 2.11 that $\tilde u \equiv v \equiv 0$ on $\partial\Omega \cap B(w, r)$ and that $\tilde u$, v are continuous in $\overline{\Omega} \cap B(w, r)$. Using this note and the weak maximum principle for solutions to the p-Laplacian, we see that $\tilde u/v \ge \lambda$ at some point ξ on Γ. Then from (2.17), (2.15), and Harnack's inequality we deduce for some s, $0 < s < r/2$, that

$$\frac{\mu[B(\zeta, s)]}{\nu[B(\zeta, s)]} \ge c^{-1} \left[\frac{u(\xi)}{v(\xi)}\right]^{p-1} \ge c^{-1} \lambda^{p-1} = \lambda' \tag{2.18}$$

where $\zeta = \zeta_1$ or $\zeta = \zeta_2$. Allowing ζ_i to vary in $B(z_i, r_2) \cap \partial\Omega$, $i = 1, 2$, we get a covering of either $B(z_1, r_2) \cap \partial\Omega$ or $B(z_2, r_2) \cap \partial\Omega$ by balls of the form $B_\zeta = B(\zeta, s)$. Assume for example that $G = B(z_1, r_2) \cap \partial\Omega$ is covered by balls of this type. Then using a standard covering argument we get a subcovering $\{B_{\zeta_n}\}$ of G such that the balls with one-fifth the diameter of the original balls but the same centers, call them $\{B^*_{\zeta_n}\}$, are disjoint. From (2.17), (2.18), and Lemma 2.14 we deduce for some $c = c(\widehat M)$ that

$$\lambda' \nu(G) \le \lambda' \nu\Big(\bigcup_n B_{\zeta_n}\Big) \le \sum_n \mu(B_{\zeta_n}) \le c \sum_n \mu(B^*_{\zeta_n}) \le c^2 \mu\big(B(w, 2r)\big). \tag{2.19}$$

From (2.15) with u, μ replaced by v, ν, (2.3) and Harnack's principle we see for some $c = c(\widehat{M})$ that

$$\nu\big(B(w,2r)\big) \leq c\nu(G) \tag{2.20}$$

which together with (2.19), (2.15) and $u\big(a_r(w)\big) = v\big(a_r(w)\big)$ implies for some $c_* = c_*(\widehat{M})$,

$$c_*^{-1}\lambda' \leq \frac{\mu\big(B(w,2r)\big)}{\nu\big(B(w,2r)\big)} \leq c_*. \tag{2.21}$$

From (2.21) we conclude the validity of Lemma 2.16. □

We observe for later use that Lemma 2.16 remains valid for suitable c_+ if $B(w, M_+^{-1}r)$ is replaced by $B(w, r/2)$, as follows easily from applying Lemma 2.16 at points in $\partial\Omega \cap B(w, r/2)$ and using Harnack's inequality. Let $H \Delta K = (H \setminus K) \cup (K \setminus H)$ be the symmetric difference of H and K. For use in Section 3 we need

Lemma 2.22. *Let $\partial\widetilde{\Omega}_1, \partial\widetilde{\Omega}_2$ be quasicircles, $w \in \partial\widetilde{\Omega}_1 \cap \partial\widetilde{\Omega}_2$ and $\tilde{u}_1$, $\tilde{u}_2$ as in Lemma 2.6 with $\tilde{u}$, Ω replaced by $\tilde{u}_i$, $\widetilde{\Omega}_i, i = 1, 2$. Let $\tilde{\mu}_i$ be the corresponding measures for $\tilde{u}_i$, $i = 1, 2$, and suppose $E \subset \partial\widetilde{\Omega}_1 \cap \partial\widetilde{\Omega}_2$ is Borel. Also suppose that $C_1 \geq 1$, $B(w, C_1^{-1}r) \cap \partial\widetilde{\Omega}_1 \subset E$, $\operatorname{diam} E \leq r$ and $d(E, \partial\widetilde{\Omega}_1 \Delta \partial\widetilde{\Omega}_2) > C_1^{-1}r$. If $Y \subset E$, then for some $C = C(\widehat{M}, C_1)$,*

$$C^{-1}\frac{\tilde{\mu}_2(Y)}{\tilde{\mu}_1(Y)} \leq \frac{\tilde{\mu}_2(E)}{\tilde{\mu}_1(E)} \leq C\frac{\tilde{\mu}_2(Y)}{\tilde{\mu}_1(Y)}$$

whenever $Y \subset E$ is Borel and $\tilde{\mu}_1(Y) \neq 0$.

Proof. In Lemma 2.22, $\widehat{M}$ is a constant for which (2.3) is valid with Ω replaced by $\widetilde{\Omega}_i$, $i = 1, 2$. The assumptions in the lemma imply that we can cover E with a bounded number of balls of radius $\frac{1}{2}C_1^{-1}r$ whose doubles do not intersect $\partial\widetilde{\Omega}_1 \Delta \partial\widetilde{\Omega}_2$. In each of these balls we can apply the rate theorem and use Harnack's inequality to get

$$C^{-1}\frac{\tilde{u}_1\big(a_r(w)\big)}{\tilde{u}_2\big(a_r(w)\big)} < \frac{\tilde{u}_1(z)}{\tilde{u}_2(z)} < C\frac{\tilde{u}_1\big(a_r(w)\big)}{\tilde{u}_2\big(a_r(w)\big)}$$

provided $d(z, E) < \frac{1}{4}C_1^{-1}r$ and $z \in \widetilde{\Omega}_1 \cap \widetilde{\Omega}_2$. Furthermore using (2.15) we get

$$C^{-1}\frac{\tilde{\mu}_1(E)}{\tilde{\mu}_2(E)} < \left(\frac{\tilde{u}_1(z)}{\tilde{u}_2(z)}\right)^{p-1} < C\frac{\tilde{\mu}_1(E)}{\tilde{\mu}_2(E)} \tag{2.23}$$

where C in (2.23) has the same dependence as in Lemma 2.22. Using (2.23) and (2.15) once again we conclude for $y \in E$, $0 < \varrho < r$,

$$\frac{\tilde\mu_1\big(B(y,\varrho)\big)}{\tilde\mu_2\big(B(y,\varrho)\big)} \approx \frac{\tilde\mu_1(E)}{\tilde\mu_2(E)}$$

so if

$$\frac{d\tilde\mu_1}{d\tilde\mu_2}(y) = \lim_{\varrho\to 0} \frac{\tilde\mu_1\big(B(y,\varrho)\big)}{\tilde\mu_2\big(B(y,\varrho)\big)}$$

then

$$\frac{d\tilde\mu_1}{d\tilde\mu_2}(y) \approx \frac{\tilde\mu_1(E)}{\tilde\mu_2(E)}.$$

From Theorem 7.15 in [R] it follows that $\tilde\mu_1$ has no singular part with respect to $\tilde\mu_2$ and vice versa. Thus by the Radon–Nikodym theorem

$$\frac{\tilde\mu_1(Y)}{\tilde\mu_2(Y)} = \frac{\displaystyle\int_Y \frac{d\tilde\mu_1}{d\tilde\mu_2}\, d\tilde\mu_2}{\tilde\mu_2(Y)} \approx \frac{\tilde\mu_1(E)}{\tilde\mu_2(E)}$$

where the proportionality constant depends on C_1, $\widehat M$, p. □

Next suppose that Ω is simply connected and $\tilde u_i$, $\tilde\mu_i$, $i = 1,2$, are as in (1.1), (1.2) with u, μ replaced by $\tilde u_i$, $\tilde\mu_i$. Then from the maximum principle for p-harmonic functions we first see for some $c = c(\tilde u_1, \tilde u_2, N)$ that $\tilde u_i \le c\tilde u_j$, $i,j \in \{1,2\}$ and thereupon from Lemma 2.14 that

$$c^{-1}\tilde\mu_1\big(B(w,r/2)\big) \le \tilde\mu_2\big(B(w,r)\big) \le c\tilde\mu_1\big(B(w,2r)\big)$$

whenever $w \in \partial\Omega$ and $0 < r < \operatorname{diam}\partial\Omega$. This inequality and basic measure theoretic arguments, similar to the above, imply that

$$\text{H-dim}\,\tilde\mu_1 = \text{H-dim}\,\tilde\mu_2. \tag{2.24}$$

Thus to estimate $\text{H-dim}\,\mu$ where u is as in (1.1), (1.2) and Ω is simply connected it suffices to find the Hausdorff dimension of either μ_1 or μ_2, corresponding to u_1, u_2, respectively, where u_1, u_2 are defined as follows. If Ω is bounded we write $\Omega = \Omega_1$, while if Ω is unbounded we put $\Omega = \Omega_2$. In the bounded case choose $\hat x$ in Ω_1 and $\hat r > 0$ so that $B(\hat x, 4\hat r) \subset \Omega_1$. In the unbounded case, choose $\hat x \in \mathbf{R}^2 \setminus \Omega$ and $\widehat R > 0$ so that $\partial B(\hat x, \widehat R/4) \subset \Omega_2$. Also,

$$\text{either } \hat r \approx d(\hat x, \partial\Omega) \text{ or } \widehat R \approx \operatorname{diam}\Omega. \tag{2.25}$$

In the bounded case, let u_1 be a weak solution to the p-Laplacian in $D_1 = \Omega_1 \setminus B(\hat x, \hat r)$ with $u_1 = 0$ on $\partial\Omega_1$ and $u_1 = 1$ on $\partial B(\hat x, \hat r)$ in the Sobolev sense. In the unbounded case, let u_2 be a weak solution to the p-Laplacian in $D_2 = \Omega_2 \cap B(\hat x, \widehat R)$ with $u_2 = 0$ on $\partial\Omega_2$ and $u_2 = 1$ on $\partial B(\hat x, \widehat R)$ in the Sobolev sense. Existence of u_1, u_2 follow from the usual variational argument (see [GT]). Let $u', D', r' = u_1, D_1, \hat r$, when Ω is bounded and $u', D', r' = u_2, D_2, \widehat R$, otherwise.

Lemma 2.26. *We have $\nabla u' \neq 0$ in D' and u' is real analytic in an open set containing $D' \cup \partial B(\hat{x}, r')$. Also if $x \in D'$ and $\partial\Omega$ is a quasicircle, then for some $c = c(\widehat{M}) \geq 1$,*

$$c^{-1} d(x, \partial\Omega)^{-1} u'(x) \leq |\nabla u'(x)| \leq c d(x, \partial\Omega)^{-1} u'(x)$$

and

$$\sum_{l,m=1}^{2} |u'_{x_l x_m}|(x) \leq c d(x, \partial\Omega)^{-2} u'(x).$$

Proof. $\nabla u' \neq 0$ in D', is proved in [L, (1.4)]. Real analyticity of u' in D' then follows from Lemma 2.9. To outline the proof of real analyticity on $\partial B(\hat{x}, r')$ suppose $\hat{x} = 0$ and $r' = \hat{r}$. Let $f(y_1, y_2) = \bigl(y_1, y_2 + \sqrt{\hat{r}^2 - y_1^2}\bigr)$ when $(y_1, y_2) \in H = B(0, \hat{r}/2) \cap \{y : y_2 > 0\}$. Then $f(H) \subset D_1$ and $f(\partial H \cap \{y : y_2 = 0\}) \subset \partial B(0, \hat{r})$. Moreover $w = 1 - u_1 \circ f$ is a weak solution to

$$\nabla \cdot \bigl(\langle A\nabla w, \nabla w\rangle^{(p/2-1)} A\nabla w\bigr) = 0 \quad \text{in } H, \tag{2.27}$$

where

$$A(y) = \begin{pmatrix} 1 & \dfrac{-y_1}{\sqrt{\hat{r}^2 - y_1^2}} \\ \dfrac{-y_1}{\sqrt{\hat{r}^2 - y_1^2}} & \dfrac{\hat{r}^2}{\hat{r}^2 - y_1^2} \end{pmatrix}$$

and ∇w is regarded as a column matrix. Also, $w = 0$ on $\partial H \cap \{y : y_2 = 0\}$ in the Sobolev sense. Note that A has real analytic coefficients which depend only on y_1. We now apply our previous program to w. That is, arguing as in [D], [L1], [T], we first show that w has Hölder continuous derivatives locally in H. Next using a boundary Harnack inequality for nondivergence form equations of Krylov and a Campanato type argument as in [Li], we see that w extends to a function with Hölder continuous derivatives in $\overline{H}$. Moreover partial derivatives of w satisfy the inequality in Lemma 2.7 with u replaced by w and $B(z,s)$, $B(z,2s)$ replaced by $\overline{H}$. Before proceeding further we note that these arguments also give boundary regularity in higher dimensions. A somewhat simpler proof of the above results, valid only in two dimensions, will be given in the first author's thesis. Second, using Schauder estimates and a bootstrap argument we get that $w \in C^\infty(\overline{H})$. Third, real analyticity of w follows (once again) from a theorem of Hopf (see [H], [F], [F1]). Finally from $w \leq 1$ we deduce that $|\nabla w| \leq c\hat{r}^{-1}$. Transforming back by f we get results for u_1 in a neighborhood of $(0, \hat{r})$. Covering $\partial B(0, \hat{r})$ by such neighborhoods we conclude that u_1 is real analytic on $\partial B(0, \hat{r})$. Moreover, for some positive c,

$$c^{-1}\hat{r}^{-1} \leq |\nabla u_1| \leq c\hat{r}^{-1}, \quad x \in B(\hat{x}, [1 + c^{-1}]\hat{r}) \setminus B(\hat{x}, [1 - c^{-1}]\hat{r}),$$

as follows for example from the analogue of Lemma 2.7 for the extension of u_1 and barrier type estimates. As in Lemma 2.9, this inequality implies

$$\sum_{l,m=1}^{2} |(u_1)_{x_l x_m}| \le c^2 \hat{r}^{-2}, \quad x \in B(\hat{x}, [1+c^{-2}]\hat{r}) \setminus B(\hat{x}, [1-c^{-2}]\hat{r})$$

provided c is large enough. From these inequalities and (2.25) we see that to complete the proof of Lemma 2.26 when $u' = u_1$ and $\partial\Omega$ is a quasicircle, it remains to consider the case when $x \in \Omega_1 \setminus B(\hat{x}, [1+c^{-2}]\hat{r})$. First suppose that $x \in \Omega_1$ and $B\big(x, d(x)\big) \cap B(\hat{x}, [1+c^{-2}]\hat{r}) = \emptyset$. Choose $y \in B\big(x, d(x)\big)$ with $u(y) = u(x)/2$. We apply the mean value theorem of elementary calculus to u restricted to the line segment with endpoints, x, y. Then from this theorem and Lemma 2.13 we deduce the existence of $\tilde{c} = \tilde{c}(\widehat{M}) \ge 4$ and z such that $y, z \in B\big(x, (1-\tilde{c}^{-1})d(x)\big)$ and

$$u_1(x)/2 = |u_1(x) - u_1(y)| \le |\nabla u_1(z)|\,|x-y|.$$

Using this equality and Lemma 2.7 we see for some positive $\hat{c}$ that if

$$t_2 = [1-(2\tilde{c})^{-1}]\, d(x,\partial\Omega), \quad t_1 = (1-\tilde{c}^{-1}) d(x,\partial\Omega),$$

then

$$\hat{c}^{-1} u_1(x)/d(x,\partial\Omega) \le \max_{B(x,t_1)} |\nabla u_1| \le \max_{B(x,t_2)} |\nabla u_1| \le \hat{c} u_1(x)/d(x,\partial\Omega). \tag{2.28}$$

From (2.28), Lemma 2.9, and an iteration type argument, we conclude that Lemma 2.26 is valid for u_1 at x. Lemma 2.26 for $x \in \Omega_1 \setminus B(\hat{x}, [1+c^{-2}]\hat{r})$ follows from the previous special case, simple connectivity of Ω_1, and once again an iteration argument using Lemma 2.9. Thus in all cases Lemma 2.26 is valid when $u' = u_1$. The proof of Lemma 2.26 for $u' = u_2$ is similar. We omit the details. □

3. Construction of snowflakes and a shift invariant ergodic measure

We begin this section by constructing "snowflake"-type regions and showing that they are quasi circles. We follow the construction in [W]. Let $\phi\colon \mathbf{R} \to \mathbf{R}$ be a piecewise linear function with $\operatorname{supp}\phi \subset]-1,1[$. Let $0 < b$, $0 < \hat{b} < \tilde{b}$, and Q a line segment (relatively open) with center a_Q, length $l(Q)$. If e is a given unit normal to Q define

$$\begin{aligned} T_Q &= \operatorname{cch}\big(Q \cup \{a_Q + b l(Q) e\}\big), \\ \widetilde{T}_Q &= \operatorname{int}\ \operatorname{cch}\big(Q \cup \{a_Q - \tilde{b} l(Q) e\}\big), \\ \widehat{T}_Q &= \operatorname{int}\ \operatorname{cch}\big(Q \cup \{a_Q - \hat{b} l(Q) e\}\big) \end{aligned}$$

where cch E, int E, denote the closed convex hull and interior of the set E. Let $e=-e_2$ and define

$$\Lambda=\{x\in T_{]-1,1[}\cup \widetilde{T}_{]-1,1[}: x_2>\phi(x_1)\},$$
$$\partial=\{x\in \mathbf{R}^2: x_1\in]-1,1[, x_2=\phi(x_1)\}.$$

We assume that

$$\partial\subset \operatorname{int}\big(T_{]-1,1[}\cup \widehat{T}_{]-1,1[}\big). \tag{3.1}$$

Suppose Ω is a domain with a piecewise linear boundary and $Q\subset\partial\Omega$ is a line segment with $T_Q\cap\Omega=\emptyset$, $\widetilde{T}_Q\subset\Omega$. Let e be the normal to Q pointing into T_Q. Form a new domain $\widetilde{\Omega}$ as follows: Let S be the affine map with $S(]-1,1[)=Q, S(0)=a_Q, S(-e_2)=e$. Let $\Lambda_Q=S(\Lambda)$ and $\partial_Q=S(\partial)$. Then $\widetilde{\Omega}\cap(T_Q\cup\widetilde{T}_Q)=\Lambda_Q$ and $\widetilde{\Omega}\setminus(T_Q\cup\widetilde{T}_Q)=\Omega\setminus(T_Q\cup\widetilde{T}_Q)$. We call this process 'adding a blip to Ω along Q'. Let Ω_0 be one of the following domains: (a) the interior of the unit square with center at the origin, (b) the equilateral triangle with sidelength one and center at the origin, (c) the exterior of the set in (a) and (d) the exterior of the set in (b). We assume that b, $\hat{b}$, $\tilde{b}$, ϕ are such that inductively we can define Ω_n as follows. If $n\ge 1$ and Ω_{n-1} has been defined, then its boundary is given as $\partial\Omega_{n-1}=E_{n-1}\cup\big(\bigcup_{\underline{G}_{n-1}}Q\big)$ where E_{n-1} is a discrete set of points and

$$\begin{aligned}
&(\alpha)\quad \text{Each } Q\in\underline{G}_{n-1} \text{ is a line segment with } T_Q\cap\Omega_{n-1}=\emptyset,\ \widetilde{T}_Q\subset\Omega_{n-1}.\\
&(\beta)\quad \text{If } Q_1,Q_2\in\underline{G}_{n-1} \text{ then } \operatorname{int}T_{Q_1}\cap\operatorname{int}T_{Q_2}=\emptyset \text{ and } \widetilde{T}_{Q_1}\cap\widetilde{T}_{Q_2}=\emptyset.\\
&(\gamma)\quad \text{If } Q_1,Q_2\in\underline{G}_{n-1} \text{ have a common endpoint, } z, \text{ then there exists}\\
&\qquad \text{an open half space } P \text{ with } z\in\partial P \text{ and}\\
&\qquad \textstyle\bigcup_{i=1}^2 \widetilde{T}(Q_i)\cup T(Q_i)\subset P\cup\{z\}.
\end{aligned} \tag{3.2}$$

To form Ω_n add a blip along each $Q\in\underline{G}_{n-1}$. Then $\partial\Omega_n=E_{n-1}\cup\big(\bigcup_{Q\in\underline{G}_{n-1}}\partial_Q\big)$ is decomposed as $E_n\cup\big(\bigcup_{Q\in\underline{G}_n}Q'\big)$ where Q' are the line segments which make up ∂_Q. In this case we say that Q' is directly descended from Q. Assume that (3.2) holds with Q, $n-1$ replaced by Q', n. Also assume for Q' directly descended from $Q\in\underline{G}_{n-1}$ that

$$\begin{aligned}
&(+)\quad T_{Q'}\cup\widetilde{T}_{Q'}\subset T_Q\cup\widetilde{T}_Q,\\
&(++)\quad T_{Q'}\cup\widehat{T}_{Q'}\subset T_Q\cup\widehat{T}_Q,\\
&(+++)\quad T_{Q'}\subset\mathbf{R}^2\setminus\Lambda_Q, \widetilde{T}_{Q'}\subset\Lambda_Q.
\end{aligned} \tag{3.3}$$

So by assumption, the induction hypothesis is satisfied and the construction can continue. By induction, we obtain Ω_n, $n=1,2,\dots$. Assuming there exists (Ω_n)

satisfying (3.2), (3.3) for $n = 1, 2, \dots$, we claim that $\Omega = \lim_{n\to\infty} \Omega_n$, where convergence is in the sense of Hausdorff distance. In fact from (3.1)–(3.3) we deduce the existence of $c \ge 1$ and θ, $0 < \theta < 1$, such that for $m \ge n \ge 0$,

$$\sup_{x\in\Omega_m} d(x, \Omega_n) + \sup_{x\in\Omega_n} d(x, \Omega_m) \le c\theta^n. \tag{3.4}$$

Let

$$\Omega = \bigcap_{n=1}^{\infty} \biggl(\bigcup_{m=n}^{\infty} \Omega_m \biggr).$$

Using (3.4) and taking limits we deduce that (3.4) holds with Ω_m replaced by Ω. Hence our claim is true. We note that (3.1)–(3.3) are akin to the 'open set condition' for existence of a self similar set (see [Ma, Chapter 4]).

Next let $\underline{G} = \bigcup_{n=1}^{\infty} \underline{G}_n$. If $Q, Q' \in \underline{G}$ we say Q' is descended from Q in n stages if there are $Q_0 = Q', Q_1, \dots, Q_n = Q$ with Q_j directly descended from Q_{j+1} for $j = 0, \dots, n-1$. If Q' is descended from Q we write $Q' \prec Q$. Also if $Q \in \underline{G}$, let $\Gamma_Q = \partial\Omega \cap [T_Q \cup \widetilde{T}_Q]$. Note from (3.1)–(3.3) that if $Q' \in \underline{G}_m$, $Q \in \underline{G}_n$, and $m > n$, then $\Gamma_{Q'} \subset \Gamma_Q$ provided $Q' \prec Q$ while otherwise $\Gamma_Q \cap \Gamma_{Q'} = \emptyset$. We prove

Lemma 3.5. *$\partial\Omega$ is a quasicircle.*

Proof. To prove Lemma 3.5 it suffices to prove that

(3.6) $\partial\Omega_m$ satisfies the Ahlfors three-point condition in (2.2) for $m = 1, 2, \dots$,

with constant M independent of m. Indeed, once (3.6) is proved we can use Lemma 2.5 to get (f_m) a sequence of quasiconformal mappings of $\mathbf{R}^2$ and $0 < k < 1$, independent of m, with

$$f_m\bigl(\partial B(0,1)\bigr) = \partial\Omega_m \quad \text{and} \quad |(f_m)_{\bar z}| \le k|(f_m)_z| \tag{3.7}$$

for all $z = x_1 + ix_2$ in $\mathbf{R}^2$. Using the fact that a subsequence of (f_m) converges uniformly on compact subsets of $\mathbf{R}^2$ to a quasiconformal $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ (see [A] or [Re]) and (3.4), we deduce that (3.7) holds with f_m, Ω_m, replaced by f, Ω. Thus Lemma 3.5 is true once we show that (3.6) holds. For this purpose note that by construction Ω_m is a piecewise linear Jordan curve with a finite number of segments. First assume z_1, z_2, z_3 lie on line segments in $\partial\Omega_m$ which are descendants of the same side in $\underline{G}_0$ and z_3 lies between z_1, z_2. Then there exists a line segment $Q_0 \in \underline{G}$ with the property that z_1, z_2, z_3 all are descendants of Q_0 and this property is not shared by any descendant of Q_0. Suppose z_i, $i = 1, 2, 3$ belong to line segments descended from Q_i where $Q_i \subset \partial_{Q_0}$ and $Q_1 \cap Q_2 = \emptyset$. Let z_i^* denote the projection of z_i on Q_i. Using (3.1)–(3.3) we see for b, $\tilde b$, small enough that

$$|z_1 - z_2| \ge |z_1^* - z_2^*|/2 \quad \text{and} \quad |z_i - z_3| \le 2|z_i^* - z_3^*| \quad \text{for } i = 1, 2, \dots.$$

These inequalities and Lipschitzness of ϕ now imply the existence of M, $1 \le M < \infty$, depending only on b, $\hat{b}$, $\tilde{b}$, and the Lipschitz norm of ϕ with

$$\frac{\max\{|z_1 - z_3|, |z_2 - z_3|\}}{|z_1 - z_2|} \le M. \tag{3.8}$$

Choosing J_1, J_2 to be arcs on $\partial\Omega_m$ connecting z_1 to z_2 and taking the supremum over all z_3 between z_1, z_2, we deduce from (3.8) that (2.2) is valid. The situation when z_1, z_2 lie on different sides in $\underline{G}_0$ is handled similarly using (3.1)–(3.3). We omit the details. From our earlier reductions we conclude first (3.6) and second Lemma 3.5. □

As for existence of Ω, Ω_n satisfying (3.2)–(3.4), Lemma 3.5, (3.6), and (3.7) we note that if ψ is any piecewise linear Lipschitz function with support contained in $]-1,1[$, then the above construction can be carried out for $\phi(x) = A^{-1}\psi(Ax)$, $x \in \mathbf{R}$, provided A is suitably large and b, $\tilde{b}$ small enough. To give some explicit examples, fix $\theta, 0 < \theta < \pi/2$, and let $\varrho = (1 + \cos\theta)^{-1}$. Define $\phi = \phi(\,\cdot\,, \theta)$ on $\mathbf{R}$ as follows.

$$\phi(x) = \begin{cases} 0 & \text{when } |x| \ge 1 - \varrho, \\ \varrho \sin\theta - |x| \tan\theta & \text{when } 0 \le |x| < 1 - \varrho. \end{cases}$$

Then the above construction can be carried out for this ϕ and suitable b, $\tilde{b}$, $\hat{b}$. If for example Ω_0 is the exterior of the unit square with center at the origin, one can show from some basic trigonometry that it suffices to choose $b = \delta$, $\hat{b} = \frac{1}{2}\sin\bigl(\frac{1}{2}\theta + \delta\bigr)$, $\tilde{b} = \frac{1}{2}\sin\bigl(\frac{1}{2}\theta + 2\delta\bigr)$ provided $\delta > 0$ is small enough (any δ with $0 < 4\delta < \pi/2 - \theta$ works for $\hat{b}$, $\tilde{b}$). The choice $\theta = \pi/3$ yields the Van Koch snowflake for $\partial\Omega$.

Next we relate our snowflake, Ω, constructed as in (3.1)–(3.8), to a certain Bernoulli shift. Following [C], [W], we number the line segements on the graph of the Lipschitz function ϕ in (3.1) over $]-1,1[$. Number these segments from left to right (so the first and last segments are subsets of $]-1,1[$). Suppose there are $l > 1$ segments in the graph of ϕ over $]-1,1[$. If $\Sigma \in \underline{G}_0$, then there is a one to one correspondence between the set $\{1,\dots,l\}$ and the line segments $Q \in \underline{G}_1$, $Q \prec \Sigma$. Using the same numbering of the graph of ϕ on the second generation of line segments we get a correspondence between $Q \in \underline{G}_2$, $Q \prec \Sigma$ and $\bigl\{(\omega_1, \omega_2) : \omega_i \in \{1,\dots,l\}\bigr\}$. If we continue in this way it is clear that we get a correspondence between all but a countable set of points on the snowflake (corresponding to the endpoints of intervals in $\underline{G}$) and the set Θ of infinite sequences:

$$\Theta = \bigl\{\omega = (\omega_1, \omega_2, \dots) : \omega_i \in \{1,\dots,l\}\bigr\}.$$

Define the shift S on Θ by

$$S(\omega) = (\omega_2, \omega_3, \dots) \quad \text{whenever } \omega = (\omega_1, \omega_2, \dots) \in \Theta. \tag{3.9}$$

Hence for ω as in (3.9),

$$S^{-1}(\omega) = \{\omega^1, \omega^2, \ldots, \omega^l\} \quad \text{where } \omega^i = (i, \omega_1, \ldots),\ 1 \le i \le l. \tag{3.10}$$

The geometric meaning of the shift S on Γ_Σ is as follows. If $Q \in \underline{G}_1, Q \prec \Sigma$, the restriction of S to Γ_Q is a composition of a rotation, translation, and dilation (i.e. a conformal affine map) which takes Q to Σ and maps the normal into T_Q to the normal into T_Σ. Thus if $n \ge 1$, $Q \in \underline{G}_n$, $Q \prec \Sigma$, then the restriction of S^n to Γ_Q takes Q to Σ and maps the normal into T_Q to the normal into T_Σ. Therefore, we can regard S as either a function defined on Θ or on $\partial\Omega$ minus a countable set. For any $x \in \Gamma_\Sigma$ which is not an endpoint of a $Q \in \underline{G}_n$, let $Q_n(x)$ be the line segment with $x \in \Gamma_{Q_n(x)}$ and $Q_n(x) \in \underline{G}_n$. Let u' be as in Lemma 2.26 defined relative to $\partial\Omega$ with $\hat{x} = 0$, $\hat{r} = 1/200$, $\widehat{R} = 200$. Then either $\Omega = \Omega_1$ or $\Omega = \Omega_2$. We write u, D for u', D' in Lemma 2.26. Let μ be the measure corresponding to u as in (1.2). We shall show later (see (3.17) or (3.28)) that μ has no point masses. Thus we can regard μ as being defined on Θ.

Lemma 3.11. *The measure μ is mutually absolutely continuous with respect to a Borel measure ν which is invariant under S.*

Proof. By 'ν invariant under S', we mean that

$$\nu(S^{-1}E) = \nu(E) \quad \text{whenever } E \subset \partial\Omega \text{ is Borel.} \tag{3.12}$$

Let

$$\nu^{(n)}(Y) = \frac{1}{n}\sum_{j=0}^{n-1} \mu(S^{-j}Y)$$

and let ν be a weak* limit point of $\{\nu^{(n)}\}$. Then if Y is a Borel subset of Γ_Σ, we have

$$\nu^{(n)}(S^{-1}Y) - \nu^{(n)}(Y) = \frac{1}{n}\big(\mu(S^{-n}Y) - \mu(Y)\big) \to 0 \quad \text{as } n \to \infty,$$

since μ is finite. Thus ν is invariant under S. To prove mutual absolute continuity, given $\varepsilon > 0$, let $E_\varepsilon = \big\{x \in \Gamma_\Sigma : d(x, \{a, b\}) < \varepsilon\big\}$ where a, b are the endpoints of Γ_Σ. We claim for some c, $c_\varepsilon = c(\varepsilon)$, that

$$\begin{aligned} &(*) \quad \text{If } Y \subset \Gamma_\Sigma \text{ and } Y \cap E_\varepsilon = \emptyset, \text{ then } c_\varepsilon^{-1}\mu(Y) \le \nu(Y) \le c_\varepsilon \mu(Y), \\ &(**) \quad \nu(E_\varepsilon) \le c\varepsilon^\alpha \text{ for some } \alpha,\ 0 < \alpha < 1. \end{aligned} \tag{3.13}$$

To prove (3.13)$(*)$ let $Q \in \underline{G}_n$ and let V be the restriction of S^n to Γ_Q. Then as noted below (3.10), V is a conformal affine map of $\mathbf{R}^2$ onto $\mathbf{R}^2$ with $V(\Gamma_Q) = \Gamma_\Sigma$. Let $\tilde{u}_2(x) = u(V^{-1}x)$, $x \in V(\overline{D})$. We note that the p-Laplacian partial differential equation is rotation, translation and dilation invariant. Thus u_2 is p-harmonic and we can apply Lemma 2.22 with $\tilde u_1, \widetilde\Omega_1 = u, \Omega$, respectively, while $\widetilde\Omega_2 = V(\Omega)$ and $E = \Gamma_\Sigma \setminus E_\varepsilon$. If $Y \subset E$ is Borel, we get for some $\tilde c_\varepsilon = c(\varepsilon) \geq 1$ that

$$\tilde c_\varepsilon^{-1} \frac{\tilde\mu_2(E)}{\mu(E)} \leq \frac{\tilde\mu_2(Y)}{\mu(Y)} \leq \tilde c_\varepsilon \frac{\tilde\mu_2(E)}{\mu(E)}. \tag{3.14}$$

Next we note for some $\lambda > 0$ that $\tilde\mu_2(X) = \lambda\mu(S^{-n}X \cap \Gamma_Q)$ whenever X is a Borel subset of Γ_Σ. Using this fact and (2.15) we deduce that

$$\tilde\mu_2(E) \approx \lambda\mu(\Gamma_Q) \quad \text{and} \quad \mu(E) \approx 1 \tag{3.15}$$

From (3.14), (3.15) it follows that

$$c_\varepsilon^{-1}\mu(Y)\mu(\Gamma_Q) \leq \mu(S^{-n}Y \cap \Gamma_Q) \leq c_\varepsilon\mu(Y)\mu(\Gamma_Q). \tag{3.16}$$

Summing (3.16) over $Q \in \underline{G}_n$, we get

$$c_\varepsilon^{-1}\mu(Y) \leq \mu(S^{-n}Y) \leq c_\varepsilon\mu(Y)$$

which in view of the definiton of ν, implies (3.13) $(*)$. To prove (3.13) $(**)$ observe from (2.15) and Harnack's inequality that for some $c = c(\widehat M) \geq 2$,

$$\mu\big(B(z,2r)\big) \leq c\mu\big(B(z,r)\big) \quad \text{and} \quad \mu\big(B(z,r)\big) \leq (1-c^{-1})\mu\big(B(z,2r)\big) \tag{3.17}$$

whenever $z \in \partial\Omega$ and $0 < r \leq \operatorname{diam}\partial\Omega$. If $Q \in \underline{G}_n$ then $(S^{-n}E_\varepsilon) \cap \Gamma_Q$ is contained in two disks of radius $\varepsilon l(Q)$ whose centers are in $\partial\Omega$. Using this fact and iterating (3.17) with $r = l(Q)$, we conclude for some $0 < \alpha < 1$ that

$$\mu[(S^{-n}E_\varepsilon) \cap \Gamma_Q] \leq c\varepsilon^\alpha\mu(\Gamma_Q). \tag{3.18}$$

Summing (3.18) over n and $Q \in \underline{G}_n$, as well as, using the definition of ν we obtain (3.13) $(**)$. To complete the proof of Lemma 3.11 note from (3.13) $(*)$ that μ, ν are proportional on $\Gamma_\Sigma \setminus E_\varepsilon$, $\Sigma \in G_0$, while (3.13) $(**)$ shows $\lim_{\varepsilon\to0}\nu(E_\varepsilon) = \lim_{\varepsilon\to0}\mu(E_\varepsilon) = 0$. These facts clearly imply that μ, ν are mutually absolutely continous on Γ_Σ. Since $\Sigma \in \underline{G}_0$ is arbitrary we conclude the validity of Lemma 3.11. □

Next if $\Sigma \in \underline{G}_0$ we say that ν is ergodic with respect to S on Γ_Σ, provided that whenever $H \subset \Gamma_\Sigma$ is Borel and $S^{-1}(H) = H$, it is true that either $\nu(\Gamma_\Sigma \setminus H) = 0$ or $\nu(H) = 0$.

Lemma 3.19. *S is ergodic with respect to ν on Σ, whenever $\Sigma \in \underline{G}_0$.*

Proof. Let $H \subset \Gamma_\Sigma$ be as above, $F = \Gamma_\Sigma \setminus H$, and E_ε as in (3.13). If $\nu(H) > 0$, we choose $\varepsilon > 0$ so small that $\nu(E_\varepsilon) < \nu(H)/4$. Let $H' = H \setminus E_\varepsilon$ and suppose $x \in F \setminus E_{2\varepsilon}$. Then for n large enough, $\Gamma_{Q_n(x)} \cap E_\varepsilon = \emptyset$. By (3.16), (3.13) with $Y = H'$, $S^{-n}H' \cap \Gamma_{Q_n(x)}$, $\Gamma_{Q_n(x)}$, we have

$$\tag{3.20} \begin{aligned} \frac{\nu(H \cap \Gamma_{Q_n(x)})}{\nu(\Gamma_{Q_n(x)})} &= \frac{\nu(S^{-n}H \cap \Gamma_{Q_n(x)})}{\nu(\Gamma_{Q_n(x)})} \\ &\geq \frac{\nu(S^{-n}H' \cap \Gamma_{Q_n(x)})}{\nu(\Gamma_{Q_n(x)})} \geq c_\varepsilon^{-3}\nu(H') \geq 3c_\varepsilon^{-3}\nu(H)/4. \end{aligned}$$

From (3.17), (3.13), and differentiation theory (see [R]), we deduce

$$\frac{\nu(H \cap \Gamma_{Q_n(x)})}{\nu(\Gamma_{Q_n(x)})} \to 0$$

for ν almost every $x \in F \setminus E_{2\varepsilon}$ (i.e., ν almost every $x \in F \setminus E_{2\varepsilon}$ is a point of ν density one). (3.20) and the above display imply that $\nu(F \setminus E_{2\varepsilon}) = 0$. Since ε can be arbitrarily small and (3.13) $(**)$ holds, we conclude that $\nu(F) = 0$. Thus ν is ergodic on each $\Sigma \in G_0$. □

Let $f(x) = -\log l(Q)$ if $x \in \Gamma_Q$ and $Q \in \underline{G}_1$. Then

$$\tag{3.21} \|f\|_{L^\infty(\nu)} \leq c < \infty,$$

where c depends only on the length of the segments on the graph of ϕ over $]-1,1[$. Armed with Lemma 3.19 we now are in a position to apply the ergodic theorem of Birkhoff and entropy theorem of Shannon–McMillan–Breiman (see [Bi] or [Sh]).

Lemma 3.22. *If $\sigma_n(x) = l\bigl(Q_n(x)\bigr)$ and $h_n(x) = \mu(\Gamma_{Q_n(x)})$, then*

$$\sigma = \lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\sigma_n(x)}, \quad h = \lim_{n\to\infty} \frac{1}{n} \log \frac{1}{h_n(x)},$$

exist almost everywhere ν and are constant.

Proof. First we note that

$$\frac{1}{n} \log \frac{1}{\sigma_n(x)} = \frac{1}{n} \sum_{j=0}^{n-1} f(S^j x)$$

so (3.21) and the ergodic theorem applied to f yields the limit involving σ_n. The other part of Lemma 3.22 follows from the entropy theorem applied to ν and (3.13) $(*)$. It should be noted though that in [Bi], [Sh], the proof of the entropy theorem is given for the shift S on the space, Ψ, of sequences: $\bigl\{(x_i)_{-\infty}^{\infty}$:

$x_i \in \{1, 2, \dots, l\}\big\}$. However, the proofs of these authors is easily adapted to our situation. For example the proof in [Sh] uses Ψ to conclude that if $g(x) = \liminf_{n\to\infty}[n^{-1}\log\nu(\Gamma_{Q_n(x)})]$, then $g(Sx) = g(x)$ for ν almost every $x \in \Psi$. In fact if $Q_n(x) \prec Q \in \underline{G}_1$, then this equality follows from (3.17), (3.13), and (3.16) with $Y = Q_n(Sx)$, $n = 1$. The rest of the proof in [Sh] is essentially unchanged if Ψ is replaced by Θ. □

As a consequence of Lemma 3.22 we have

Lemma 3.23. *For μ almost every $x \in \partial\Omega$, we have*

$$\lim_{t\to 0}\frac{\log\mu[B(x,t)]}{\log t} = h/\sigma = \text{H-dim}\,\mu.$$

Proof. Let $x \in \partial\Omega$ and suppose x is not an endpoint of any Γ_Q with $Q \in G$. Given t, $0 < t < 1$, choose n a nonnegative integer so that $l\big(Q_{n+1}(x)\big) \le t < l\big(Q_n(x)\big)$. Then

$$t \approx l\big(Q_n(x)\big) \text{ since } l\big(Q_{n+1}(x)\big) \approx l\big(Q_n(x)\big) \tag{3.24}$$

where the proportionality constants are independent of n. Also from the doubling property of μ in (3.17) we have for some $c = c(\widehat{M})$ that

$$\left|\ln\frac{\mu[\Gamma_{Q_n(x)}]}{\mu[B(x,t)]}\right| \le c. \tag{3.25}$$

Using (3.24), (3.25) and Lemmas 3.11, 3.22 we deduce for μ almost every $x \in \partial\Omega$, that

$$h/\sigma = \lim_{n\to\infty}\frac{\log h_n(x)}{\log\sigma_n(x)} = \lim_{t\to 0}\frac{\log\mu[B(x,t)]}{\log t}.$$

Finally $h/\sigma = \text{H-dim}\,\mu$ follows from the definition of $\text{H-dim}\,\mu$ and basic measure theoretic type arguments. □

Next given $Q \in \underline{G}$, let $\widehat{T}_Q$ be as defined above (3.1). Given n a positive integer, put

$$O_n = \bigcup_{Q\in\underline{G}_n}\widehat{T}_Q \text{ and put } \widehat{\Omega}_n = \Omega_n \setminus \overline{O}_n \text{ when } n = 1, 2, \dots. \tag{3.26}$$

For use in Section 4 we note that

$$\begin{aligned} &(\alpha) \quad \widehat{\Omega}_n \subset \Omega_m \text{ for } m \ge n \text{ so } \widehat{\Omega}_n \subset \Omega, \\ &(\beta) \quad \partial\widehat{\Omega}_n \text{ is a } k\text{-quasicircle with } k \text{ independent of } n. \end{aligned} \tag{3.27}$$

(3.27) (α) follows from (3.1)–(3.3). (3.27) (β) is proved in the same way as the corresponding statement for Ω_n (see (3.6)–(3.8)). Recall that $\hat{x} = 0$, $\hat{r} = 1/200$,

$\widehat{R} = 1/200$ in the definition of u, D. Define $\widehat{D}_n$, $\hat{u}_n$ relative to $\widehat{\Omega}_n$ in the same way as D, u were defined relative to Ω. Then $\hat{u}_n$, $n = 1, 2, \ldots$, is the p-harmonic function in $\widehat{D}_n$ with boundary values 1 on either $\partial B(0, 1/200)$ or $\partial B(0, 200)$ and 0 on $\partial\widehat{\Omega}_n$ in the continuous sense. Continuity of $\hat{u}_n$, $n = 1, 2, \ldots$, on the closure of $\widehat{D}_n$ follows from Lemma 2.11.

Let $\hat{\mu}_n$, $n = 1, 2, \ldots$, be the measure corresponding to $\hat{u}_n$, as in (1.2). From (3.17) we see for some $0 < \lambda < 1 < \lambda'$, that if $\tilde{x} \in \partial\widehat{\Omega}_m$, $0 < t < \varrho < 1/100$, and $\tilde{y} \in \partial\widehat{\Omega}_m \cap B(\tilde{x}, \varrho)$, then

$$\hat{\mu}_n[B(\tilde{x}, \varrho)]\left(\frac{t}{\varrho}\right)^{\lambda'} \le \hat{\mu}_n[B(\tilde{y}, t)] \le c\hat{\mu}_n[B(\tilde{x}, \varrho)]\left(\frac{t}{\varrho}\right)^{\lambda}. \tag{3.28}$$

Observe that (3.28) also holds with $\hat{u}_n$, $\hat{\mu}_n$ replaced by u, μ. From this observation with $\varrho = 1/200$, we deduce that

$$h > 0 \tag{3.29}$$

and since $l\big(Q_{n+1}(x) \approx l(Q_n(x)\big)$ (with constants depending on the definition of ϕ above (3.1)), we also have

$$\sigma > 0. \tag{3.30}$$

4. Proof of Theorem 1

In this section we complete the program outlined in the introduction and get Theorem 1. Let $\widehat{T}_Q$ be as defined above (3.1) and let $\tau(Q)$ be the union of the two closed line segments joining the endpoints of Q to $a_Q - \hat{b}l(Q)e$. From (3.2), (3.3) and the definition of $\widehat{\Omega}_n$ in (3.26) we see that $\tau(Q)$, $\tau(Q')$ have disjoint interiors whenever Q, Q' are distinct intervals in $\underline{G}_n$ and

$$\partial\widehat{\Omega}_n = \bigcup_{Q \in \underline{G}_n} \tau(Q), \quad n = 1, 2, \ldots. \tag{4.1}$$

Also let $\hat{\mu}_n$, $\hat{u}_n$, be as defined after (3.27). From the maximum principle for p-harmonic functions and (3.27) we see that $\hat{u}_n \le u$ in $\widehat{D}_n$. Put

$$\mu_n = \frac{\hat{\mu}_n}{\hat{\mu}_n(\partial\widehat{\Omega}_n)},$$
$$u_n = \frac{\hat{u}_n}{\hat{\mu}_n(\partial\widehat{\Omega}_n)} \quad \text{for } n = 1, 2, \ldots.$$

From (2.15) and the above facts we see for some $c = c(\widehat{M}) \ge 1$ that

$$u_n \le cu \quad \text{in } \widehat{D}_n. \tag{4.2}$$

From the definition of μ_n, (4.2), (3.27), (3.28) and once again (2.15) we conclude that

$$\mu_n(\partial\widehat{D}_n) = 1 \quad \text{and} \quad \mu_n\big(\tau(Q)\big) \le c\mu(\Gamma_Q) \tag{4.3}$$

whenever $Q \in \underline{G}_n$, $n = 1, 2, \ldots$. We first prove

Lemma 4.4. *If h is as in Lemma 3.22, then*

$$\lim_{n\to\infty}\left[\frac{1}{n}\sum_{Q\in \underline{G}_n}\mu_n\big(\tau(Q)\big)\log\frac{1}{\mu_n\big(\tau(Q)\big)}\right]=h.$$

Proof. From Lemma 3.22, (3.29), and Egoroff's theorem we see for given ε, $0<\varepsilon\le 1/4$, that there exists $N=N(\varepsilon)$ such that

$$(4.5)\qquad (1-\varepsilon)h<\frac{1}{n}\log\frac{1}{\mu(\Gamma_{Q_n(x)})}<(1+\varepsilon)h$$

when $n\ge N$ for all $Q=Q_n(x)\in \underline{G}_n$, except for a set $\bigcup_{Q\in A_n}\Gamma_Q$ of μ-measure $<\varepsilon$ where $A_n\subset \underline{G}_n$. It follows from (4.3), (4.5) that

$$(4.6)\qquad \begin{aligned}\frac{1}{n}\sum_{Q\in \underline{G}_n\setminus A_n}\mu_n\big(\tau(Q)\big)\log\frac{1}{\mu_n\big(\tau(Q)\big)}&\ge\frac{1}{n}\sum_{Q\in \underline{G}_n\setminus A_n}\mu_n\big(\tau(Q)\big)\log\frac{1}{c\mu(\Gamma_Q)}\\&\ge -c/n+(1-\varepsilon)h\sum_{Q\in \underline{G}_n\setminus A_n}\mu_n\big(\tau(Q)\big)\\&\ge -c'/n+(1-c'\varepsilon)h\end{aligned}$$

for $c'=c'(k)$, large enough. From (4.6) and arbitrariness of ε we obtain

$$(4.7)\qquad \liminf_{n\to\infty}\left[\frac{1}{n}\sum_{Q\in \underline{G}_n}\mu_n\big(\tau(Q)\big)\log\frac{1}{\mu_n\big(\tau(Q)\big)}\right]\ge h.$$

To obtain an estimate from above for the sum in (4.7) we note from (3.28) for μ, μ_n with $\varrho=1/200$ and (4.3) that for some $c=c(k)\ge 1$,

$$(4.8)\qquad c^{-1}n\le\log\frac{1}{c\mu\big(\Gamma(Q)\big)}\le\log\frac{1}{\mu_n\big(\tau(Q)\big)}\le cn$$

whenever $n=2,3,\dots$, and $Q\in \underline{G}_n$. Also note for $0<\lambda\le 1/4$ that $\lambda\mu(\Gamma_Q)<\mu_n\big(\tau(Q)\big)$ except for a set $E=\bigcup_{Q\in B_n}\tau(Q)$ with $\mu_n(E)\le\lambda\mu(\partial\Omega)\le c\lambda$. Using this weak type estimate, (4.5), and (4.8) we deduce

$$(4.9)\qquad \begin{aligned}\frac{1}{n}\sum_{Q\in \underline{G}_n\setminus B_n}\mu_n(\tau(Q))\log\frac{1}{\mu_n(\tau(Q))}&<\frac{1}{n}\sum_{Q\in \underline{G}_n\setminus B_n}\mu_n\big(\tau(Q)\big)\log\frac{1}{\lambda\mu(\Gamma_Q)}\\&<(1+\varepsilon)h-(1/n)\log\lambda\\&\quad+n^{-1}\sum_{Q\in A_n}\mu_n\big(\tau(Q)\big)\log\big(1/\mu\big(\Gamma_Q\big)\big)\\&\le(1+\varepsilon)h-(1/n)\log\lambda+c\varepsilon.\end{aligned}$$

Also from (4.8) and our weak type estimate we have

$$\frac{1}{n}\sum_{Q\in B_n}\mu_n\big(\tau(Q)\big)\log\frac{1}{\mu_n(\tau(Q))}\leq c\lambda. \tag{4.10}$$

Choosing $\lambda = 1/n$ in (4.9), (4.10) and using the arbitrariness of ε we see that

$$\limsup_{n\to\infty}\bigg[\frac{1}{n}\sum_{Q\in \underline{G}_n}\mu_n\big(\tau(Q)\big)\log\frac{1}{\mu_n\big(\tau(Q)\big)}\bigg]\leq h. \tag{4.11}$$

Combining (4.7), (4.11) we conclude that Lemma 4.4 is valid. □

Next we note from a Schwarz reflection type argument that if $Q\in \underline{G}_n$ and $x\in \tau(Q)$ is not an endpoint of either of the two line segments making up $\tau(Q)$, then u_n has a p-harmonic extension to a neighborhood of x. From p-harmonicity, it follows as in Lemma 2.7 that this extension has Hölder continuous partial derivatives in a neighborhood of x. This fact, (1.1), (1.2) for u_n, μ_n, and an easy limiting argument imply that at x

$$d\mu_n = |\nabla u_n|^{p-1}(x)\,dH^1x. \tag{4.12}$$

Moreover from the doubling property of $\mu_n, n = 2,\dots,$ (see (3.28)) and (4.8) it follows that μ_n has no point masses at the endpoints of $\tau(Q)$ whenever $Q\in\underline{G}_n$. In view of (4.1), we see that (4.12) holds for μ_n almost every $x\in\partial\widehat{\Omega}_n$. Next let $l\big(\tau(Q)\big)$ be the length of $\tau(Q)$ whenever $Q\in\underline{G}_n$.

Lemma 4.13. *If $Q\in\underline{G}_n$, then for some $c=c(k,\phi)\geq 1$,*

$$\sum_{Q\in\underline{G}_n}\int_{\tau(Q)}\bigg|\log\bigg(\frac{l\big(\tau(Q)\big)|\nabla u_n|^{p-1}(x)}{\mu_n\big(\tau(Q)\big)}\bigg)\bigg||\nabla u_n|^{p-1}\,dH^1x\leq c.$$

Proof. To prove this lemma we use results from [K] where examples are constructed of p-harmonic functions $w=w(\,\cdot\,,p)$ in the region $S(\alpha)$, $0<\alpha<\pi$, where if $x_1=\varrho\cos\theta$, $x_2=\varrho\sin\theta$, in polar coordinates ϱ, θ, $|\theta|<\pi$, then

$$\begin{aligned} S(\alpha) &= \big\{(\varrho,\theta):\varrho>0 \text{ and } |\theta|<\alpha\big\},\\ w(\varrho,\theta) &= \varrho^{\lambda}f(\theta) \text{ in } S(\alpha),\\ w(1,0) &= 1 \text{ and } w(\varrho,\pm\alpha)=0. \end{aligned}$$

Moreover $f\in C_0^\infty[-\alpha,\alpha]$ and for some $c\geq 1$,

$$c^{-1}\lambda^2\theta\leq|f'(\theta)|\leq c\lambda^2\theta. \tag{4.14}$$

The relation between λ, α is given by (see the remark in [K]),

$$1-\frac{(\lambda-1)\sqrt{p-1}}{\sqrt{\lambda^2(p-1)-\lambda(p-2)}}=2\alpha/\pi. \tag{4.15}$$

Let $Q\in \underline{G}_n$ and suppose that $\widehat{Q}$, Q' are the distinct closed line segments in $\tau(Q)$. If $\widehat{Q}\cap Q'=\{z\}$, let $\widehat{S}$ be the open sectorial region with vertex z,

$$\widehat{S}\cap \widehat{D}_n\cap B\big(z,l(\widehat{Q})/2\big)\neq\emptyset \quad\text{and}\quad \widehat{Q},Q'\subset\partial\widehat{S}.$$

After a rotation and translation we assume, as we may, that $z=0$ and $\widehat{S}=S(\theta_0)$ for some $\pi/2<\theta_0<\pi$. For fixed p, $\widehat{Q}$, Q', let w be the corresponding p harmonic function in $S(\theta_0)$. Given $\varepsilon>0$, let $t=3l(\widehat{Q})/4$ and put

$$b_n^{p-1}=t^{p-2}\mu_n(\widehat{Q})=t^{p-2}\int_{\widehat{Q}}|\nabla u_n|^{p-1}\,dH^1x.$$

From (2.15) with $\tilde\mu$, $\tilde u$ replaced by μ_n, u_n, and Harnack's inequality, we see that $u_n(t,0)\approx b_n$. This fact, (3.1)–(3.3), and Lemma 2.16 with Ω, $\tilde u$, v, replaced by $S(\theta_0)$, u_n, w, imply that

$$t^{-\lambda}w\approx u_n/b_n \qquad\text{in } S(\theta_0)\cap\overline{B}(0,t). \tag{4.16}$$

From smoothness of u_n, w in $\widehat{Q}\cup S(\theta_0)\cap B(0,t)$ and (4.16) it follows that

$$t^{-\lambda}|\nabla w(x)|\approx|\nabla u_n(x)|/b_n \quad\text{for } x\in\widehat{Q}\cap\overline{B}(0,t). \tag{4.17}$$

Proportionality constants may depend on k, ϕ but are independent of $n=1,2,\dots$. From (4.17), (4.14), (4.15), we see for some $c=c(k,\phi)$ that

$$\begin{aligned}\int_{\widehat{Q}\cap\overline{B}(0,t)}&\big|\log\big[(t|\nabla u_n|/b_n)^{p-1}\big]\big||\nabla u_n|^{p-1}\,dH^1x\\ &\leq c(b_n/t^\lambda)^{p-1}\int_0^t\log\Big(\frac{ct}{\varrho}\Big)\varrho^{(\lambda-1)(p-1)}\,d\varrho\leq c^2t^{2-p}b_n^{p-1}\leq c^3\mu_n(\widehat{Q}).\end{aligned}$$

Here we have also used $\lambda>(p-2)/(p-1)$ when $p>2$, as we see from (4.15). Thus,

$$\int_{\widehat{Q}\cap B(0,t)}\bigg|\log\bigg(\frac{l(\widehat{Q})|\nabla u_n|^{p-1}}{\mu_n(\widehat{Q})}\bigg)\bigg|\,|\nabla u_n|^{p-1}\,dH^1x\leq c\mu_n(\widehat{Q}) \tag{4.18}$$

for some $c = c(k, \phi)$. Let q denote the other endpoint of $\widehat{Q}$. If $x \in \widehat{Q} \setminus B(0,t)$, observe from (3.1)–(3.3) and Lemma 2.16 that we can apply the rate theorem to the above functions in $\widehat{\Omega}_n \cap B(x, |x-q|/c)$ for some $c = c(k, \phi)$. Doing this we get

$$|\nabla u_n(x)| \approx u_n(x^*)|x-q|^{-1}$$

where $x^* \in B(x, |x-q|/c)$ and $d(x^*, \partial\hat{\Omega}_n) \geq |x-q|/c^2$. Using this fact, (2.15), and (3.28), we see that

$$\begin{aligned}
&\int_{\widehat{Q}\setminus B(0,t)} \left|\log\left(\frac{l(\widehat{Q})|\nabla u_n|^{p-1}}{\mu_n(\widehat{Q})}\right)\right| |\nabla u_n|^{p-1}\, dH^1 x \\
&\quad\leq \int_{\widehat{Q}\setminus B(0,t)} \left|\log\left(\frac{cl(\widehat{Q})\mu_n[B(q, 2|x-q|)]}{|x-q|\mu_n(\widehat{Q})}\right)\right| |\nabla u_n|^{p-1}\, dH^1 x \\
&\quad\leq c\mu_n(\widehat{Q}) + c\int_{\widehat{Q}\setminus B(0,t)} \log\left(\frac{2t}{|x-q|}\right)|\nabla u_n|^{p-1}\, dH^1 x \\
&\quad\leq c\mu_n(\widehat{Q}),
\end{aligned}$$

where we have written the last integral as a sum over $\widehat{Q} \cap \big[B\big(q, 2^{-m}l(\widehat{Q})\big) \setminus B\big(q, 2^{-m-1}l(\widehat{Q})\big)\big]$, $m = 2, 3, \ldots$, and then used (3.28) to make estimates. Combining this display with (4.18) we have

$$\int_{\widehat{Q}} \left|\log\left(\frac{l(\widehat{Q})|\nabla u_n|^{p-1}}{\mu_n(\widehat{Q})}\right)\right| |\nabla u_n|^{p-1}\, dH^1 x \leq c\mu_n(\widehat{Q}).$$

A similar inequality holds for Q'. Since by (3.28)

$$\mu_n\big(\tau(Q)\big) \approx \mu_n(\widehat{Q}) \approx \mu_n(Q')$$

with proportionality constants depending on k, ϕ, we can sum the above for $\widehat{Q}$, Q' over $Q \in G_n$ to get Lemma 4.13. ▫

Our final lemma before the proof of Theorem 1 is

Lemma 4.19. *For fixed p, $1 < p < \infty$,*

$$\eta = \lim_{n\to\infty} n^{-1} \int_{\partial\widehat{\Omega}_n} |\nabla u_n|^{p-1} \log|\nabla u_n|\, dH^1 x$$

exists. If $\eta > 0$ then H-dim $\mu < 1$ *while if $\eta < 0$, then* H-dim $\mu > 1$.

Proof. First from Lemma 3.22, (3.30), and an argument similar to the one used in Lemma 4.4, we see that

$$\lim_{n\to\infty}\left[\frac{1}{n}\sum_{Q\in\underline{G}_n}\mu_n\big(\tau(Q)\big)\log l\big(\tau(Q)\big)\right]=-\sigma. \tag{4.20}$$

From (4.1) and Lemma 4.13 we have for $n=1,2,\dots,$

$$\left|(p-1)\int_{\partial\widehat{\Omega}_n}|\nabla u_n|^{p-1}\log|\nabla u_n|\,dH^1x-\sum_{Q\in\underline{G_n}}\mu_n\big(\tau(Q)\big)\log\frac{\mu_n\big(\tau(Q)\big)}{l\big(\tau(Q)\big)}\right|\le c. \tag{4.21}$$

Dividing this inequality by n, letting $n\to\infty$, and using Lemma 4.4, (4.20), we conclude that the limit exists in Lemma 4.19. In fact,

$$(p-1)\eta=\sigma-h=\sigma[1-h/\sigma]=\sigma(1-\text{H-dim}\,\mu).$$

Clearly this equality gives the last sentence in Lemma 4.19. □

We now prove Theorem 1. Note from Lemma 2.26 and the arguments leading to (4.12), (4.17) that u_n is infinitely differentiable and $\nabla u_n\neq 0$ in the closure of $\widehat{D}_n$, except at endpoints of segments in $\partial\widehat{\Omega}_n$. Let Z_n denote the finite set of endpoints of segments on $\partial\widehat{\Omega}_n$. Let t_0 be 1/2 of the length of the smallest segment contained in $\partial\widehat{\Omega}_n$. If $z\in Z_n$, $x\in\partial B(z,t)\cap\partial\widehat{\Omega}_n$ and $0<t<t_0$, then as in (4.17), we get an estimate for $|\nabla u_n(x)|$ in terms of $t_0^{-\lambda}$, $u(x_0)$, where $x_0\in\partial B(z,t_0)\cap\Omega_n$ with $d(x_0,\partial\Omega_n)\approx t_0$. Using the Schwarz reflection argument once again to extend u_n, and $C^{1,\sigma}$ results for the p-Laplacian (see Lemma 2.7), we deduce that this estimate also holds in the closure of $B(x,t/c)\cap\widehat{\Omega}_n$ provided c is large enough. This fact and Lemma 2.26 for the extension of u_n, imply that on the closure of $\partial B(z,t)\cap\widehat{\Omega}_n$ we have for sufficiently small $t>0$, some $\lambda>\max\{0,(p-2)/(p-1)\}$, and $c=c\big(k,\phi,t_0,u(x_0)\big)$,

$$\begin{aligned} &t u_n|\nabla u_n|^{p-3}\sum_{i,j=1,}^{2}|(u_n)_{x_ix_j}|+t|\nabla u_n|^{p-1}\big|\log|\nabla u_n|\big|\\ &\qquad\le ct\ln(1/t)|\nabla u_n|^{p-1}\\ &\qquad\le ct\ln(1/t)t^{(\lambda-1)(p-1)}\to 0\quad\text{as } t\to 0.\end{aligned} \tag{4.22}$$

Let b_{ik}, $1\le i,k\le 2$, be as in (1.14). Let

$$v=\log|\nabla u_n|\quad\text{in } \widetilde{O}_n=\widehat{D}_n\setminus\bigcup_{z\in Z_n}\overline{B}(z,t).$$

Let L be the divergence form operator in (1.11). As in the discussion following (1.20) we apply the divergence theorem in $\widetilde{O}_n$ to the vector field whose ith component, $i = 1, 2$, is

$$u_n \sum_{k=1}^{2} b_{ik} v_{x_k} - v \sum_{k=1}^{2} b_{ik} (u_n)_{x_k}.$$

Such an application is permissible, since u_n, v are smooth in the closure of $\widetilde{O}_n$ (see Lemma 2.26). We get

$$\int_{\tilde{O}_n} u_n L v \, dx = \int_{\partial \tilde{O}_n} \sum_{i,k=1}^{2} b_{ik} \xi_i [u_n v_{x_k} - v (u_n)_{x_k}] \, dH^1 x \tag{4.23}$$

where ξ denotes the outer unit normal to $\partial \widetilde{O}_n$. Next observe from (1.14), (4.22) that for some $c = c(k, \phi)$,

$$\begin{aligned} &\sum_{z \in Z_n} \int_{\partial B(z,t) \cap \partial \tilde{O}_n} \sum_{i,k=1}^{2} |b_{ik}| \big[|v| \, |\nabla u_n| + u_n |\nabla v| \big] \, dH^1 x \\ &\quad \leq \sum_{z \in Z_n} c \int_{\partial B(z,t) \cap \partial \tilde{O}_n} \bigg[|\log |\nabla u_n|| \, |\nabla u_n|^{p-1} \\ &\qquad + u_n |\nabla u_n|^{p-3} \sum_{i,j=1}^{2} |(u_n)_{x_i x_j}| \bigg] \, dH^1 x \\ &\quad \to 0 \quad \text{as } t \to 0. \end{aligned} \tag{4.24}$$

Letting $t \to 0$ in (4.23), we see from (4.24) that

$$\int_{\widehat{D}_n} u_n L v \, dx = \int_{\partial \widehat{D}_n} \sum_{i,k=1}^{2} b_{ik} \xi_i [u_n v_{x_k} - v (u_n)_{x_k}] \, dH^1 x. \tag{4.25}$$

Observe from (1.18), (1.19) that if $p \neq 2$, $1 < p < \infty$, then at $x \in \widehat{D}_n$

$$(p-2)^{-1} L v \approx |\nabla u_n|^{p-4} \sum_{i,j=1}^{2} [(u_n)_{x_i x_j}]^2 \tag{4.26}$$

where the proportionality constant depends only on p. Also from Lemma 2.26, the fact that $u_n = 0$, $\xi = -\nabla u_n |\nabla u_n|^{-1}$ on $\partial \widehat{\Omega}_n \setminus Z_n$, and the definition of (b_{ik}), we see for some $c = c(\widehat{M})$ that

$$\begin{aligned} &\int_{\partial \widehat{D}_n} \sum_{i,k=1}^{2} b_{ik} \xi_i [u_n v_{x_k} - v (u_n)_{x_k}] \, dH^1 x \\ &\qquad = (p-1) \int_{\partial \widehat{\Omega}_n} |\nabla u_n|^{p-1} \log |\nabla u_n| \, dH^1 x + E \end{aligned} \tag{4.27}$$

where $|E| \le c$. From (4.25)–(4.27) and Lemma 4.19 we see that in order to prove Theorem 1 it suffices to show

$$\liminf_{n\to\infty}\left(n^{-1}\int_{\widehat{D}_n} u_n|\nabla u_n|^{p-4}\sum_{i,j=1}^{2}[(u_n)_{x_ix_j}]^2(x)\,dx\right) > 0. \tag{4.28}$$

To prove (4.28) let $1 \le m < n$ and suppose that $Q \in \underline{G}_m$. Choose $Q' \in \underline{G}_m$ with the same endpoint (say z) as Q and the property that Q, Q' are distinct direct descendants of the same $Q'' \in \underline{G}_{m-1}$. Let β' be the angle between the rays drawn from z through Q, Q', as measured from within Ω_m. If $s = \frac{1}{100}\min\{l(Q), l(Q')\}$, note from (3.1)–(3.3) that there exists a sectorial region $\widetilde{S}$ with vertex z, angle opening β, $0<\beta<2\pi$, and $\widetilde{S}\cap B(z,10s)\cap\Omega_m \ne \emptyset$. Also,

$$\begin{aligned}
&\text{(a)}\quad \beta,\beta' \text{ lie in the same component of } (0,2\pi)\setminus\{\pi\}.\\
&\text{(b)}\quad \text{If } \beta'<\pi, \text{ then } \beta'<\beta \text{ and } \partial\widetilde{S}\cap B(z,10s)\cap\Omega_l=\emptyset \text{ for } l\ge m.\\
&\text{(c)}\quad \text{If } \beta'>\pi, \text{ then } \beta<\beta' \text{ and } \partial\widetilde{S}\cap B(z,10s)\subset\Omega_l \text{ for } l\ge m.
\end{aligned}\tag{4.29}$$

We assume as we may that $z=0$ and $\widetilde{S}=\{(\varrho,\theta): 0<\varrho,\ |\theta|<\beta/2\}$. If (b) in (4.29) is valid, then from (3.1)–(3.3) we deduce for some $\theta_0,\delta>0$, that

$$S'=\{(\varrho,\theta): 0<\varrho\le 5s,\ |\theta-\theta_0|<\delta\}$$

satisfies

$$S'\subset S\cap[\widetilde{T}_Q\setminus\widehat{T}_Q]. \tag{4.30}$$

Here δ depends on our choice of constants in (3.1)–(3.3) and ϕ but is independent of $m=1,2,\dots$. Let w, λ, be as defined above (4.14) relative to $S(\beta/2)$ and write $u_n(\varrho,\theta)$ for $u_n(x)$ when $x\in\widehat{\Omega}_n$ has polar coordinates ϱ, θ. If (b) holds then from the maximum principle for p-harmonic functions and Lemma 2.13, we get upon comparing boundary values of $u_n/u_n(s,\theta_0)$ and $w/w_n(s,\theta_0)$,

$$\frac{u_n(\varrho,\theta_0)}{u_n(s,\theta_0)}\le c\left(\frac{\varrho}{s}\right)^{\lambda},\quad 0<\varrho\le s, \tag{4.31}$$

where $\lambda>1$, as follows from (4.15). From (4.31) and Lemma 2.26 we deduce that

$$\frac{|\nabla u_n|(\varrho,\theta_0)}{|\nabla u_n|(s,\theta_0)}\le c^*\left(\frac{\varrho}{s}\right)^{\lambda-1} \tag{4.32}$$

where c^* is independent of m. Taking logarithms, we see that there exists $\eta<1/2$ such that if $\varrho=\eta s$, then

$$\log|\nabla u_n|(\varrho,\theta_0)-\log|\nabla u_n|(s,\theta_0)\le -1. \tag{4.33}$$

Applying the mean value theorem from calculus and using Lemmas 2.7, 2.26, we find from (4.33) for some $\tilde{x}$ with polar coordinates ϱ, θ_0, $\eta s < \varrho < s$, that

$$cs \sum_{i,j=1}^{2} \frac{|(u_n)_{x_i x_j}|(\tilde{x})}{|\nabla u_n|(\tilde{x})} \geq 1 \tag{4.34}$$

where $c = c(k, \phi)$. From (4.34), Lemma 2.26, and Lemma 2.9, we see first that (4.34) is valid in $B(\tilde{x}, s/c) \subset S'$ provided $c = c(k, \phi)$ is large enough and thereupon from (2.15), Lemma 2.26, that

$$c \int_{B_Q} u_n |\nabla u_n|^{p-4} \sum_{i,j=1}^{2} |(u_n)_{x_i x_j}|^2 \, dx \geq \mu_n\big(B(z, 10l(Q))\big). \tag{4.35}$$

Here $B_Q = B\big(\tilde{x}, (s/2c)\big)$ and we have used the fact that $l(Q) \approx l(Q')$ (since Q, Q', are direct descendants of Q''). If (c) holds a similar argument applies only now $\lambda < 1$ and

$$S' \cap S = \emptyset, \quad S' \subset \widetilde{T}_Q \setminus \widehat{T}_Q. \tag{4.36}$$

From (4.36) and the maximum principle for p-harmonic functions we get as in (4.31),

$$c \frac{u_n(\varrho, 0)}{u_n(s, 0)} \geq \left(\frac{\varrho}{s}\right)^{\lambda}, \quad 0 < \varrho \leq s,$$

for some $\lambda < 1$, $c \geq 1$, and thereupon from Harnack's inequality that

$$c^2 \frac{u_n(\varrho, \theta_0)}{u_n(s, \theta_0)} \geq \left(\frac{\varrho}{s}\right)^{\lambda} \quad \text{for } 0 < \varrho \leq s \text{ and } c \text{ large.} \tag{4.37}$$

Arguing as following (4.31), we still get (4.35) for a ball with a radius and center having the above properties. Carrying out this procedure for each $Q \in \underline{G}_m$ and $1 \leq m < n$, we obtain $\{B_Q : Q \in \underline{G}_m,\ 1 \leq m < n\}$, satisfying (4.35). We claim that each point in Ω_n lies in at most $c = c(k, \phi)$ of the balls in $\{B_Q : Q \in \Omega_m,\ 1 \leq m < n\}$. To prove this claim we note from (3.1)–(3.3) that if $1 \leq m < j < n$, $Q \in \underline{G}_m, Q' \in \underline{G}_j$, then $B_Q \cap B_{Q'} = \emptyset$ unless $Q' \prec Q$. If $Q' \prec Q$, then from (3.1)–(3.3) we see that $B_{Q'}$ has radius, $r_{Q'}$, with

$$r_{Q'} \leq c^{m-j+1} l(Q).$$

Since

$$c_+ d(B_Q, \widehat{T}_Q) \geq l(Q) \quad \text{and} \quad d(B_{Q'}, \widehat{T}_Q) \leq c_+ r_{Q'}$$

where $c_+ = c_+(k,\phi)$, we see that our claim is true. From our claim, (4.35), (3.28), and (3.1)–(3.3) we conclude that

$$
\begin{aligned}
c^3 \int_{\widehat{\Omega}_n} u_m |\nabla u_m|^{p-4} \sum_{i,j=1}^{2} |u_{x_i x_j}|^2\, dx &\geq c^2 \sum_{m=1}^{n-1} \sum_{Q \in \underline{G}_m} \int_{B_Q} u_m |\nabla u_m|^{p-4} \sum_{i,j=1}^{2} |u_{x_i x_j}|^2\, dx \\
&\geq c \sum_{m=1}^{n-1} \mu_n(\partial \widehat{\Omega}_n) \geq n. \qquad (4.38)
\end{aligned}
$$

Thus (4.28) is true and the proof of Theorem 1 is now complete. □

5. Proof of Theorem 2

To begin the proof of Theorem 2, let

$$J = \left\{ (x,y) : |x| = \tfrac{1}{2},\ -\tfrac{1}{2} \leq y \leq \tfrac{1}{2},\ \text{and } |y| = \tfrac{1}{2},\ -\tfrac{1}{2} \leq x \leq \tfrac{1}{2} \right\}$$

be the unit square with center at 0 and side length one. Let O_J be the open set bounded by J and suppose that $J_1, J_2, \dots, J_l$ are translated and dilated copies of J with

$$J_i \cap J_j = \emptyset, \quad i \neq j, \quad \text{and} \quad J_i \subset O_J \qquad (5.1)$$

for $1 \leq i \leq l$. Let O_{J_i}, $1 \leq i \leq l$, be the open set bounded by J_i and put $\underline{G}_0 = \{J\}$, $\underline{G}_1 = \{J_i,\ 1 \leq i \leq l\}$. By induction, if $\underline{G}_n$ has been defined and $Q \in \underline{G}_n$ let Q_i, $1 \leq i \leq l$, be a translation and dilation of J_i, chosen so that if Q is mapped onto J by a composition of a translation and a magnification, then this mapping also sends Q_i to J_i, $1 \leq i \leq l$. As in Section 3 we write $Q_i \prec Q$, $1 \leq i \leq l$, and put

$$
\begin{aligned}
\underline{G}_{n+1} &= \{Q_i : Q_i \prec Q \text{ for some } Q \in \underline{G}_n\}, \\
\underline{G} &= \bigcup_{n=0}^{\infty} \underline{G}_n
\end{aligned}
$$

Let O_Q be the open set bounded by $Q \in \underline{G}_n$ and set

$$
\begin{aligned}
E_n &= \bigcup_{Q \in G_n} \overline{O}_Q, \quad n = 1, \dots, \\
E &= \bigcap_{n=1}^{\infty} E_n, \\
\Omega_n &= B(0,2) \setminus E_n, \quad n = 1, \dots, \\
\Omega &= B(0,2) \setminus E.
\end{aligned}
$$

Then Ω is the Cantor set referred to in Theorem 2. We claim that

(5.2) Ω, Ω_n are uniform domains with constant $\widehat{M}$ in (2.3), independent of n.

We prove this claim only for Ω. If $x, y \in \Omega$ let n be the smallest integer such that $x, y \notin \bigcup_{Q \in \underline{G}_n} \overline{O}_Q$. Draw the line l' connecting x to y. If l' intersects $Q \in \underline{G}_n$, let w_1, w_2 be the first and last points of intersection of l' with Q. We can replace the line segment connecting w_1 to w_2 by line segments $\subset Q$, in such a way that the resulting curve connects w_1 to w_2 and satisfies (2.3). Continuing in this manner we get $\sigma \subset \overline{\Omega}_n$ satisfying (2.3) with w_1, w_2 replaced by x, y. Thus claim (5.2) is true and consequently Lemma 2.13, (2.15) are valid since $p > 2$. To prove Lemma 2.16 for $p > 2$, let $\tilde{u}, v$ be the p-harmonic functions in this lemma and $w \in \partial\Omega$. Given r, $0 < r < 1$, we see from (5.1) and the definition of Ω that there exists $Q \in \underline{G}_n$ for some positive integer n with $w \in O_Q \subset B(w, r)$ and

$$r \approx l(Q) \approx d(\partial\Omega, Q). \tag{5.3}$$

Here $l(Q)$ denotes the length of Q and the proportionality constants are independent of $w \in \partial\Omega$ as well as r, $0 < r < 1$. From (5.2), Harnack's inequality, we deduce first that

$$c^{-1} \leq \frac{\tilde{u}}{v} \leq c \quad \text{on } Q. \tag{5.4}$$

Second from (5.3), (5.4) and the weak maximum principle for p harmonic functions, we get Lemma 2.16. □

Next given $p > 2$, let $u \in W^{1,p}(\Omega)$ be the p-harmonic function in $\Omega = B(0,2) \setminus E$ with continuous boundary values 0 on E and 1 on $\partial B(0,2)$. As in Section 2 we see that existence of u follows from the usual variational argument and Lemma 2.11. Let μ be the measure associated with u as in (1.2). Also suppose $\tilde{u}$ is another positive p-harmonic function in $\Omega \cap N$ ($N =$ a neighborhood of $\partial\Omega$) with boundary value 0 on $\partial\Omega$ in the Sobolev sense. Let $\tilde{\mu}$ be the measure associated with $\tilde{u}$ as in (1.2). Using (2.15) and arguing as in (2.24) we deduce first that $\mu \approx \tilde{\mu}$ and thereupon that it suffices to prove Theorem 2 for fixed $p > 2$ and u, μ as above. If $Q \in \underline{G}$, let $\Gamma_Q = O_Q \cap \partial\Omega$. We can now repeat the arguments in Sections 3 and 4, essentially verbatim. In fact the notation was chosen so that we can define the shift S as in (3.9), (3.10), and identify it with the following mapping: If $Q \in \underline{G}_1$, then S restricted to Q is the translation-magnification which takes Γ_Q onto $E = \Gamma_J$. Using Lemma 2.16 and arguing as in (3.12)–(3.18) we get Lemma 3.11. Arguing as in (3.20)–(3.21) we obtain first Lemma 3.19 and then Lemma 3.22. Finally arguing as in (3.24)–(3.25) we get Lemma 3.23. □

For fixed $p > 2$ and n a nonnegative integer, let $\hat{u}_n$ be the p harmonic function in Ω_n with continuous boundary values 0 on ∂E_n and 1 on $\partial B(0,2)$. Let $\hat{\mu}_n$ be the measure corresponding to $\hat{u}_n$ as in (1.2) with u replaced by $\hat{u}_n$. We observe first that (3.28) is valid and second that (3.29), (3.30) hold. As in Section 4 put

$$\mu_n = \frac{\hat{\mu}_n}{\hat{\mu}_n(E_n)},$$
$$u_n = \frac{\hat{u}_n}{\hat{\mu}_n(E_n)}.$$

Arguing as in Section 4 we obtain (4.2), (4.3) with $\tau(Q)$, $\widehat{\Omega}_n$ replaced by Q, Ω_n. Also arguing as in (4.5)–(4.11) we get Lemma 4.4 with $\tau(Q)$ replaced by Q. Moreover, arguing as in (4.14)–(4.18) we get first Lemma 4.13 (Q replacing $\tau(Q)$) and then Lemma 4.19. Armed with Lemma 4.19 we can apply the divergence theorem to the vector field below (4.22). In this case there is an additional complication because ∇u_n can vanish at points of Ω_n. To handle this complication we first show that

$$\nabla u_n \text{ vanishes at a finite number of points in } \Omega_n. \tag{5.5}$$

Indeed if $Q \in \underline{G}_n$, then for small $t > 0$ we see from the maximum principle for p-harmonic functions that there is a simply connected component U of $\{x : u_n(x) < t\}$ with $\overline{O}_Q \subset U$. Then $D_Q = U \setminus \overline{O}_Q$ is doubly connected and as in Lemma 2.26, it follows that $\nabla u_n \neq 0$ in D_Q. Since $\underline{G}_n$ has finite cardinality, we conclude that there exists a neighborhood N containing E_n with $\nabla u_n \neq 0$ in $N \cap \Omega_n$. Also as in Lemma 2.26, we see that u_n has a real analytic extension to a neighborhood of $\partial B(0,2)$ with $|\nabla u_n| \approx 1$ on $\partial B(0,2)$. From these facts and Lemma 2.8 we conclude that (5.5) is true.

Now suppose $w \in \Omega_n$ and $\nabla u_n(w) = 0$. Choose $r_0 > 0$ so small that $\nabla u_n \neq 0$ in $B(w, r_0) \setminus \{w\}$. Then as noted in the display below (2.10), $v = \log |\nabla u_n|$ satisfies a uniformly elliptic partial differential equation in divergence form when $\nabla u_n \neq 0$, so if $0 < 6r < r_0$, then by the usual Caccioppoli type estimate,

$$\int_{B(w,4r)\setminus B(w,3r)} |\nabla v|^2 \, dx \leq c r^{-2} \int_{B(w,5r)\setminus B(w,2r)} v^2 \, dx \leq c^2 A(r)^2 \tag{5.6}$$

where $A(r)$ denotes the essential supremum of $|v|$ in $B(w,6r) \setminus B(w,r)$. Using Hölder's inequality we deduce from (5.6) that

$$\int_{B(w,4r)\setminus B(w,3r)} |\nabla v| \, dx \leq c_- r A(r) \tag{5.7}$$

where $c_- = c_-(w, u_n)$ is independent of r. From (5.7) and a weak type estimate we conclude for some $\hat{r} \in (3r, 4r)$ that

$$\int_{\partial B(w,\hat{r})} |\nabla v| \, dH^1 \leq c_-^2 A(r). \tag{5.8}$$

Furthermore using the factorization theorem mentioned after Lemma 2.7 and well-known properties of analytic functions, quasiconformal mappings, we see that

$$A(r) \approx \log r \quad \text{as } r \to 0. \tag{5.9}$$

From Lemma 2.7 we also see that

$$|\nabla u_n| \le c r^{\sigma} \quad \text{on } \partial B(w, \hat{r}). \tag{5.10}$$

Let $\{w_i\}_1^m$ denote the zeros of ∇u_n in Ω_n. Let $r = r(w_i)$, $\hat{r} = \hat{r}(w_i)$, $1 \le i \le m$, be as above and put $\Omega_n' = \Omega_n \setminus \bigcup_{i=1}^m \bar{B}\big(w_i, \hat{r}(w_i)\big)$. We repeat the argument following (4.22) to obtain (4.25) with $\widehat{D}_n$ replaced by Ω_n'. Let ξ be the outer unit normal to $\partial\Omega_n'$. Using (5.8)–(5.10) and arguing as in (4.27) we get

$$\begin{aligned} &\int_{\partial\Omega_n'} \sum_{i,k=1}^{2} b_{ik}\xi_i[u_n v_{x_k} - v(u_n)_{x_k}]\, dH^1 x \\ &\qquad = (p-1)\int_{\partial E_n} |\nabla u_n|^{p-1} \log|\nabla u_n|\, dH^1 x + F \end{aligned} \tag{5.11}$$

where

$$\begin{aligned} |F| &\le c + \sum_{i=1}^{m} \int_{\partial B(w_i, \hat{r}(w_i))} \big[|\nabla u_n|^{p-1}|v| + u_n|\nabla u_n|^{p-2}|\nabla v|\big]\, dH^1 x \\ &\le c + r^{\sigma(p-2)} \log r. \end{aligned} \tag{5.12}$$

Since $p > 2$ we can let $r \to 0$ in (5.11) and conclude in view of (5.12), as in (4.28), that it suffices to show

$$\liminf_{n\to\infty} \left(n^{-1} \int_{\Omega_n} u_n |\nabla u_n|^{p-4} \sum_{i,j=1}^{2} [(u_n)_{x_i x_j}]^2(x)\, dx \right) > 0 \tag{5.13}$$

in order to prove Theorem 2. To do this we first claim there exists $\tilde{c}$, a positive integer, depending only on p and the initial choice of J_i, $1 \le i \le l$, such that if $m < n - 3\tilde{c}$, $j = m + 2\tilde{c}$, we have,

$$\nabla u_n(w) = 0 \text{ for some } w \in O_Q \setminus E_j \text{ whenever } Q \in \underline{G}_m. \tag{5.14}$$

To prove claim (5.14) recall that $u_n \equiv 0$ on E_n. Given $Q \in \underline{G}_m$, let $q = m + \tilde{c}$, and choose $Q' \in \underline{G}_q$, with $Q' \prec Q$. Note from Lemma 2.13 that for $\tilde{c}$ large enough,

$$t_1 = 2 \max_{\bar{O}_{Q'}} u_n < \min_{Q} u_n \tag{5.15}$$

and

$$t_2 = 2 \max_{E_j \cap O_{Q'}} u_n < \min_{\partial E_{q+1} \cap O_{Q'}} u_n. \tag{5.16}$$

Let $D(s) = \{x : u_n(x) < s\}$. From (5.16) and the maximum principle for p-harmonic functions we see that $D(t_2) \cap O_{Q'}$ splits into $\beta \geq l > 1$ components having a non-empty intersection with $E_j \cap O_{Q'}$. If $\nabla u_n = 0$ on $\partial D(t_2) \cap O_{Q'}$, then (5.14) is valid since $\partial D(t_2) \cap O_{Q'} \subset O_Q \setminus E_j$. Otherwise, it follows from the implicit function theorem that there exists $\varepsilon = \varepsilon(t_2) > 0$ for which $D(s)$, $s \in (t_2 - \varepsilon, t_2 + \varepsilon)$, has exactly β components which have a non-empty intersection with $E_j \cap O_{Q'}$. Let τ be the least upper bound of the set of all t such that $D(s)$ has exactly β components which intersect $E_j \cap O_{Q'}$ whenever $s \in (t_2, t)$. Note that $\tau \leq t_1$ since $O_{Q'} \subset D(t_1)$ (so $D(t_1)$ has only one component containing points in $O_{Q'}$). From this note and (5.15) we conclude that $\partial K \subset O_Q \setminus E_j$ whenever K is a component of $D(\tau)$ with $K \cap E_j \cap O_{Q'} \neq \emptyset$. Also there must exist $\widehat{K}$ a component of $D(\tau)$ with $\widehat{K} \cap E_j \cap O_{Q'} \neq \emptyset$ and $\nabla u_n(w) = 0$ for some $w \in \partial \widehat{K}$. In fact, otherwise, we could repeat the argument given for $D(t_2)$ and contradict the maximality of $D(\tau)$. Thus claim (5.14) is true. We also observe from (5.15) and the mean value theorem that

$$\hat{c}\, l(Q) |\nabla u_n|(x) \geq \max_Q u_n \tag{5.17}$$

for some $x \in O_Q \setminus E_q$. Let σ be a smooth curve connecting x to w in (5.14) with length $\leq c' l(Q)$ and $\sigma \subset O_Q \setminus E_j$ ($\hat{c}$, c' have the same dependence as $\tilde{c}$ above (5.14)). Then $v(y) \to -\infty$ as $y \to w$ on σ. Thus starting from x there exists a first point z such that $|v(x) - v(z)| = 1$. From Lemma 2.7, (5.17), and our choice of σ we deduce for some $c_+ \geq 1$, having the same dependence as $\tilde{c}$,

$$c_+^{-1} l(Q) \leq |z - x| \leq c_+ l(Q). \tag{5.18}$$

Applying the mean value theorem to an arclength parametrization of σ we get $y = y_Q$ on the arc of σ connecting x to z with

$$l(Q)^{-1} u_n(x) \approx |\nabla u_n| \quad \text{and} \quad c\, l(Q) |\nabla u_n|^{-1} \sum_{i,j=1}^{2} |(u_n)_{y_i y_j}| \geq 1 \tag{5.19}$$

at y. From (5.19), Lemma 2.7, we see that

$$\tfrac{1}{2} |\nabla u_n|(y) \leq |\nabla u_n|(\hat{y}) \leq 2 |\nabla u_n|(y)$$

when $\hat{y} \in B\big(y, l(Q)/c\big)$ provided c is large enough. Lemma 2.9 can now be applied to conclude that (5.19) is valid on $B_Q = B(y_Q, r_Q) \subset Q \setminus E_{j+1}$ with

$$r_Q \approx l(Q). \tag{5.20}$$

Again all constants in (5.19), (5.20) have the same dependence as $\tilde{c}$. From (5.19), (5.20), Harnack's inequality and (2.15), we conclude as in (4.35) that

$$c\int_{B_Q} u_n|\nabla u_n|^{p-4}\sum_{i,j=1}^{2}|(u_n)_{x_ix_j}|^2\,dx \geq \mu_n(O_Q). \tag{5.21}$$

We do this for each $Q\in\underline{G}_m$ and $1\leq m<n-3\tilde{c}$. We obtain, $\Theta=\{B_Q: Q\in\underline{G}_m,\ 1\leq m<n-3\tilde{c}\}$, satisfying (5.21). Since $B_Q\subset O_Q\setminus E_{j+1}$ when $Q\in\underline{G}_m$, we see (as in the argument following (4.37)) that each point in Ω_n lies in at most c members of Θ. Using this fact and summing (5.21), we get (5.13). The proof of Theorem 2 is complete. □

6. Proof of Theorem 3

To begin the proof of Theorem 3, fix $p\neq 2$, $1<p<\infty$. We note from (2.24) that it suffices to prove Theorem 3 for u_1, u_2 as defined below (2.25). In fact, we just prove Theorem 3 for u_1, since the proof for u_2 is unchanged. We assume, as we may, that $\hat{x}=0$, $\hat{r}=1$ in (2.25), since otherwise we translate and dilate Ω. Thus if $u=u_1$, then u is continuous in the closure of $D=\Omega\setminus\overline{B}(0,1)$ with $u\equiv 1$ on $\partial B(0,1)$ and $u\equiv 0$ on $\partial\Omega$. Also $B(0,4)\subset\Omega$, $\partial\Omega$ is a k-quasicircle, and u is a solution to the p-Laplacian partial differential equation in D. If $v(x)=\log|\nabla u(x)|$, $x\in D$, let $w(x)=\max(v,0)$ when $1<p<2$ and $w(x)=\max(-v,0)$ when $p>2$. Following Makarov (see [M], [P, Chapter 8, Section 5]), we first prove

Lemma 6.1. *Let m be a nonnegative integer. There exists $c_+=c_+(k)\geq 1$ such that for $0<t<1$,*

$$\int_{\{x:u(x)=t\}}|\nabla u|^{p-1}w^{2m}\,dH^1x\leq c_+^{m+1}m![\log(2/t)]^m.$$

Proof. Put $h(x)=\max\bigl(w(x)-c',0\bigr)$, $x\in D$. Here c' is chosen so large that $h\equiv 0$ in $B(0,2)\cap D$. Existence of c' follows from Lemma 2.26. Extend h continuously to Ω by putting $h\equiv 0$ in $\overline{B}(0,1)$. Let $\Omega(t)=\Omega\setminus\{x:u(x)\leq t\}$, whenever $0<t<1$ and let $L,(b_{ik})$, be as in (1.13), (1.14). We note that $h^2\in W^{2,\infty}\bigl(\Omega(t)\bigr)$. Also from (1.14), (1.20) we deduce for H^2 almost every $x\in\Omega(t)$,

$$\begin{aligned}L(h^{2m})(x)&=\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}[b_{ij}(x)(h^{2m})_{x_j}(x)]\\&\leq 2m(2m-1)h^{2m-2}(x)\sum_{i,j=1}^{2}b_{ij}(x)h_{x_i}(x)h_{x_j}(x)\\&\leq 2m(2m-1)p|\nabla u|^{p-2}(x)h^{2m-2}(x)|\nabla h|^2(x).\end{aligned}\tag{6.2}$$

Next we apply the divergence theorem in $\Omega(t)$ to the vector field whose ith component for $i = 1, 2$ is

$$(u-t)\sum_{j=1}^{2} b_{ij}(h^{2m})_{x_j} - h^{2m}\sum_{j=1}^{2} b_{ij}u_{x_j}.$$

Using (6.2) and our choice of c' we deduce from Lemma 2.26, as in (4.27), that

$$(p-1)\int_{\{x:u(x)=t\}} |\nabla u|^{p-1}h^{2m}\, dH^1x \le 2m(2m-1)p\int_{\Omega(t)} u|\nabla u|^{p-2}h^{2m-2}|\nabla h|^2\, dx. \tag{6.3}$$

We note from Lemma 2.26 that for some $c = c(k) \ge 1$,

$$|\nabla h| \le c d(x, \partial\Omega)^{-1} \quad \text{and} \quad d(x, \partial\Omega)^{-1}u(x) \approx |\nabla u(x)| \tag{6.4}$$

for H^2 almost every $x \in D$. Using (6.4) in (6.3) and the coarea formula (see [Fe, Section 3.2]), we obtain with $p' = p/(p-1)$,

$$\begin{aligned} I_m(t) &= \int_{\{x:u(x)=t\}} |\nabla u|^{p-1}h^{2m}\, dH^1x \\ &\le 2m(2m-1)p'\int_{\Omega(t)} u|\nabla u|^{p-2}h^{2m-2}|\nabla h|^2\, dx \\ &= 2m(2m-1)p'\int_t^1 \tau\Big(\int_{\{x:u(x)=\tau\}} |\nabla u|^{p-3}h^{2m-2}|\nabla h|^2\, dH^1x\Big)\, d\tau \\ &\le 2m(2m-1)p'c\int_t^1 \Big(\int_{\{x:u(x)=\tau\}} |\nabla u|^{p-1}h^{2m-2}\, dH^1x\Big)\tau^{-1}\, d\tau \\ &= 2m(2m-1)p'c\int_t^1 I_{m-1}(\tau)\tau^{-1}\, d\tau. \end{aligned} \tag{6.5}$$

Observe that

$$\int_{\{x:u(x)=\tau\}} |\nabla u|^{p-1}\, dH^1x = \text{constant} \tag{6.6}$$

for $0 < \tau < 1$ as follows easily from $\nabla\cdot(|\nabla u|^{p-2}\nabla u) = 0$ and the divergence theorem. From (6.6) we have trivially, for some $c_* = c_*(k)$,

$$I_0(t) \le c_* \quad \text{for } 0 < t < 1. \tag{6.7}$$

By induction if $m \ge 1$ and

$$I_{m-1}(t) \le c_*^m(m-1)!\log^{m-1}(2/t), \quad 0 < t < 1, \tag{6.8}$$

then from (6.5), (6.8), we obtain upon integrating that (6.8) holds with $m-1$ replaced by m provided c_* is suitably large. Thus by induction (6.8) is true for $m = 1, 2, \ldots$. Since $w \le h + c'$, we conclude from (6.8) and simple estimates that Lemma 6.1 is valid. □

Dividing the display in Lemma 6.1 by $(2c_+)^m m! \log^m(2/t)$ and summing we see for $0 < t < 1$ that

$$\int_{\{x:u(x)=t\}} |\nabla u|^{p-1} \exp\left[\frac{w^2}{2c_+ \log(2/t)}\right] dH^1x \leq 2c_+. \tag{6.9}$$

Let

$$\lambda(t) = \sqrt{4c_+ \log(2/t)}\,\sqrt{\log(-\log t)} \quad \text{for } 0 < t < e^{-2},$$
$$F(t) = \{x : u(x) = t \text{ and } w(x) \geq \lambda(t)\}.$$

Then from (6.9) and weak type estimates we deduce

$$\int_{F(t)} |\nabla u|^{p-1}\, dH^1x \leq 2c_+[\log(1/t)]^{-2}. \tag{6.10}$$

Next for a fixed and large, we define Hausdorff measure (denoted σ) with respect to

$$\gamma(r) = \begin{cases} re^{a\lambda(r)} & \text{when } 1 < p < 2, \\ re^{-a\lambda(r)} & \text{when } p > 2. \end{cases}$$

That is, for fixed $0 < \delta < e^{-2}$ and $E \subseteq \mathbf{R}^2$, let $L(\delta) = \{B(z_i, s_i)\}$ be such that $E \subseteq \bigcup B(z_i, s_i)$ and $0 < s_i < \delta$, $i = 1, 2, \ldots$. Set

$$\phi_\delta(E) = \inf_{L(\delta)}\left(\sum_i \gamma(s_i)\right).$$

Then

$$\sigma(E) = \lim_{\delta\to 0} \phi_\delta(E).$$

We claim for $a = a(k)$ large enough, that

$$\mu \text{ is absolutely continuous with respect to } \sigma \text{ when } 1 < p < 2. \tag{6.11}$$

To prove this claim let $K \subset \partial\Omega$ be a Borel set and suppose that $\sigma(K) = 0$. Let K_1 be the subset of all $x \in K$ with

$$\limsup_{r\to 0} \frac{\mu\big(B(x,r)\big)}{\gamma(r)} \leq 2. \tag{6.12}$$

Then from the definition of σ, it is easily shown that $\mu(K_1) = 0$. Thus to prove (6.11) it suffices to show $\mu(E) = 0$ when E is Borel and is equal μ almost everywhere to the set of all points in $\partial\Omega$ for which (6.12) is false. For this purpose given $r_0 > 0$, small, we use a well-known covering argument to get $\{B(x_i, r_i)\}$ pairwise disjoint with $r_i \leq r_0$,

$$\mu[B(x_i, 5r_i)] \geq \gamma(5r_i) \quad \text{and} \quad E \subset \bigcup B(x_i, 5r_i). \tag{6.13}$$

Let $t_m = 2^{-m}$ for $m = 4, 5, \dots$. From Lemmas 2.13, 2.26 and (2.15) we see there exists $y_i \in B(x_i, r_i/2)$ with $u(y_i) = t_m$ for some $m = m(i)$, $d(y_i, \partial\Omega) \approx r_i$, and

$$\mu[B(x_i, r_i)]/r_i \approx [u(y_i)/d(y_i, \partial\Omega)]^{p-1} \approx |\nabla u(y)|^{p-1} \tag{6.14}$$

whenever $y \in B\big(y_i, d(y_i, \partial\Omega)/2\big)$. We conclude from (6.13), (6.14) for some $\tilde{c} = \tilde{c}(k) \geq 1$ that

$$w(y) = \log|\nabla u(y)| \geq a\lambda(5r_i)/\tilde{c} \quad \text{on } B\big(y_i, d(y_i, \partial\Omega)/2\big). \tag{6.15}$$

Note that

$$H^1\big[B\big(y_i, d(y_i, \partial\Omega)/2\big) \cap \{x : u(x) = t_m\}\big] \geq d(y_i, \partial\Omega)/2 \tag{6.16}$$

as we see from the maximum principle for p-harmonic functions, a connectivity argument and basic geometry. Finally from Lemma 2.13, Harnack's inequality, (3.28) we find for some $\beta = \beta(k)$, $0 < \beta < 1$, $\bar{c} = \bar{c}(k)$, that

$$r_i \leq \bar{c} t_m^{\beta} \leq \bar{c}^2 r_i^{\beta^2}. \tag{6.17}$$

Using (6.14)–(6.17) and the doubling property of μ (see (3.28)) we conclude for a large enough that

$$\mu[B(x_i, 5r_i)] \leq c \int_{F(t_m) \cap B(x_i, r_i)} |\nabla u|^{p-1} \, dH^1 x. \tag{6.18}$$

From (6.18), disjointness of $\{B(x_i, r_i)\}$, and (6.10) it follows for c large enough that

$$\begin{aligned} \mu(E) &\leq \bigcup_i \mu[B(x_i, 5r_i)] \\ &\leq c \sum_{m=m_0}^{\infty} \int_{F(t_m)} |\nabla u|^{p-1} \, dH^1 x \leq c^2 \sum_{m=m_0}^{\infty} m^{-2} \leq c^3 m_0^{-1}, \end{aligned} \tag{6.19}$$

where $2^{-m_0 \beta} = \bar{c} r_0^{\beta^2}$. Since r_0 can be arbitrarily small we see from (6.19) and the remark after (6.12) that claim (6.11) is true. Theorem 3 follows for $1 < p < 2$ from claim (6.11) since σ is absolutely continuous with respect to $H^{1-\varepsilon}$ measure for each $\varepsilon > 0$.

Finally to prove Theorem 3 for $p > 2$, we show there exists a Borel set $\widehat{K} \subset \partial\Omega$ with

$$\mu(\widehat{K}) = \mu(\partial\Omega) \quad \text{and} \quad \sigma(\widehat{K}) \le 100\mu(\widehat{K}). \tag{6.20}$$

Since $H^{1+\varepsilon}$ measure is absolutely continuous with respect to σ for each $\varepsilon > 0$ it follows from (6.20) that Theorem 3 is true when $p > 2$. The proof of (6.20) is similar to the proof of (6.11). Let $\widehat{K}$ be the set of all $x \in \partial\Omega$ with

$$\limsup_{r \to 0} \frac{\mu\big(B(x,r)\big)}{\gamma(r)} \ge 2. \tag{6.21}$$

From the definition of σ, (6.21), and a Vitali covering type argument (see [Ma, Chapter 2]) it follows easily that $\sigma(\widehat{K}) \le 100\mu(\widehat{K})$. Thus to prove (6.20) it suffices to show $\mu(\widehat{E}) = 0$ when $\widehat{E}$ is Borel and equal σ almost everywhere to the set of all points in $\partial\Omega$ for which (6.21) is false. For this purpose given $r_0 > 0$, small, we argue as above to first get $\{B(x_i, r_i)\}$ pairwise disjoint with $r_i \le r_0$,

$$\mu[B(x_i, 5r_i)] \le \gamma(5r_i) \quad \text{and} \quad \widehat{E} \subset \bigcup B(x_i, 5r_i).$$

Next we repeat the argument after (6.13) leading to (6.18). We obtain

$$\mu[B(x_i, 5r_i)] \le c \int_{F(t_m) \cap B(x_i, t_i)} |\nabla u|^{p-1} \, dH^1 x$$

(t_m as defined above). Finally as in (6.19) we get

$$\mu(\widehat{E}) \le c m_0^{-1} \to 0 \quad \text{as } r_0 \to 0.$$

Hence (6.20) is true and the proof of Theorem 3 is complete. ▫

7. Closing remarks

We now discuss some open problems arising from Theorems 1–3.

(1) We would like to know if the rate theorem in Lemma 2.16 is valid for Ω an NTA domain $\subset \mathbf{R}^n$ and $1 < p < \infty$, as is the case when $p = 2$. The argument in the proof of Lemma 2.16 cannot be used in $\mathbf{R}^n$, $n \ge 3$. In $\mathbf{R}^2$, this argument works for somewhat more general domains than domains bounded by quasicircles. For example the same argument would work for a disk with a slit. However if we only assume that Ω is an NTA-domain our proof no longer works. For example, let $x_{i,j} = (i2^{-j}, 2^{-j})$ for $-2^{j+1} \le i \le 2^{j+1}$, $j = 1, 2, \ldots$. Put $\Delta = \bigcup_{i,j} B(x_{i,j}, 2^{-j-2})$ and $\Omega = \{x \in \mathbf{R}^2 : x_2 > 0,\ |x| < 4\} \setminus \bar{\Delta}$. It is easily shown that Ω is an NTA domain. If we proceed as in the proof of Lemma 2.16,

we can construct curves between points in $I_1 = B\big((1,0),1/2\big) \cap \{x : x_2 = 0\}$ and $I_2 = B\big((-1,0),1/2\big) \cap \{x : x_2 = 0\}$ and obtain as in (2.18)–(2.21) that

$$\mu(O) \geq \lambda' \nu(O)$$

for some open set O containing either I_1 or I_2. However this inequality does not lead to a contradiction since (2.15) only gives an estimate for

$$\frac{\mu\big[B\big((\pm 1,0),1/2\big)\big]}{\nu\big[B\big((\pm 1,0),1/2\big)\big]}$$

and $O \subset \{x : |x_2| < \varepsilon\}$ is possible. In fact, for each j we have by doubling that $\mu\big[B\big((i2^{-j},0),2^{-j-2}\big)\big] < c\mu\big[B(x_{i,j},2^{-j-2})\big]$. Since the projections of the balls $B\big((i2^{-j},0),2^{-j-2}\big)$, j fixed, cover one fourth of $(-2,2)$ and are equally spaced, we find (again using doubling) that

$$\mu\big[\{x : x_2 = 0\} \cap \big(-\tfrac{3}{2},\tfrac{3}{2}\big)\big] < c\mu\Big[\bigcup_i B(x_{i,j},2^{-j-2})\Big].$$

Summing this inequality over $1 \leq j \leq n$ we get

$$n\mu\big(\{x : x_2 = 0\} \cap \big(-\tfrac{3}{2},\tfrac{3}{2}\big)\big) < c\mu(\Delta).$$

Letting $n \to \infty$ it follows that $\mu\big(\{x : x_2 = 0\} \cap \big(-\tfrac{3}{2},\tfrac{3}{2}\big)\big) = 0$. A similar argument holds for ν. Thus,

$$\mu[I_1] \geq \lambda' \nu[I_1]$$

is valid for any $\lambda' > 0$.

(2) Given boundedness of the ratio in Lemma 2.16, we would like to know if this ratio is also Hölder continuous (as is the case when $p = 2$).

(3) As for Theorem 1, we would like to know the exact value of H-dim μ when $\partial\Omega$ is the Van Koch snowflake and $p \neq 2$. One can also ask for a given p, what the supremum ($p < 2$) or infimum ($p > 2$) is of H-dim μ taken over the class of quasicircles or even simply connected domains. If this question is too hard, perhaps one can obtain estimates for H-dim μ as $p \to 1$ and $p \to \infty$? At $p = 1$, the p-Laplacian degenerates into the mean curvature equation, so for a given Ω, a natural conjecture would be that H-dim $\mu \to$ the Hausdorff dimension of $\partial\Omega$ as $p \to 1$. Is H-dim μ continuous and/or decreasing as a function of p when $\partial\Omega$ is the Van Koch snowflake? Are the p-harmonic measures defined on each side of a snowflake mutually singular? The answer is yes when $p = 2$ (see [BCGJ]).

(4) Is it always true for $p > 2$ that H-dim $\mu <$ Hausdorff dimension of $\partial\Omega$ when $\partial\Omega$ is as in Theorem 2? The $p = 2$ case was handled in [Ba].

(5) Does Lemma 2.26 generalize to simply connected domains? If so, then our proof scheme in Theorem 3 could be used to get an analogue of Makarov's theorem for $1 < p < \infty$. That is, Theorem 3 would generalize to simply connected domains.

(6) Does the Jones–Wolff theorem alluded to in the introduction hold for $p > 2$ in all planar domains?

(7) In the proof of Theorem 3 we showed for $1 < p < 2$ that μ is absolutely continuous with respect to γ where γ is defined below (6.10). In the case $p = 2$ Makarov proves absolute continuity with respect to a smaller measure and shows by way of example that his result is essentially best possible. His methods also imply that μ is concentrated on a set of σ finite H^1 measure (see [P, Chapter 6, Section 5] or [W1]) which is considerably better than our result for μ when $2 < p$. Can one, at least for quasicircles, obtain the full strength of the results in [M] when $1 < p < \infty$?

(8) If p is fixed, $1 < p < \infty$, give a criterion for which the p-harmonic measures defined on both sides of a quasicircle (or more generally a Jordan domain), are mutually absolutely continuous. Necessary and sufficient conditions for $p = 2$ are given in [BCGJ].

(9) There are numerous related problems for $p = 2$ which one can try to find analogues of when $p \neq 2$ (see [CM]).

(10) Let u be a positive weak solution to some divergence form partial differential equation in a neighborhood of $\partial\Omega$ with $u \equiv 0$ on $\partial\Omega$ in the Sobolev sense. Then the existence of a measure μ satisfying an integral equality, similar to (1.2), is essentially equivalent to showing that powers of u satisfy a basic Caccioppoli inequality. Thus μ corresponding to u as above, exists for a large class of divergence form partial differential equations. For which divergence form partial differential equations can one prove analogues of Theorems 1–3? Regarding this question we note that the invariance of the p-Laplacian under rotation, translation, and dilation was of crucial importance in the proof of Theorems 1 and 2. We also made important use in Theorems 1–3 of the fact that u and its derivatives both satisfy (1.11). This property is also somewhat special for the p-Laplacian, as one sees from considering smooth solutions u to

$$\nabla \cdot [f(|\nabla u|^2)\nabla u] = 0$$

(f smooth on $(0,\infty)$). As in (1.11) we see that if $\zeta = \langle \nabla u, \eta \rangle$ and $\nabla u(x) \neq 0$, then

$$\tilde{L}\zeta = \nabla \cdot \left[2f'(|\nabla u|^2)\langle \nabla u, \zeta \rangle \nabla u + f(|\nabla u|^2)\nabla\zeta \right] = 0$$

at x. Moreover,

$$\tilde{L}u = \nabla \cdot [2f'(|\nabla u|^2)|\nabla u|^2 \nabla u]$$

at x and this equation is only obviously zero if $f(t) = at^\lambda$ for some real a, λ. Thus our theorems may be special for the p-Laplacian.

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