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NON-ACCESSIBLE CRITICAL POINTS OF CERTAIN RATIONAL FUNCTIONS WITH CREMER POINTS

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Abstract. Let R be a rational function with a completely invariant (super)attracting Fatou component. We show that R has a non-accessible critical point in its Julia set, provided that R has a Cremer fixed point with the small cycles property. This extends Kiwi's result stating that the same is true for polynomials with Cremer fixed points.

1. Introduction

Cremer fixed points of rational functions are irrationally indifferent fixed points in the Julia set. In the case of polynomial maps, the presence of Cremer fixed points forces the Julia set to be not locally connected, while rational maps with Cremer fixed points may have a locally connected Julia set (see [13]). When the Julia set fails to be locally connected, it is of interest to determine which points in the Julia set are accessible from the complement of the Julia set.

Repelling periodic points belong to the Julia set and are always accessible, as they are the landing points of at least one periodic external ray [9]. Since a Cremer fixed point belongs to the Julia set, every neighborhood of it contains infinitely many repelling periodic points, but there is no guarantee that any neighborhood of the Cremer point contains infinitely many periodic orbits, or even one entire orbit. When every neighborhood of a Cremer fixed point contains infinitely many cycles, we say that the Cremer point has the *small cycles property*, or that is *approximated by small cycles*. According to Yoccoz [16], Cremer points of quadratic polynomials are always approximated by small cycles. It is not known whether this is true for polynomials of higher degree, or for arbitrary rational functions. According to Perez-Marco [12], rational functions with Cremer periodic points always have some non-accessible points in their Julia sets.

Kiwi [8] has shown that polynomials with a Cremer fixed point having the small cycles property have a non-accessible critical point in their Julia set. We show that the same is true in the case of a rational function with a completely invariant (super)attracting Fatou component. We prove the following results:

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Theorem 4.1. *Let R be a rational function of degree at least two with a completely invariant (super)attracting Fatou component D. Assume that R has a Cremer fixed point z_0. Let C be the connected component of $\overline{\mathbf{C}} \setminus D$ containing the Cremer point z_0. Then there exists a polynomial-like map $(R; U_1, U_2)$ with $C \subset U_1$, whose filled Julia set $K_{(R;U_1,U_2)}$ coincides with C.*

Theorem 4.2. *Let R be a rational function of degree at least two with a completely invariant (super)attracting Fatou component. Assume that R has a Cremer fixed point that is approximated by small cycles. Then R has a critical point which is not accessible from the complement of the Julia set.*

The paper is organized as follows. In Section 2 we introduce some definitions and collect some results from holomorphic dynamics, which will be used in the sequel. For general background in complex dynamics we refer to [1], [3], [9].

In Section 3 we summarize Kiwi's results for polynomials [8]. He first proved that polynomials with a Cremer periodic point and with a connected Julia set have a non-accessible critical value, provided that the Cremer fixed point has the small cycles property. For polynomials with disconnected Julia sets, Kiwi's idea was to extract a polynomial-like mapping which after *straightening* according to Douady and Hubbard [4] becomes a polynomial with a connected Julia set. Then he applied his first result to obtain a non-accessible critical point [8].

In Section 4 we prove our results for rational functions which have a completely invariant (super)attracting component of the Fatou set. Following Kiwi's idea, we show that we can extract a polynomial-like mapping which after *straightening* becomes a polynomial with a connected Julia set, and so it has a non-accessible critical point in the Julia set.

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2. Cremer fixed points and accessible points

Consider a rational function R of degree at least 2 with a fixed point z_0 of multiplier λ. The multiplier is equal to $R'(z_0)$ when z_0 is finite and is equal to $S'(0)$ where $S(z) = 1/R(1/z)$ when $z_0 = \infty$. By definition, a fixed point is either *attracting*, *repelling* or *indifferent* according as the multiplier satisfies $|\lambda| < 1$, $|\lambda| > 1$ or $|\lambda| = 1$. If λ is a root of unity, we call the corresponding fixed point *parabolic* or *rationally indifferent.* For $\lambda = e^{2\pi i\zeta}$, with ζ real and irrational, we call the fixed point *irrationally indifferent.* We say that an irrationally indifferent fixed point is a *Cremer point* if it belongs to the Julia set, and is a *Siegel point* otherwise.

A fixed point z_0 is said to have the *small cycles property* [9] if every neighborhood of z_0 contains infinitely many periodic orbits. According to Yoccoz [16], Cremer fixed points of quadratic polynomials always have the small cycles property. Yoccoz's theorem raises the question as to whether every Cremer point of a holomorphic germ has small cycles. The answer was provided by Perez-Marco [10] who showed that for ζ not satisfying a certain diophantine condition, there exists a germ with multiplier $\lambda = e^{2\pi i\zeta}$ that has a Cremer point but no small cycles.

It is not known whether Cremer fixed points of polynomials of arbitrary degree always have small cycles. Also, it is not known whether any rational function can have Cremer points without small cycles. For a detailed discussion of the local dynamics near an irrationally indifferent fixed point see [11].

A point z in the Julia set is called *accessible* from a Fatou component V if there exists a path γ contained in V that ends at z, i.e.

$$\gamma\colon [0,1) \longrightarrow V, \quad \text{and} \quad \gamma(r) \to z, \quad r \to 1.$$

When z is not accessible from any Fatou component we say that z is a *non-accessible point.* We note that accessibility is completely invariant under the dynamics: z is accessible from V if and only if $R(z)$ is accessible from $R(V)$.

Consider a monic polynomial map $P\colon \mathbf{C} \longrightarrow \mathbf{C}$ of degree $d \geq 2$. The complement of the super-attracting basin of infinity, that is the set of all the points $z \in \mathbf{C}$ with bounded forward orbit under P is called the *filled Julia set of* P and is denoted by K_P.

Assume that K_P is connected. From Boetcher's theorem [1], [3], we know that the complement of the closed unit disk $\bar{\Delta}$ is conformally isomorphic to the complement of the filled Julia set K_P. Moreover, there exists a conformal isomorphism:

$$\phi\colon \mathbf{C} \setminus \bar{\Delta} \longrightarrow \mathbf{C} \setminus K_P,$$

such that

$$\begin{aligned} P \circ \phi(z) &= \phi(z^d), \\ \phi(z) &= z + \mathscr{O}(1), \quad z \to \infty. \end{aligned}$$

The question of which points in J_P are accessible from $\mathbf{C} \setminus K_P$ is closely related to the boundary behavior of ϕ.

The set

$$R_t = \phi\big(\{re^{2\pi i t} : r \in (1,\infty)\}\big)$$

is called the *external ray with angle* $t \in \mathbf{R}$. Here the angle t is measured in fractions of a full turn, and not in radians. We say that an external ray R_t *lands* at z, if

$$\lim_{r \searrow 1} \phi(re^{2\pi i t}) = z.$$

We see that the landing points of external rays are accessible from $\mathbf{C} \setminus K_P$. The converse is also true: every point accessible from $\mathbf{C} \setminus K_P$ is the landing point of some external ray (Lindelöf [14], [15]).

The closure $\overline{R_t}$ of an external ray R_t is called a *closed ray.* If R_t lands, then the closure $\overline{R_t}$ is the union of the external ray and its landing point. Otherwise, if R_t fails to land, $\overline{R_t}$ is the union of the external ray and a non-trivial connected subset of the Julia set.

Let $P^{\circ n}$ denote the n-fold iterate of P. An external ray R_t is called *periodic* if $P^{\circ n}(R_t) = R_t$ for some integer n. In particular, if $P(R_t) = R_t$, we say that R_t is a *fixed ray.* Periodic rays always land, and their landing points are parabolic or repelling periodic points. Conversely, parabolic and repelling periodic points are the landing points of at least one periodic ray (see [7], [9]).

3. Kiwi's results for polynomials

For polynomials with connected Julia sets, we know that every point that is accessible from the complement of the filled Julia set, is the landing point of some external ray (Lindelöf, [14], [15]). For the same case, Kiwi [8] proved a stronger result that states the existence of a non-accessible critical value with the property that, if an external ray accumulates to it, this external ray also accumulates to the Cremer fixed point.

Theorem 3.1 (Kiwi, [8]). *Let P be a polynomial with connected Julia set J_P and with a Cremer fixed point z_0 that is approximated by small cycles. Then there exists a critical value $v \in J_P$ which is not accessible from $\mathbf{C} \setminus J_P$ such that: if $v \in \overline{R}_t$ for some external ray R_t, then also $z_0 \in \overline{R}_t$.*

When the Julia set is disconnected, Kiwi's idea is to extract a polynomial-like mapping which after *straightening* becomes a Cremer polynomial with a connected Julia set. Before we can explain how this is done, we need some definitions and results about polynomial-like mappings (see [4]).

Definition. Let $U_1 \subset \mathbf{C}$ and $U_2 \subset \mathbf{C}$ be bounded simply connected domains with smooth boundaries such that $\overline{U}_1 \subset U_2$. We say that $(f; U_1, U_2)$ is a *polynomial-like map of degree d* if $f\colon \overline{U}_1 \to \overline{U}_2$ is a d-fold branched covering, where $d \geq 2$, which is holomorphic in U_1.

Definition. The *filled Julia set* $K_{(f;U_1,U_2)}$ of $(f; U_1, U_2)$ is the set of points in U_1 for which the forward iterates of f are well-defined:

$$K_{(f;U_1,U_2)} = \bigcap f^{-n}(\overline{U}_1).$$

The *Julia set* $J_{(f;U_1,U_2)}$ is the boundary of $K_{(f;U_1,U_2)}$.

A polynomial-like mapping can be extended to the complex plane in such a way that it is quasiconformally conjugate to a polynomial of the same degree.

Theorem 3.2 (The straightening theorem, Douady–Hubbard, [4]). *If $(f; U_1, U_2)$ is a polynomial-like map of degree d, then there exists a quasiconformal map $\phi\colon \mathbf{C} \to \mathbf{C}$ and a polynomial Q of degree d such that $\phi \circ f = Q \circ \phi$ on $\overline{U}_1$. Moreover, $\phi(K_{(f;U_1,U_2)}) = K_Q$.*

For polynomials with disconnected Julia sets, Kiwi proved the following results.

Theorem 3.3 (Kiwi, [8]). *Let P be a polynomial of degree at least 2 and let C be a connected component of its filled Julia set K_P such that $P(C) = C$. If C is not a singleton consisting of a repelling fixed point, then there exists a polynomial-like mapping $(f; U_1, U_2)$ such that $C = K_{(f;U_1,U_2)}$.*

Theorem 3.4 (Kiwi, [8]). *Let P be a polynomial with a Cremer fixed point z_0 that is approximated by small cycles. Then there exists a critical point of P in the Julia set which is not accessible from the complement of the Julia set.*

4. Our results

We now consider a rational function R of degree two or more with a completely invariant (super)attracting Fatou component. Recall that a set D is *completely invariant* under the map R means that z belongs to D if and only if $R(z)$ belongs to D.

We extend Kiwi's Theorem 3.3 as follows.

Theorem 4.1. *Let R be a rational function of degree at least two with a completely invariant (super)attracting Fatou component D. Assume that R has a Cremer fixed point z_0. Let C be the connected component of $\overline{\mathbf{C}} \setminus D$ containing the Cremer point z_0. Then there exists a polynomial-like map $(R; U_1, U_2)$ with $C \subset U_1$ such that $C = K_{(R;U_1,U_2)}$.*

Proof. Let d be the degree of R and let $A = \overline{\mathbf{C}} \setminus D$. We will treat the attracting and superattracting components in the same way, and simply call them "attracting". By replacing R with $M^{-1} \circ R \circ M$ for a suitable Möbius transformation M that may depend on the invariant component D, we may assume that the attracting fixed point is located at infinity.

We will work with the generalized Green's function of the attracting component D, as introduced in [6]. It is known that the Julia set J has positive Hausdorff dimension, and thus it has positive logarithmic capacity ([2], [5]). It follows that $\partial D = J$ also has positive logarithmic capacity, and thus D has a generalized Green's function $g(z) = g(z, \infty)$ with pole at ∞. The function g is positive and harmonic in $D \setminus \{\infty\}$, and can be extended to a non-negative subharmonic function in $\mathbf{C}$ in the standard way. Hence g is continuous and vanishes in $A = \overline{\mathbf{C}} \setminus D$. Moreover, all the boundary points of D are regular for the Dirichlet's problem on D.

Also, we have

$$g\big(R(z), \infty\big) = \sum_{R(w)=\infty} g(z, w),$$

where the sum is taken over all $w \in \overline{\mathbf{C}}$ with $R(w) = \infty$ with due count of multiplicity. Note that if $R(w) = \infty$ then $w \in D$, that there are d terms in the sum, and that $w = \infty$ occurs at least once in the sum since $R(\infty) = \infty$ (see [6] for the proofs).

Since there are countably many zeros of the gradient of g in D, there exist only countably many positive numbers $t > 0$ such that the corresponding level sets $L_t = \{z \in D : g(z) = t\}$ of g contain zeros of the gradient of g. Also it follows that there exist only countably many positive numbers t such that the image $R(L_t) = \{R(z) : z \in D,\ g(z) = t\}$ contains zeros of the gradient of g. We call $t > 0$ a *regular value* for the Green's function g if neither L_t nor $R(L_t)$ contains any zero of the gradient of g.

Choose $t > 0$ such that L_t does not contain any zeros of the gradient of g. Then L_t consists of finitely many Jordan curves since there is no branching. Indeed, L_t is closed, since g is continuous on $\mathbf{C}$, and it is bounded, since $g(z,\infty)$ tends to ∞ as z tends to ∞. Also, $L_t \cap \partial D = \emptyset$, since $g(z) \to 0$, as $z \to \zeta \in \partial D$.

If γ_t is a component of L_t, so a Jordan curve, then, recalling that g is subharmonic in $\mathbf{C}$, we know that the maximum of g on the closed Jordan domain enclosed by γ_t is attained on γ_t, so we have $g < t$ inside γ_t. For $z \in \gamma_t$, we have

$$g\big(R(z),\infty\big) = \sum_{R(w)=\infty} g(z,w) > g(z,\infty),$$

since $w = \infty$ occurs at least once in the sum, and the continuous function

$$\sum_{R(w)=\infty} g(z,w) - g(z,\infty)$$

has a positive minimum on the compact set γ_t. Thus, $R(L_t)$ lies outside γ_t, and we have $g > t$ outside all the closed curves γ_t.

Let us denote by $V(t)$ the connected component of $\{z \in \mathbf{C} : g(z) < t\}$ containing the Cremer point z_0 of R. Recall that C denotes the connected component of $A = \overline{\mathbf{C}} \setminus D$ containing the Cremer point z_0. We claim that

$$C = \bigcap_{t>0 \text{ regular for } g} V(t).$$

Clearly, C is contained in each $V(t)$, since $g = 0$ on A, so also on C. So, $C \subset \bigcap V(t)$. To prove the equality, let us assume the contrary, that there exists a point z in $\bigcap V(t)$ with $z \notin C$. Then $g(z) = 0$ so that $z \in A = \overline{\mathbf{C}} \setminus D$. Let C_1 be the connected component of A containing z. Since $z \notin C$, we know that C and C_1 are distinct components of A. Then there exists a Jordan curve $\gamma \subset D$ that separates C and C_1. Now on $\gamma \subset D$, the Green's function g attains a positive minimum δ_0, so $g \geq \delta_0 > 0$ on γ. Choose t with $0 < t < \delta_0$. Then we can find two distinct components γ_t and $\gamma_t{}'$ of the level set L_t, situated in the two distinct components of $\mathbf{C} \setminus \gamma$. So we have $C_1 \cap V(t) = \emptyset$ (for they are separated by γ), in contradiction with our assumption that $z \in C_1 \bigcap V(t)$. Thus the equality holds.

In particular, if a critical point of R is contained in $V(t)$ for all regular values t of g, then it belongs to C. It follows that there exists a regular value t_0 of g such that all the critical points of R contained in $V(t_0)$ belong to C. Indeed, let $c_1, c_2, \ldots, c_n$ denote the critical points of R. If $c_k \notin C$ for some k, there exists a regular value t_k of g such that $c_k \notin V(t_k)$. Let t_0 be a regular value of g with $0 < t_0 \leq \min\{t_j : c_j \notin C\}$. Now $V(t_0) \subset \bigcap_{c_j \notin C} V(t_j)$, and we conclude that all the critical points of R contained in $V(t_0)$ belong to C.

Choose t_0 as above and such that for any $z \in D$ which is a zero of the gradient of g we have $R(z) \notin L_{t_0}$.

Let γ_2 be the component of L_{t_0} whose interior contains the Cremer point z_0, and let U_2 be the Jordan domain inside γ_2 (so that $C \subset U_2$). We claim that $R(C) = C$, and so we may let U_1 denote the component of $R^{-1}(U_2)$ that contains C, and γ_1 denote its boundary. Then γ_1 is mapped by R onto γ_2 and outside γ_1, for if $g\big(R(z)\big) = t_0$ then $g(z) < g\big(R(z)\big)$, so $g(z) < t_0$. Thus U_1 and U_2 are Jordan domains containing C such that $\overline{U}_1 \subset U_2$.

We still need to verify that $R(C) = C$. It is easy to see that $R(C) \subset C$. Indeed, first note that $R(C)$ is connected, since C is connected and R is continuous. So $R(C)$ is a connected subset of A, C is the connected component of A containing the Cremer fixed point z_0, $R(z_0) = z_0$, hence $R(C) \subset C$. Now we show that $C \subset R(C)$. Recall that

$$C = \bigcap_{t>0 \text{ regular for } g} V(t),$$

so C is a compact connected subset of $\overline{\mathbf{C}}$. Then we know that $R^{-1}(C)$ has at most d components, and each is mapped by R onto C (see [2]). Say $R^{-1}(C) = C_1 \cup C_2 \cup \cdots \cup C_l$ so that $R(C_j) = C$ for all j with $1 \leq j \leq l$. We know that $R(C) \subset C$, and so by applying R^{-1} we get

$$C \subset R^{-1}\big(R(C)\big) \subset R^{-1}(C) = C_1 \cup C_2 \cup \cdots \cup C_l \subset A.$$

Since C is connected, we get that $C \subset C_k \subset A$, for some k. But C is a connected component of A, so we must have $C = C_k$, and hence $R(C) = C$.

The proper holomorphic map

$$R\colon U_1 \longrightarrow U_2, \quad U_1 \overset{\text{onto}}{\longrightarrow} U_2, \quad \partial U_1 \longrightarrow \partial U_2$$

is a k-fold branched covering. Here k is the degree of R as a covering map of C onto C, all other inverse images of C being excluded.

Note that $R\colon U_1 \longrightarrow U_2$ is not a conformal isomorphism. Assuming R is a conformal isomorphism of U_1 onto U_2, the Denjoy–Wolff fixed point theorem ([15]) applied to the holomorphic map $S = R^{-1}\colon U_2 \longrightarrow U_1$ shows that there exists a unique Denjoy–Wolff fixed point $\alpha \in U_2$, with $S(\alpha) = \alpha$ and $|S'(\alpha)| < 1$. Then α belongs to $\overline{U}_1$, and since S maps U_1 into a smaller domain U_3 with $\overline{U}_3 \subset U_1$, we have $\alpha \in U_1$. The Denjoy–Wolff theorem states the uniqueness of the fixed point α in U_1. But R and so S has a Cremer fixed point z_0 in U_1, and thus we must have $\alpha = z_0$, a contradiction since z_0 is a Cremer fixed point of R and α is a repelling fixed point of R. Hence, R is not a conformal isomorphism.

The above remark forces $\deg(R \mid U_1) \geq 2$, so $(R; U_1, U_2)$ is a polynomial-like map of degree at least two.

Let

$$K_{(R;U_1,U_2)} = \bigcap_{n\geq 0} R^{-n}(U_1)$$

be the filled Julia set of $(R; U_1, U_2)$.

Note that $C \subset K_{(R;U_1,U_2)}$, since $R(C) = C$. As we have remarked before, the critical points of the polynomial-like map $(R; U_1, U_2)$ are contained in C, therefore they never escape from U_1 ($c \in C$, $R(C) = C \Rightarrow R^n(c) \in C \subset U_1$, for all $n \geq 0$). This means that all the critical points of $(R; U_1, U_2)$ belong to $K_{(R;U_1,U_2)}$. Therefore the set $K_{(R;U_1,U_2)}$ is connected (by Proposition 2 in [4]). It follows that

$$C = K_{(R;U_1,U_2)},$$

since $C \subset K_{(R;U_1,U_2)} \subset A$ and C is a connected component of A. □

We now extend Kiwi's Theorem 3.4 to obtain the following result.

Theorem 4.2. *Let R be a rational function of degree at least two with a completely invariant (super)attracting Fatou component. Assume that R has a Cremer fixed point that is approximated by small cycles. Then R has a critical point which is not accessible from the complement of the Julia set.*

Proof. Just as in the Theorem 4.1, let C be the connected component of $A = \overline{\mathbf{C}} \setminus D$ that contains the Cremer fixed point z_0. It follows from the proof of Theorem 4.1, that $R(C) = C$, and that we can extract a polynomial-like map $(R; U_1, U_2)$. After *straightening* $(R; U_1, U_2)$ using the Douady–Hubbard theorem, we obtain a polynomial Q and a homeomorphism ϕ of $\mathbf{C}$ such that $\phi \circ R = Q \circ \phi$ on $\overline{U}_1$ and $K_Q = \phi(C)$. Then K_Q is connected, hence the Julia set $J_Q = \partial K_Q = \phi(\partial C)$ is connected.

Every neighborhood $U \subset U_1$ of z_0 contains infinitely many cycles of R, hence every neighborhood of $\phi(z_0)$ contains infinitely many cycles of Q. Thus, $\phi(z_0)$ is a Cremer fixed point of Q that is approximated by small cycles. By Theorem 3.1 for polynomials with connected Julia sets, we find a critical point $c \in J_Q$ of Q that is not accessible from $\mathbf{C} \setminus J_Q$. Since $c \in J_Q = \phi(\partial C)$, it follows that

$\phi^{-1}(c) \in \partial C \subset J_R$, so $\phi^{-1}(c)$ is a critical point of R that belongs to $\partial C \subset J_R$. Also, paths in $\mathbf{C} \setminus J_Q$ correspond under ϕ^{-1} to paths in $\mathbf{C} \setminus \partial C$. Thus, the critical point $\phi^{-1}(c)$ of R is not accessible from $\mathbf{C} \setminus \partial C \supset \mathbf{C} \setminus J_R$. □

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