Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 31, 2006, 61–70

BANACH SPACES WHOSE BOUNDED SETS ARE BOUNDING IN THE BIDUAL

Humberto Carrión, Pablo Galindo, and Mary Lilian Lourenço

Universidade de São Paulo, Instituto de Matemática e Estatística
Dep. de Matemática, Caixa Postal 66281, 05315-970 São Paulo, Brazil; leinad@ime.usp.br

Universidad de Valencia, Facultad de Matemáticas
Departamento de Análisis Matemático, ES-46100 Burjasot, Valencia, Spain; galindo@uv.es

Universidade de São Paulo, Instituto de Matemática e Estatística
Dep. de Matemática, Caixa Postal 66281, 05315-970 São Paulo, Brazil; mllouren@ime.usp.br

Abstract. We discuss necessary conditions for a Banach space to satisfy the property that its bounded sets are bounding in its bidual space. Apart from the classic case of c_0, we prove that, among others, the direct sum $c_0(l_2^n)$ is another example of spaces having such property.

A subset B of a complex Banach space X is said to be *bounding* if every entire function defined on X is bounded on B. Such a set is also a limited set (see below) and, of course, every relatively compact set is bounding. If X is such that every bounding subset is relatively compact, then X is called a *Gelfand–Phillips* space and there is some literature devoted to the topic. Gelfand–Phillips spaces are characterized also as those whose sequences converging against entire functions are norm convergent [7]. B. Josefson [12] and T. Schlumprecht [15] found simultaneously and independently examples of complex Banach spaces containing limited non-bounding (hence non-relatively compact) sets.

By $H(X)$ we denote the space of entire functions defined on X, that is, functions that are Fréchet differentiable at every point of X. $H_b(X)$ is the subspace of all $f \in H(X)$ such that f is bounded on bounded sets; these are the so-called entire functions of bounded type.

Bounding sets are related as well to the extension of holomorphic functions to a larger space in the following way: *Let $E \subset F$ be complex Banach spaces. Every bounded subset of E is bounding in F if, and only if, every $f \in H(E)$ having a holomorphic continuation to F is of bounded type.* In the particular instance of $F = E^{**}$ it is known that every $f \in H_b(E)$ has a holomorphic extension to E^{**}. Therefore, in this case the only holomorphic functions which may be extended to

---

2000 Mathematics Subject Classification: Primary 46G20.

The second author was supported by Projects AE-2003-0392 (Universidad de Valencia) and BFM 2003-07540 (DGI. Spain) and the third author in part by CAPES, Brazil, Research Grant AEX0287/03-2.

the bidual are those of bounded type if, and only if, every bounded set in E is bounding in E^{**}.

As far as we know there is essentially one known case ($E = c_0$) where the above mentioned (equivalent) conditions hold. This was proved among other results by B. Josefson in his 1978 paper [11]. Clearly, no reflexive infinite-dimensional Banach space satisfies those conditions and neither the conditions hold in general for quasi-reflexive Banach spaces, i.e., spaces of finite codimension in its bidual, the conditions may not hold: Consider, for instance, the Q-reflexive (see [1]) and quasi-reflexive Tsirelson–James space T_J^*. Its bidual space is separable and hence there the bounding sets are relatively compact (see [5, 4.26]). So the bounded sets in T_J^* are not bounding in $(T_J^*)^{**}$.

Our aim is to study this situation and to provide further examples of Banach spaces whose bounded sets are bounding in the bidual. So we consider the class $\mathscr{F}$ of Banach spaces

$$\mathscr{F} = \{F : \text{every bounded set } B \subset F \text{ is a bounding set in } F^{**}\}.$$

We show that such a class $\mathscr{F}$ is stable under products and quotients, but not under closed subspaces; however the closed subspaces of c_0 do belong to the class. In the second part of the article we prove that $c_0(l_2^n)$ the predual of C. Stegall's example of a Schur space whose dual lacks the Dunford–Pettis property [16] is a Banach space whose bounded sets are bounding in the bidual. Our argument does not rely on Josefson's paper and so provides another proof of the fact that c_0 is in the class we are dealing with.

1. Generalities

A subset B of a complex Banach space X is bounding if, and only if, $\lim_m \|P_m\|_B^{1/m} = 0$ for all entire functions $f = \sum_m P_m$, where $\sum_m P_m$ represents the Taylor series of $f \in H(X)$ at the origin ([6, Lemma 4.50]). We will use this characterization in Section 2.

A subset A of a Banach space is called *limited* if weak* null sequences in the dual space converge uniformly on A. The unit ball of an infinite-dimensional Banach space is not a limited set by the Josefson–Nissenzweig theorem. An operator $T: X \to Y$ is said to be limited if it maps the unit ball of X onto a limited set in Y. It turns out that T is limited if, and only if, T^* is weak*-to-norm sequentially continuous. As we mentioned, each bounding set is a limited one, therefore if $E \in \mathscr{F}$, then the canonical inclusion $E \to E^{**}$ is a limited operator.

Recall that a Banach space X has the property (P) of Pełczyński if for every $K \subset X^*$ that is not weakly relatively compact, there exists a weakly Cauchy series $\sum x_n$ in X such that $\inf_n \|x_n\|_K > 0$. Every $C(K)$ space has the (P) property (see [17, III.D.33]).

We refer to [5] for infinite-dimensional holomorphy background.

Proposition 1.1. *If the canonical inclusion $E \to E^{**}$ is a limited operator, then E^* is a Schur space. Conversely, if E^* is a Schur space and* (i) *E^{**} is a Grothendieck space or* (ii) *E has the (P) property, then the canonical embedding of E into E^{**} is limited.*

Proof. If the canonical inclusion $E \to E^{**}$ is a limited operator, then the natural projection $\pi_{E^*}\colon E^{***} \to E^*$ is weak*-to-norm sequentially continuous. Therefore the identity on E^* as a composition of the canonical embedding in E^{***} and the natural projection is weak-to-norm sequentially continuous, i.e., E^* is a Schur space. For the converse statement (i), note that $\pi_{E^*}\colon E^{***} \to E^*$ is weak-weak sequentially continuous, hence weak*-to-norm sequentially continuous by the assumptions on E^* and E^{**}. And for the (ii) case, $\pi_{E^*}\colon E^{***} \to E^*$ is weak*-to-weak sequentially continuous by [10, Proposition 3.6, p. 131]. ▫

Note. *If the canonical inclusion $E \to E^{**}$ is a limited operator, then $H_b(E) = H_{wu}(E)$.* This is so because then E^* is a Schur space, hence E has the Dunford–Pettis property and does not contain copies of l_1.

In the sequel, we aim at studying whether important classes of Banach spaces belong to the class $\mathscr{F}$.

Firstly, we observe that for no (isomorphic) infinite-dimensional dual space X^*, the canonical inclusion $X^* \to X^{***}$ can be limited, since otherwise by composing it with the restriction to X operator, $\varrho\colon X^{***} \to X^*$, it would turn out that the identity on X^* would be a limited operator. Therefore, no dual space belongs to $\mathscr{F}$.

When does a $C(K)$-space belong to $\mathscr{F}$? The above proposition shows that in this setting of Banach spaces of continuous functions the condition of E^* being a Schur space is equivalent to the bounded sets in E being bounding in the bidual: Indeed, if $C(K)^*$ is a Schur space, then $C(K)$ does not contain a copy of l_1, hence by [13] its bidual is isomorphic to $l_\infty(I)$ for some set I. But then $C(K)^{**}$ is a Grothendieck space, so the bounded sets in $C(K)$ are limited in $C(K)^{**}$ and furthermore bounding there because of [11, Theorem 1].

Since $c_0 \in \mathscr{F}$ and it is an M-ideal in l^∞ one wonders whether there are any connections between $\mathscr{F}$ and the class of Banach spaces E which are an M-ideal in E^{**}. We are to see that there is no a clear connection.

On one hand, for the Lorentz sequence space $d(w,1)$ ([10, p. 103]), its predual $E = d_*(w,1)$ is known to be an M-ideal in its bidual ([10, III 1.4]). However, the unit basis of $d(w,1)$ is weakly null, so $d(w,1)$ is not a Schur space. Hence, it turns out that the canonical inclusion $E \to E^{**}$ is not a limited operator. Thus, not every bounded set in E is bounding in E^{**}. Another (simpler) example is the following: According to [10, III Examples 1.4(f) and (g)], for $1 < p < \infty$, the space $K(l_p)$ of compact linear operators of l_p, is an M-ideal in its bidual. However $K(l_p)^*$ is not a Schur space since $K(l_p)$ does not have the Dunford–Pettis property since it contains a complemented copy of the reflexive space l_p.

On the other hand, the condition of E being an M-ideal in E^{**} is not necessary for their bounded sets being bounding in E^{**}. The Bourgain–Delbaen Banach space Y—introduced in [2]—is shown there to satisfy $Y^* \approx l_1$ and not to contain any copy of c_0. Thus $Y^{**} \approx l_\infty$ is a Grothendieck space and Proposition 1.1 applies. Further, again by Josefson's result the limited sets in Y^{**} are bounding sets, so it turns out that the bounded sets in Y are bounding in Y^{**}. However, Y is not an M-ideal in its bidual, since if it were, then it would be isomorphic to c_0 as a consequence of [10, III Theorem 3.11]. We also mention in passing that Y fails the (P) property.

Corollary 1.2. *If the Banach space E is an M-ideal in its bidual, then the canonical inclusion $E \to E^{**}$ is a limited operator if, and only if, E^* is a Schur space.*

Proof. Since any Banach space which is an M-ideal in its bidual has the (P) property ([10, III Theorem 3.4]), it suffices to apply Proposition 1.1. ▫

Is the class $\mathscr{F}$ stable by closed subspaces? The Bourgain–Delbaen space Y also shows that $\mathscr{F}$ is not stable by closed subspaces since, as Haydon [9] proved, Y contains a subspace isomorphic to some l_p, which clearly does not belong to $\mathscr{F}$.

Recall ([10, III Theorem 1.6]) that every subspace M of $K(l_p)$ is also an M-ideal in its bidual and those subspaces which embed into c_0 satisfy that M^* are Schur spaces ([14, p. 415]). Moreover, every closed subspace E of c_0 is also an M-ideal in its bidual, has the Dunford–Pettis property and does not contain any copy of l_1, hence E has the (P) property and E^* is a Schur space. Therefore, the canonical inclusion $E \to E^{**}$ is a limited operator. So the arising question is: Does every closed subspace of c_0 have its bounded sets bounding in the bidual? Actually this question has a positive answer which reduces to Josefson's result that $c_0 \in \mathscr{F}$.

Proposition 1.3. *The bounded sets in every closed subspace E of c_0 are bounding in E^{**}.*

Proof. To begin with, note that since E has the DP property and does not contain any copy of l_1, $H_b(E) = H_{\mathrm{wu}}(E)$, the space of entire functions on E which are weakly uniformly continuous on bounded sets. Moreover E^* is a separable space, hence by Corollary 12 in [3], $H_{\mathrm{wu}}(E) = H_{\mathrm{wsc}}(E)$, the space of weakly sequentially continuous entire functions on E.

Let f be entire on E with extension, $\bar{f}$, to E^{**}. If $f \notin H_b(E)$, f is not weakly sequentially continuous, thus there is a sequence $(x_n) \subset E$ weakly convergent to $a \in E$ such that $\big(f(x_n)\big)$ does not converge to $f(a)$. By considering the function $g(x) = f(x+a)$ we may assume that $a = 0$. Clearly, (x_n) cannot be a norm null sequence, hence by the Bessaga–Pełczyński selection principle (see [4, Chapter V, p. 46]) there is a subsequence of (x_n) which is a basic sequence equivalent to a block basic sequence taken with respect to the unit vector basis of c_0, and further such subsequence spans a Banach subspace X of E which is complemented in c_0 along a projection p (see [8, Proposition 249]). There is no loss of generality assuming that the sequence (x_n) is such a subsequence. Now $\bar{f}|_{X^{**}}$ is an extension to X^{**} of the function $f|_X$, and therefore, $\bar{f}|_{X^{**}} \circ p^{**}$ is an extension of $f|_X \circ p$ to l_∞. Further, since $c_0 \in \mathscr{F}$, it follows that $f|_X \circ p$ belongs to $H_b(c_0)$, so it has to be a weakly sequentially continuous function and this is prevented by the choice of $(x_n) \subset X$. □

We end this section with another positive result.

Proposition 1.4. *$\mathscr{F}$ is stable by products and quotients.*

Proof. (a) Let $E, F \in \mathscr{F}$. Let $f \in H(E^{**} \times F^{**})$ and consider a bounded set in $E \times F$ which we may suppose to be $A \times B$ with $A \subset E$ and $B \subset F$ both bounded sets. We check that the collection $\{f(x, \cdot)\}_{x \in A} \subset H(F^{**})$ is τ_0 bounded: Indeed, for any compact subset K of F^{**}, the collection $\{f(\,\cdot\,, y)\}_{y \in K} \subset \big(H(E^{**}), \tau_0\big)$ is bounded, hence is τ_δ bounded ([5, 2.44, 2.46]). In addition, since A is bounding in E^{**}, $\|f\|_A = \sup\{|f(x)| : x \in A\}$ defines a τ_δ continuous seminorm in $H(E^{**})$ by [5, 4.18], so we have

$$\sup_{x \in A} \sup_{y \in K} |f(x, y)| = \sup_{y \in K} \sup_{x \in A} |f(x, y)| = \sup_{y \in K} \|f(\,\cdot\,, y)\|_A < \infty,$$

as we wanted. Now, since B is bounding in F^{**}, we have that $\{f(x, \cdot)\}_{x \in A}$ is bounded for the $\|\cdot\|_B$ seminorm, hence $\{|f(x, y)|\}_{x \in A, y \in B}$ is bounded and so $A \times B$ is shown to be bounding in $E^{**} \times F^{**}$.

(b) If F is a quotient of a Banach space $E \in \mathscr{F}$, then F belongs to $\mathscr{F}$ as well: Let $q\colon E \to F$ be the quotient mapping and $f \in H(F)$ with a holomorphic extension g to F^{**}. Then $g \circ q^{tt}$ is a holomorphic function in E^{**} which extends $f \circ q$. Therefore, $f \circ q \in H_b(E)$ and since any bounded subset $A \subset F$ is contained in $q(B)$ for some bounded set B, it follows that f is bounded in A. □

2. The example $c_0(l_2^n)$

In this section we consider $X = c_0(l_2^n)$. X^* enjoys the Schur property, yet its dual fails the Dunford–Pettis property, and it was the first example of a space with the Dunford–Pettis property whose dual lacks it, a fact discovered by C. Stegall [16]. Thus in view of Proposition 1.1 we found it suitable to explore whether X belongs to $\mathscr{F}$.

We now set some notation. Consider a partition of $\mathbf{N}$ by defining for each $n \in \mathbf{N}$ the interval

$$I_n = \left[\frac{n(n-1)}{2} + 1, \frac{n(n+1)}{2}\right] \cap \mathbf{N}.$$

Stegall's example is

$$X^* := \left\{(x_i) \in l^\infty : \sum_{n=1}^{\infty} \Big(\sum_{i \in I_n} |x_i|^2\Big)^{1/2} < \infty\right\}.$$

The canonical predual of X^* is the space

$$X := \left\{ (x_i) \in l^\infty : \lim_{k\to\infty} \sup_{n\geq k} \left(\sum_{i\in I_n} |x_i|^2 \right)^{1/2} = 0 \right\}.$$

Observe that X coincides with the closed linear span in X^{**} of the unit vectors $\{e_j = (\delta_{nj})_{n\in\mathbf{N}},\ j\in\mathbf{N}\}$.

The dual of X^* can be represented by

$$X^{**} = \left\{ (x_i) \in l^\infty : \sup_{n\in\mathbf{N}} \left(\sum_{i\in I_n} |x_i|^2 \right)^{1/2} < \infty \right\}.$$

Let S be a subset of $\mathbf{N}$. A sequence of complex numbers $x = (x_i)$ is said to have *support* in S if $x_i = 0$ for each $i \notin S$. If, moreover, x belongs to a sequence space E, then we write $x \in E_S$.

$B(E)$ denotes the closed unit ball of E whatever the normed space E be.

For $f\colon E \to \mathbf{C}$ we denote $\|f\|_S = \sup_{x\in B(E_S)} |f(x)|$.

Let $(p_i)_{i\in\mathbf{N}}$ be a strictly increasing sequence of natural numbers. For each $i \in \mathbf{N}$, let $\alpha_{p_i+1}, \dots, \alpha_{p_{i+1}}$ be scalars at least one of which is nonzero, and let $u_i = \sum_{j=p_i+1}^{p_{i+1}} \alpha_j e_j$ be a block sequence taken from $(e_i)_{i\in\mathbf{N}}$. The sequence $(u_i)_{i\in\mathbf{N}}$ is called *totally disjoint* with respect to $(I_n)_{n\in\mathbf{N}}$ whenever $p_i \in I_{n_i}$, $p_{i+1} \in I_{n'_i}$ and $n_i \neq n'_i$.

We remark that a block sequence $(u_i)_{i\in\mathbf{N}}$ may be written as $u_i = \sum_{j\in U_i} \alpha_j e_j$, where $U_i = \bigcup_{j\in F_i} I_j$ and F_i is a finite subset of $\mathbf{N}$. Further, if $p_i \in I_s$, $p_{i+1} \in I_r$ such that $I_r \cap I_s = \emptyset$ we can choose the subsets F_i such that $\max F_i < \min F_{i+1}$ and in this case $(u_i)_{i\in\mathbf{N}}$ is a totally disjoint sequence.

In the sequel we consider a sequence $(U_n)_{n\in\mathbf{N}}$ of subsets of $\mathbf{N}$ such that for each n

(i) $U_n = \bigcup_{j\in F_n} I_j$, where F_n is a finite subset of $\mathbf{N}$ and
(ii) $\max F_n < \min F_{n+1}$.

Condition (ii) implies $U_n \cap U_m = \emptyset$ for each $m \neq n$.

It is not difficult to show that if the sequence $(u_i)_{i\in\mathbf{N}}$, $u_i = \sum_{j\in U_i} \alpha_j e_j$, is totally disjoint and $\sup_{i\in\mathbf{N}} \|u_i\| \leq C$ for some positive constant C, then the partial sums of the series $\sum \theta_i u_i$ are bounded by $C \cdot \sup_i |\theta_i|$ and so the formal series $\sum \theta_i u_i$ defines an element in X^{**}.

To prove the next theorem we will make use of the following result which extends a similar one from Dineen ([6, p. 299]) valid for l_∞.

Lemma 2.1. *Let Q be a continuous polynomial in X^{**} such that $Q(0) = 0$. Let $N' \subset \mathbf{N}$ be an infinite set. Then for each $\varepsilon > 0$ there is an infinite subset $S \subset N'$ such that*

$$\|Q\|_{\cup_{n\in S} U_n} < \varepsilon.$$

Proof. First we consider a homogeneous polynomial P. Suppose that there is an $\varepsilon > 0$ such that for each infinite subset $S \subset N'$ we have

$$\|P\|_{\cup_{n\in S} U_n} > \varepsilon.$$

Let $(S_i)_i$ be a disjoint partition of N' into an infinite number of infinite sets. Then

$$\|P\|_{\cup_{n\in S_i} U_n} > \varepsilon \quad \text{for all } i \in \mathbf{N}.$$

Thus, given $i \in \mathbf{N}$ there is an x_i in $X^{**}_{\cup_{n\in S_i} U_n}$ with $\|x_i\| \le 1$ such that $|P(x_i)| > \varepsilon$.

Now, by Lemma 1.9 of [6] we have that

$$\sup_{|\theta_i|=1} \left| P\left(\sum_{i=1}^{l} \theta_i x_i\right)\right|^2 \ge \sum_{i=1}^{l} |P(x_i)|^2 > l\varepsilon^2.$$

Since the sequence (x_i) has pairwise disjoint support, it follows that for $|\theta_i| = 1$, $i = 1, \dots, l$ and any $l \in \mathbf{N}$, the combination $\sum_{n=1}^{l} \theta_i x_i \in B(X^{**})$. So, for all $l \in \mathbf{N}$ we have

$$\|P\|^2 \ge \sup_{|\theta_i|=1} \left| P\left(\sum_{i=1}^{l} \theta_i x_i\right)\right|^2 \ge \sum_{i=1}^{l} |P(x_i)|^2 > l\varepsilon^2.$$

This leads to a contradiction when we let $l \to \infty$.

Finally, let Q be an arbitrary continuous polynomial with $Q(0) = 0$. Then there are homogeneous continuous polynomials P_j such that $Q = \sum_{j=0}^{m} P_j$ and, by the above, we may find inductively infinite subsets $S_j \subset S_{j-1} \subset N'$, $j = 1, \dots, m$ such that $\|P_j\|_{\cup_{n\in S_j} U_n} < \varepsilon/m$. Consequently, $\|Q\|_{\cup_{n\in S_m} U_n} < \varepsilon$. ▫

Theorem 2.2. *Let $(u_i)_{i\in\mathbf{N}}$ be a totally disjoint sequence in $B(X^{**})$ given by $u_i = \sum_{j\in U_i} \alpha_j e_j$ for each i. Then $A = \{u_i : i \in \mathbf{N}\}$ is a bounding set in X^{**}.*

Proof. Suppose the set A is not bounding in X^{**}. Then there exists an entire function $f = \sum_m P_m \in H(X^{**})$ such that f is not bounded in A. By Lemma 4.50 in [6] this implies the existence of $\delta > 0$ and strictly increasing subsequences $(m_n) \subset \mathbf{N}$, $(\gamma_n) \subset \mathbf{N}$ such that

$$|P_{\gamma_n}(u_{m_n})|^{1/\gamma_n} > \delta \quad \text{for all } n \in \mathbf{N}.$$

We will show that this inequality cannot hold by using an inductive argument.

Given the polynomial $y \in X^{**} \mapsto P_{\gamma_1}(u_{m_1} + y) - P_{\gamma_1}(u_{m_1})$, Lemma 2.1 shows that there is an infinite subset $S_1 \subset S_0 := \{m_n : n \in \mathbf{N}\}$ such that

$$|P_{\gamma_1}(u_{m_1} + y) - P_{\gamma_1}(u_{m_1})| \le \left(\frac{\delta}{\gamma_1!}\right)^{\gamma_1} \quad \text{for all } y \in B(X^{**}_{\cup_{n\in S_1} U_n}).$$

Since S_1 is an infinite subset of S_0 there is $m_j \in S_1$ with $j > 1$. Without loss of generality we may suppose that $j = 2$. Thus $u_{m_2} \in B(X^{**}_{\cup_{n\in S_1} U_n})$. Put $\theta_1 = 1$. Now, by Lemma 1.9(b) in [6], there is $\theta_2 \in \mathbf{C}$ with $|\theta_2| = 1$ such that

$$|P_{\gamma_2}(\theta_1 u_{m_1} + \theta_2 u_{m_2})|^2 \geq |P_{\gamma_2}(\theta_1 u_{m_1})|^2 + |P_{\gamma_2}(u_{m_2})|^2 \geq \delta^{2\gamma_2}.$$

The next step is to consider the polynomial $y \in X^{**} \mapsto P_{\gamma_2}(\theta_1 u_{m_1} + \theta_2 u_{m_2} + y) - P_{\gamma_2}(\theta_1 u_{m_1} + \theta_2 u_{m_2})$. Again Lemma 2.1 shows that there is an infinite subset $S_2 \subset S_1$, such that

$$|P_{\gamma_2}(\theta_1 u_{m_1} + \theta_2 u_{m_2} + y) - P_{\gamma_2}(\theta_1 u_{m_1} + \theta_2 u_{m_2})| \leq \left(\frac{\delta}{\gamma_2!}\right)^{\gamma_2}$$

for all $y \in B(X^{**}_{\cup_{n\in S_2} U_n})$. Hence for all $y \in B(X^{**}_{\cup_{n\in S_2} U_n})$,

$$\begin{aligned} |P_{\gamma_2}(\theta_1 u_{m_1} + \theta_2 u_{m_2} + y)| &\geq |P_{\gamma_2}(\theta_1 u_{m_1} + \theta_2 u_{m_2})| - \left(\frac{\delta}{\gamma_2!}\right)^{\gamma_2} \\ &\geq \delta^{\gamma_2} - \left(\frac{\delta}{\gamma_2!}\right)^{\gamma_2} \geq \left(\frac{\delta}{2}\right)^{\gamma_2}. \end{aligned}$$

Arguing by induction we obtain a decreasing sequence (S_j) of infinite subsets of $\mathbf{N}$, and a sequence $(\theta)_i$, $\theta_1 = 1$, in the unit sphere of $\mathbf{C}$ such that

$$\left|P_{\gamma_j}\left(\sum_{k=1}^{j} \theta_k u_{m_k} + y\right)\right| \geq \left(\frac{\delta}{2}\right)^{\gamma_j} \quad \text{for all } y \in B(X^{**}_{\cup_{n\in S_j} U_n}).$$

We set $a = (a_l) \in X^{**}$ defined by $a_l = \theta_k \alpha_l e_l$ if $l \in U_{m_k}$ for some $k \in \mathbf{N}$, and $a_l = 0$ otherwise. That is, a is the formal series $\sum_{k=1}^{\infty} \theta_k u_{m_k}$.

Since the sets U_k are disjoint, the vectors u_k have disjoint supports, so $a \in B(X^{**})$ and also $a - \sum_{k=1}^{j} \theta_k u_{m_k} \in B(X^{**}_{\cup_{n\in S_j} U_n})$. Then from the latter inequality we conclude

$$|P_{\gamma_j}(a)|^{1/\gamma_j} = \left|P_{\gamma_j}\left(\sum_{k=1}^{j} \theta_k u_{m_k} + \left(a - \sum_{k=1}^{j} \theta_k u_{m_k}\right)\right)\right|^{1/\gamma_j} \geq \frac{\delta}{2}.$$

Since $\{a\}$ is a compact set in X^{**}, it follows that $\lim_{j\to\infty} |P_{\gamma_j}(a)|^{1/\gamma_j} = 0$. A contradiction. □

To prove the next theorem we need the following lemma.

Lemma 2.3. *Let $(u_i)_{i\in\mathbf{N}}$ be a totally disjoint bounded sequence given by $u_i = \sum_{j=p_i+1}^{p_{i+1}} \alpha_j e_j$ for each $i \in \mathbf{N}$. Then there exists a projection $Q\colon X \to \overline{\mathrm{span}[u_i]}$.*

Proof. Let $(u'_i)_{i\in\mathbf{N}}$ be a sequence in X^* such that $\|u'_i\| = 1, u'_i u_i = 1$ for each $i \in \mathbf{N}$ and $u'_i(e_j) = 0$ for $j \notin \{p_i + 1, \dots, p_{i+1}\}$.

If we define $Q(x) = \sum_{i=1}^{\infty} u'_i(x)u_i$ for each $x = (x_j) \in X$, since $u'_i(x) = u'_i\big(\sum_{j=p_i+1}^{p_{i+1}} x_j e_j\big)$, we have that

$$|u'_i(x)| \le \Big\| \sum_{j=p_i+1}^{p_{i+1}} x_j e_j \Big\| \le 2\Lambda \Big\| \sum_{i=1}^{\infty} \sum_{j=p_i+1}^{p_{i+1}} x_j e_j \Big\| \le 2\Lambda \|x\|,$$

where Λ is the basis constant for $(e_n)_{n\in\mathbf{N}}$. Then

$$\Big\| \sum_{i=1}^{\infty} u'_i(x)u_i \Big\| = \Big\| \sum_{i=1}^{\infty} u'_i\Big(\sum_{j=p_i+1}^{p_{i+1}} x_j e_j \Big) u_i \Big\| \le \sup_i \Big| u'_i\Big(\sum_{j=p_i+1}^{p_{i+1}} x_j e_j \Big) \Big| \le 2\Lambda \|x\|.$$

So Q is a continuous projection. □

Theorem 2.4. *Every bounded subset of X is bounding in X^{**}.*

Proof. Notice that X^* is separable. So, by a result of [3], it is sufficient to prove that every weakly compact subset of X is bounding in X^{**}. So by Eberlein's theorem, it suffices to show that every weakly convergent sequence $(x_i) \subset X$ is a bounding set in X^{**}. We may assume that (x_i) is weakly null since bounding sets are also bounding after translation.

Suppose that (x_i) is not a bounding subset in X^{**}. Then, there is an entire function $f = \sum_{n\in\mathbf{N}} P_n \in H(X^{**})$ which is unbounded on the set $\{x_i : i \in \mathbf{N}\}$. Thus, by Lemma 4.50 in [6] there exist a subsequence of (x_i) (which we are going to denote in the same way), a subsequence (P_{γ_i}), and $\delta > 0$ such that

$$|P_{\gamma_i}(x_i)|^{1/\gamma_i} > \delta.$$

Since (x_i) is a weakly null non null sequence, we find by the Bessaga–Pełczynski selection principle, a subsequence (x_{k_i}) equivalent to a basic block sequence taken from (e_i). That is, there is a strictly increasing sequence $(p_i) \subset \mathbf{N}$ such that $(x_{k_i}) \approx (u_i)$, where $u_i = \sum_{j=p_i+1}^{p_{i+1}} \alpha_j e_j$ for $i \in \mathbf{N}$. Since (p_i) is strictly increasing we may assume, passing to subsequences if necessary, that (u_i) is totally disjoint, and henceforth a bounding set in X^{**}.

Since (x_{k_i}) is equivalent to (u_i), there exists an isomorphism T from $[u_i,\ i \in \mathbf{N}]$ onto $[x_{k_i},\ i \in \mathbf{N}]$ such that $T(u_i) = x_{k_i}$ for each $i \in \mathbf{N}$. Let T^{**}: $[u_i,\ i \in \mathbf{N}]^{**} \longrightarrow [x_{k_i},\ i \in \mathbf{N}]^{**}$ be the double transpose of T and let Q^{**}: $X^{**} \longrightarrow [u_i,\ i \in \mathbf{N}]^{**}$ be the double transpose of the projection defined in Lemma 2.3. Since $f \in H(X^{**})$ we have that $f \circ T^{**} \circ Q^{**} \in H(X^{**})$ and $f \circ T^{**} \circ Q^{**}|_{[u_i]}$ coincides with $f \circ T \circ Q$.

Finally, since $\{u_i : i \in \mathbf{N}\}$ is a bounding subset in X^{**} by Theorem 2.2, we get that $\lim_i |P_{\gamma_{k_i}} \circ T^{**} \circ Q^{**}(u_i)|^{1/\gamma_{k_i}} = 0$ for each $i \in \mathbf{N}$, but $\lim_i |P_{\gamma_{k_i}} \circ T^{**} \circ Q^{**}(u_i)|^{1/\gamma_{k_i}} = \lim_i |P_{\gamma_{k_i}} \circ T \circ Q(u_i)|^{1/\gamma_{k_i}} = \lim_i |P_{\gamma_{k_i}}(x_{k_i})|^{1/\gamma_{k_i}}$. So, $\lim_i |P_{\gamma_{k_i}}(x_{k_i})|^{1/\gamma_{k_i}} = 0$. This contradicts our assumption and completes the proof. □

Acknowledgments. Part of this paper was done while the third-named author was visiting the Departamento de Análisis Matemático at University of Valencia. She thanks this department for its hospitality and specially Professor Pablo Galindo.

The question we have dealt with arose some years ago after a cup of coffee with Professor Seán Dineen to whom we are warmly grateful.

We thank the referee for his valuable suggestions which led to improvements of the paper.

References

[1] Aron, R., and S. Dineen: Q-Reflexive Banach spaces. - Rocky Mountain J. Math. 27:4, 1997, 1009–1025.

[2] Bourgain, J., and F. Delbaen: A class of special $\mathscr{L}_\infty$ spaces. - Acta Math. 145, 1981, 155–176.

[3] Carrión, H.: Entire functions on Banach spaces with a separable dual. - J. Funct. Anal. 189, 2002, 496–514.

[4] Diestel, J.: Sequences and Series in Banach Spaces. - Springer, New York, 1984.

[5] Dineen, S.: Complex Analysis on Locally Convex Spaces. - North-Holland Math. Stud. 57, 1981.

[6] Dineen, S.: Complex Analysis on Infinite Dimensional Spaces. - Springer, 1999.

[7] Galindo, P., L. A. Moraes, and J. Mujica: Weak holomorphic convergence and bounding sets in Banach spaces. - Proc. Roy. Irish Acad. 98A:2, 1998, 153–157.

[8] Habala, P., P. Hajek, and V. Zizler: Introduction to Banach Spaces. - Charles University, Prague, 1997.

[9] Haydon, R.: Subspaces of the Bourgain–Delbaen space. - Studia Math. 139, 2000, 275–293.

[10] Harmand, P., D. Werner, and W. Werner: M-ideals in Banach Spaces and Banach Algebras. - Lecture Notes in Math. 1547, Springer, 1993.

[11] Josefson, B.: Bounding subsets of $l_\infty(A)$. - J. Math. Pures Appl. 57, 1978, 397–421.

[12] Josefson, B.: A Banach space containing non-trivial limited sets but no non-trivial bounding set. - Israel J. Math. 71, 1990, 321–327.

[13] Pełczyński, A., and Z. Semadeni: Spaces of continuous functions (III). - Studia Math. 18, 1959, 211–222.

[14] Saksman, E., and H.-O. Tylli: Structure of subspaces of the compact operators having the Dunford–Pettis property. - Math. Z. 232, 1999, 411–425.

[15] Schlumprecht, T.: A limited set that is not bounding. - Proc. Roy. Irish Acad. 90A, 1990, 125–129.

[15] Stegall, C.: Duals of certain spaces with the Dunford–Pettis property. - Notices Amer. Math. Soc. 19, 1972, 799.

[16] Wojtasczyk, P.: Banach Spaces for Analysts. - Cambridge Studies in Advanced Mathematics, Cambridge Univ. Press, 1991.

Received 14 October 2004