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GENERALIZED MEAN POROSITY AND DIMENSION

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Abstract. We define the class of weakly mean porous sets and prove a sharp dimension estimate for the sets in this class. By using this geometric tool, we establish an essentially sharp dimension bound for the boundaries of generalized Hölder domains and John domains.

1. Introduction

In this paper we consider the following problem. Suppose that we are given the growth condition

$$k_\Omega(x_0, x) \le \phi\left(\frac{d(x,\partial\Omega)}{d(x_0,\partial\Omega)}\right) + C_0 \tag{1}$$

on the quasihyperbolic metric k_Ω of a domain Ω, where ϕ is a decreasing function and x_0 is a fixed point in Ω. Under which conditions on the function ϕ can we prove a dimension estimate for the boundary $\partial\Omega$, and what is the sharp dimension estimate in this case?

Let us comment on the history of this problem. Recall that a domain Ω satisfying condition (1) with the function $\phi(t) = C\log(1/t)$ is called a Hölder domain (see e.g. [SS1]). It is well known that for a Hölder domain $\Omega \subset \mathbf{R}^n$ we have the estimate $\dim_H(\partial\Omega) < n$. This was proven by Smith and Stegenga [SS2] using ideas of Jones and Makarov [JM]. They established this result by applying Marcinkiewicz integrals. Later Koskela and Rohde [KR] proved a sharp extension of this result using a different technique. They introduced the concept of mean porosity and, as an application of this concept, they proved the sharp dimension estimate for the boundary of a Hölder domain. In this paper, we define a generalized version of mean porosity and, by applying this concept, we will prove an essentially sharp dimension estimate for the boundary of a domain Ω satisfying condition (1) with some decreasing function ϕ. Note that since the Hausdorff dimension of a Hölder domain can be arbitrarily close to n, one cannot

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hope for an estimate on the usual Hausdorff dimension with weaker assumptions. Instead we will use Carathéodory's construction of measure and dimension, which allows us to measure the size of a set with much finer scale. For example, if a domain $\Omega \subset \mathbf{R}^n$ satisfies condition (1) with $\phi(t) = \bigl(\log(1/t)\bigr)^s$, then we will obtain a dimension bound when $s \le n/(n-1)$, whereas the boundary can have positive volume when $s > n/(n-1)$. In particular, for $s = n/(n-1)$, we prove that $H^h(\partial\Omega) = 0$ for the gauge function $h(t) = t^n\bigl(\log(1/t)\bigr)^C$.

Notice that the geometric problem introduced above can be considered also from the viewpoint of uniform continuity of quasiconformal mappings. Indeed, if $f\colon B^n \to \mathbf{R}^n$ is a uniformly continuous quasiconformal mapping defined in the unit ball with a modulus of continuity ψ, then the image domain $f(B^n)$ satisfies condition (1) with a corresponding function ϕ (see Section 5). For conformal mappings in the plane, the sharp condition for the function ψ implying $m_2\bigl(\partial f(B^2)\bigr) = 0$ is already known by [JM]. We will prove an extended result for quasiconformal mappings in $\mathbf{R}^n$ with $n \ge 2$ and, moreover, we will prove an essentially sharp dimension estimate for $\partial f(B^n)$.

Another question, related to our main problem, concerns John domains. It is well known that the Hausdorff dimension of the boundary of a usual c-John domain is strictly smaller than n, see [T], [MV], [KR]. But what can be said about the size of the boundary of a φ-John domain (see Section 6 for definition) with some function φ that is not linear? We will give a precise answer to this question in Chapter 6.

We obtain the results above by establishing a sharp dimension bound for sets satisfying a certain porosity condition. Roughly speaking, we require that, if we consider dyadic annuli $A_k(x)$, $k = 1, 2, \ldots$, centered at some point $x \in E$, then at least half of the annuli contain λ "holes" of size α. Here λ and α are some functions depending on the scale k. Moreover, we require that these cubes or "holes" can be picked for each point x from a single disjoint collection of cubes in the complement of E that does not depend on the point x. Thus our porosity condition is not strictly pointwise (as porosity conditions are in general). Also note that our porosity condition does not permit scaling. Nevertheless, our definition of generalized mean porosity works well from the viewpoint of our applications.

The paper is organized as follows. After establishing some notation and definitions in Section 2, we introduce the porosity condition in Section 3 and prove also the basic dimension estimate. Section 4 contains an application of generalized mean porosity to the domains satisfying a quasihyperbolic growth condition. In Section 5 we prove a corresponding result for the boundaries of image domains under uniformly continuous quasiconformal mappings. We discuss the properties of φ-John domains in Section 6 and, finally, in Section 7 we construct examples of sets showing the sharpness of the dimension estimates proven in this paper.

2. Notation and definitions

Throughout this paper we denote by $\mathbf{R}^n$, $n \ge 1$, the euclidean space of dimension n. The Lebesque measure of a set $E \subset \mathbf{R}^n$ is denoted by $|E|$, although we sometimes write $m_n(E)$ to emphasize the dimension n. We denote the euclidean distance between two points $x, y \in \mathbf{R}^n$ by $d(x,y)$. We define a neighborhood of E by $E + r := \{z \in \mathbf{R}^n : d(z,E) < r\}$, where $r > 0$ and $d(z,E) = \inf\{d(z,x) : x \in E\}$.

We set $\mathbf{Z}^+ := \{1,2,3,\dots\}$. For $x \in \mathbf{R}^n$ we denote by $A_k(x)$ the set

$$A_k(x) = \{y \in \mathbf{R}^n : 2^{-k} < |x-y| < 2^{-k+1}\},$$

where $k \in \mathbf{Z}^+$. We denote by $\sharp I$ the number of elements in the set I.

For a cube $Q \subset \mathbf{R}^n$ we denote by $l(Q)$ the edge length and by $d(Q)$ the diameter of Q. The radius of a ball $B \subset \mathbf{R}^n$ is denoted by $r(B)$. We denote by pB, $p > 0$, a ball with the same center as B but with radius $pr(B)$. We write $B^n \subset \mathbf{R}^n$ for the unit ball centered at the origin with radius 1. We denote the unit sphere by S^{n-1}.

Let $\gamma \subset \mathbf{R}^n$ be an injective curve and let $x, y \in \gamma$. We denote by $\gamma(x,y)$ the subcurve of γ connecting y to x. We write $l(\gamma)$ for the euclidean length of the curve γ.

Let $f\colon]0,1[\to \mathbf{R}$. We write $\int_0 f(t)\,dt = \infty$, if $\int_0^r f(t)\,dt = \infty$ for arbitrarily small $r > 0$.

We denote by $C(\,\cdot\,)$ various positive constants that depend only on the quantities written in paranthesis.

2.1. Carathéodory's construction. Let h be a function defined for all $t \ge 0$, monotonic increasing for $t \ge 0$, positive for $t > 0$ and continuous from the right for all $t \ge 0$. Define $h(G)$ for an open nonempty set $G \in \mathbf{R}^n$ by $h(G) = h\big(d(G)\big)$, where $d(G)$ is the diameter of G in the euclidean metric, and $h(\emptyset) = 0$.

Now the set function

$$H^h(E) = \limsup_{\delta \to 0} H^h_\delta(E),$$

where

$$H^h_\delta(E) = \inf\biggl\{\sum_{i=1}^{\infty} h(B_i) : E \subset \bigcup_{i=1}^{\infty} B_i,\ d(B_i) \le \delta\biggr\},$$

is a measure on $\mathbf{R}^n$. It is called the Haudorff measure corresponding to the premeasure h, or simply h-measure. The construction of H^h is called Carathéodory's construction in [F].

Following C. A. Rogers [R, p. 78], we write

$$g \prec h,$$

and say that g corresponds to a smaller generalized dimension than h, if

$$h(t)/g(t) \to 0 \quad \text{as} \quad t \to 0.$$

Note also the following result (see [R, p. 79]). Let f, g, h be premeasures such that $f \prec g \prec h$. If $0 < H^g(E) < \infty$, then $H^h(E) = 0$ and $H^f(E) = \infty$.

3. Generalized mean porosity

We define the generalized mean porosity as follows.

Definition 3.1. Let $E \subset \mathbf{R}^n$ be a compact set. Let $\alpha\colon]0,1[\longrightarrow]0,1[$ be a continuous function such that

$$\frac{\alpha(t)}{t} \text{ is an increasing function} \tag{2}$$

and let $\lambda\colon \mathbf{Z}^+ \longrightarrow \mathbf{Z}^+$ be a function. Let $\mathscr{Q}$ be a collection of pairwise disjoint cubes $Q_i \subset \mathbf{R}^n \setminus E$. We define for each such a collection $\mathscr{Q}$ and for every $k \in \mathbf{Z}^+$ a function

$$\chi_k^{\mathscr{Q}}(x) = \begin{cases} 1, \text{ if one can find cubes } Q_i^k(x) \in \mathscr{Q},\ i = 1, \dots, \lambda(k), \\ \text{such that } Q_i^k(x) \subset A_k(x) \text{ and } l\big(Q_i^k(x)\big) \ge \alpha(2^{-k}) \text{ for all } i; \\ 0, \text{ otherwise.} \end{cases}$$

Let

$$S_j^{\mathscr{Q}}(x) = \sum_{k=1}^{j} \chi_k^{\mathscr{Q}}(x).$$

We say that a set E is weakly mean porous with parameters (α, λ), if there exists a collection $\mathscr{Q}$ as above and an integer $j_0 \in \mathbf{Z}^+$ such that

$$\frac{S_j^{\mathscr{Q}}(x)}{j} > \frac{1}{2} \tag{3}$$

for all $x \in E$ and for all $j \ge j_0$.

There are two principal differences between generalized mean porosity and other porosities. Firstly, it is not a pointwise property, since the collection $\mathscr{Q}$ does not depend on the point x. Secondly, it does not permit scaling, since the parameters α and λ depend on the scale. In particular, $\alpha(t)/t$ is not necessarily a constant (as it is in the usual mean porosity). For example, let us consider a set $E \subset \mathbf{R}^2$ which is weakly mean porous with parameters $\alpha(t) = ct/\log(1/t)$ for small t and $\lambda(k) = k$. Fix a point $x \in E$. Roughly speaking, half of the annuli $A_k(x)$, $k = 1, 2, \dots$, now contain k disjoint "holes" $Q_i^k(x) \in \mathscr{Q}$ with side lengths at least $c2^{-k}/k$. In this section we will prove that this implies $m_2(E) = 0$ and even $H^h(E) < \infty$ for the gauge function $h(t) = t^2\big(\log(1/t)\big)^C$ with some $C > 0$.

In Definition 3.1 the property (2) can be described as follows. We require, that as one reduces the scale, the size of the "holes" does not increase in proportion to the scale. Note that when $\alpha(t)/t$ is a constant and $\lambda(k) \equiv 1$ our definition is equivalent with the definition of mean porosity in [KR]. Indeed, for a mean porous set E, we can take $\mathscr{Q}$ to be the collection of the Whitney decompositions of all cubes in the Whitney decomposition of $\mathbf{R}^n \setminus E$. The fact that this collection satisfies condition (3) is shown in the proof of [KR, Theorem 2.1].

The parameter $\lambda(k)$ controls the number of "holes" in each annulus. It is important for our applications that we use a general λ that allows us to use the contribution from several "small" holes in a single set $A_k(x) \setminus E$.

We could also define the porosity condition of Definition 3.1 in a pointwise way (i.e. allow the collection $\mathscr{Q}$ to depend on the point x), as porosity conditions are defined in general. Then, however, we could prove a dimension estimate for the set E only in the case that $\lambda(k)$ is bounded from above. We do not know whether it is possible to prove a sharp dimension bound for sets satisfying such a pointwise porosity condition with an unbounded parameter λ. However, in our applications we will find the collection $\mathscr{Q}$ independently of x, and thus Definition 3.1 works well for us.

The constant $\frac{1}{2}$ in condition (3) plays a technical role only and could be replaced with any positive constant without essential effect on the dimension estimates. In fact, if we replace it with a constant $\kappa > 0$, then the constant $C(n)$ in Corollary 3.5 is replaced with $\kappa C(n)$. Note also that our porosity condition is uniform in the sense that j_0 is independent of x.

In order to prove a dimension estimate for weakly mean porous sets we need the following well-known consequence of the Hardy–Littlewood maximal theorem, see [Bo]. Here χ_E denotes the characteristic function of the set E.

Lemma 3.2. *Let $\mathscr{B}$ be a collection of balls $B \subset \mathbf{R}^n$ and let $p \geq 1$. Then*

$$\int_{\mathbf{R}^n} \Big(\sum_{B \in \mathscr{B}} \chi_{pB}(x) \Big)^k \, dx \leq (C_1 k p^n)^k \int_{\mathbf{R}^n} \Big(\sum_{B \in \mathscr{B}} \chi_B(x) \Big)^k \, dx$$

for all $k \geq 1$, where $C_1 = C_1(n)$.

Next we introduce the main result of this section. It is an estimate on the size of weakly mean porous sets.

Theorem 3.3. *There exists a positive constant $C(n)$ such that whenever $E \subset \mathbf{R}^n$ is a weakly mean porous set with parameters (α, λ), then $H^h(E) < \infty$ for each premeasure h, which satisfies*

$$h(2^{-j}) \leq M 2^{-jn} \exp\Big(C(n) \inf_{I_j} \Big\{ \sum_{k \in I_j} \frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n} \Big\} \Big) \tag{4}$$

for all $j > j_0$ with some positive constant M. Here the infimum is taken over all index sets I_j that satisfy

$$I_j = \bigcup_{i=1}^{j} I_i \text{ with } I_i \subset I_{i+1} \subset \{1, 2, \dots, i+1\} \text{ so that } \frac{\sharp I_i}{i} > \frac{1}{2} \text{ for all } j_0 \leq i \leq j.$$

Proof. Let $Q_0 = \{(x_1, \dots, x_n) : -1 \leq x_j \leq 1\}$. We can assume that E is a subset of the cube Q_0. If this is not the case, we can subdivide E into a finite number of compact sets E_j so that each set fits into the cube Q_0. We can also assume that $Q \subset E+1$ for all $Q \in \mathscr{Q}$.

Let $j > j_0$, and for each $k \le j$ let $N(k)$ be the smallest integer such that

$$N(k) \ge \frac{2^{-j}\alpha(2^{-k})}{2^{-k}\alpha(2^{-j})}.$$

By property (2), $N(k) \ge N(k+1)$. Now we define $\mathscr{Q}_j$ by subdividing the cubes of the collection $\mathscr{Q}$ in the following way: If $Q \in \mathscr{Q}$ and there is $1 < k \le j$ such that $\alpha(2^{-k}) \le l(Q) < \alpha(2^{-k+1})$, then each edge of the cube Q is divided into $N(k)$ parts. As for a cube Q with $l(Q) \ge \alpha(2^{-1})$, divide each edge into $N(1)$ parts. Hence Q is subdivided into $N(k)^n$ cubes that have edge lengths of at least $2^{-k}\alpha(2^{-j})/2\cdot 2^{-j}$. Let $\mathscr{Q}_j$ be the collection of cubes acquired in this manner from the cubes $Q \subset \mathscr{Q}$ with $l(Q) \ge \alpha(2^{-j})$.

Denote the largest ball $B \subset Q$ by $B(Q)$. Let

$$\mathscr{B}_j = \{B(Q) : Q \in \mathscr{Q}_j\}.$$

Let $x \in E + 2^{-j}$. We choose $x' \in E$ such that $d(x,x') < 2^{-j}$. Let $k < j$ satisfy $\chi_k(x') = 1$. By Definition 3.1 there are cubes $Q_i \in \mathscr{Q}$, $i = 1,\dots,\lambda(k)$, in the annulus $A_k(x')$ such that $\alpha(2^{-k}) \le l(Q_i)$. Hence from the annulus $A_k(x')$ we find disjoint balls $B_i \in \mathscr{B}_j$, $i = 1,\dots,\lambda(k)N(k)^n$, such that

$$r(B_i) \ge \frac{1}{4}\frac{2^{-k}\alpha(2^{-j})}{2^{-j}}.$$

Let I_j consist of all the indices $k \le j$ for which $\chi_k(x') = 1$. Then, by Definition 3.1, the index set I_j satisfies

$$I_j = \bigcup_{i=1}^{j} I_i \text{ with } I_i \subset I_{i+1} \subset \{1,2,\dots,i+1\} \text{ so that } \frac{\sharp I_i}{i} > \frac{1}{2} \text{ for all } j_0 \le i \le j,$$

where the number of indices in the set I_i is denoted by $\sharp I_i$.

By enlarging the balls $B \in \mathscr{B}_j$ we have that

$$\begin{aligned}\sum_{B\in\mathscr{B}_j} \chi_{C_1(n)\frac{2^{-j}}{\alpha(2^{-j})}B}(x) &\ge \inf_{I_j}\Big\{\sum_{k\in I_j}\lambda(k)N(k)^n\Big\}\\ &\ge \left(\frac{2^{-j}}{\alpha(2^{-j})}\right)^n \inf_{I_j}\Big\{\sum_{k\in I_j}\frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n}\Big\},\end{aligned}$$

when the constant $C_1(n)$ is large enough. Hence we have the estimate

$$\frac{1}{G_j}\left(\frac{\alpha(2^{-j})}{2^{-j}}\right)^n \sum_{B\in\mathscr{B}_j}\chi_{C_1(n)\frac{2^{-j}}{\alpha(2^{-j})}B}(x) \ge 1, \tag{5}$$

where

$$G_j = \inf_{I_j} \left\{ \sum_{k\in I_j} \frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n} \right\}.$$

Next we use inequality (5) to estimate the Lebesgue measure of a neighborhood of E. For all $0 < t < 1$ and $Q > 0$ we have that

$$\begin{aligned}
|E + 2^{-j}| \exp\left(\frac{G_j}{Q}\right) &\le \int_{E+2^{-j}} \sum_{i\ge 0} \frac{1}{i!} \frac{G_j^i}{Q^i}\,dx \\
&\le |E+1| \Bigg(\sum_{0\le i<1/t} \frac{1}{i!} \frac{G_j^i}{Q^i} \Bigg) \\
&\quad + \sum_{i\ge 1/t} \frac{1}{i!} \frac{G_j^i}{Q^i} \int_{\mathbf{R}^n} \Bigg(\frac{1}{G_j} \left(\frac{\alpha(2^{-j})}{2^{-j}} \right)^n \\
&\quad \times \sum_{B\in\mathscr{B}_j} \chi_{C_1(n)\frac{2^{-j}}{\alpha(2^{-j})}B}(x) \Bigg)^{ti} dx.
\end{aligned}$$

By combining Lemma 3.2, inequality $i^i \le e^i i!$ and Hölder's inequality we thus deduce that

$$\begin{aligned}
|E + 2^{-j}| \exp\left(\frac{G_j}{Q}\right) &\le |E+1| \Bigg(\sum_{0\le i<1/t} \frac{1}{i!} \frac{G_j^i}{Q^i} \Bigg) \\
&\quad + \sum_{1/t\le i} \frac{1}{i!} \frac{G_j^{(1-t)i}}{Q^i} \Bigg(C_2(n) t i C_1(n)^n \left(\frac{\alpha(2^{-j})}{2^{-j}} \right)^n \\
&\quad \times \left(\frac{2^{-j}}{\alpha(2^{-j})} \right)^n \Bigg)^{ti} \int_{\mathbf{R}^n} \Bigg(\sum_{B\in\mathscr{B}_j} \chi_B(x) \Bigg)^{ti} \\
&\le |E+1| \Bigg(\sum_{0\le i<1/t} \frac{1}{i!} \frac{G_j^i}{Q^i} + \sum_{1/t\le i} \frac{G_j^{(1-t)i} \big(C_3(n) t i\big)^{ti}}{Q^i i!} \Bigg) \\
&\le |E+1| \Bigg(\sum_{0\le i<1/t} \frac{1}{i!} \frac{G_j^i}{Q^i} + \sum_{1/t\le i} \frac{G_j^{(1-t)i} (i! e^i)^t \big(C_3(n) t\big)^{ti}}{Q^i i!} \Bigg) \\
&\le |E+1| \Bigg(\sum_{0\le i<1/t} \frac{1}{i!} \frac{G_j^i}{Q^i} + \sum_{1/t\le i} \frac{t^{ti} G_j^{(1-t)i} \big(C_3(n) e\big)^{ti}}{Q^i (i!)^{1-t}} \Bigg)
\end{aligned}$$

$$\begin{aligned}
&\le |E+1|\biggl(\sum_{0\le i<1/t}\frac{1}{i!}\frac{G_j^i}{Q^i}+\biggl(\sum_{1/t\le i}t^{ti/t}\biggr)^t\\
&\qquad\times\biggl(\sum_{1/t\le i}\frac{G_j^i\bigl(C_3(n)e\bigr)^{ti/(1-t)}}{Q^{i/(1-t)}i!}\biggr)^{1-t}\biggr)\\
&\le |E+1|\biggl(\sum_{0\le i<1/t}\frac{1}{i!}\frac{G_j^i}{Q^i}+\biggl(\frac{1}{1-t}\biggr)^t\\
&\qquad\times\exp\biggl(G_j\bigl(C_3(n)e\bigr)^{t/(1-t)}\biggl(\frac{1-t}{Q^{1/(1-t)}}\biggr)\biggr)\biggr)\\
&\le M_0(n)\exp\biggl(\frac{G_j}{2Q}\biggr),
\end{aligned}$$

when we choose $t=\frac{1}{2}$, a constant $M_0(n)$ big enough and a constant $Q=C_3(n)e$. By the previous calculations we arrive at

$$|E+2^{-j}|\exp\biggl(\frac{G_j}{2Q}\biggr)\le M_0(n),$$

and hence

$$|E+2^{-j}|\exp\bigl(C(n)G_j\bigr)\le M_0(n), \tag{6}$$

where $C(n)=1/2C_3(n)e$.

The desired dimension estimate follows from inequality (6) by a standard calculation using the Besicovich covering theorem. We show this in the following.

Let $\mathscr{A}$ be the collection of all the balls of radii 2^{-j} with centers in the set E. By the Besicovich covering theorem we can choose balls $B_i\in\mathscr{A}$, $i=1,\dots,m_j$, such that $E\subset\bigcup_{i=1}^{m_j}B_i$ and

$$\sum_{i=1}^{m_j}\chi_{B_i}(x)<P(n) \tag{7}$$

for all $x\in\mathbf{R}^n$. By (6) and (7) we have that

$$\frac{M_0(n)}{\exp\bigl(C(n)G_j\bigr)}\ge|E+2^{-j}|\ge m_j\Omega_n(2^{-j})^n\frac{1}{P(n)},$$

and hence

$$m_j\le\frac{M_0(n)P(n)2^{jn}}{\Omega_n\exp\bigl(C(n)G_j\bigr)}.$$

Let h be a premeasure satisfying (4). Then we obtain the estimate

$$\begin{aligned}H^h(E) &\le \limsup_{j\to\infty}\{m_j h(2^{-j})\}\\ &\le \limsup_{j\to\infty}\{m_j M 2^{-jn}\exp\bigl(C(n)G_j\bigr)\}\\ &\le \limsup_{j\to\infty}\{R(n)M2^{jn}\exp\bigl(-C(n)G_j\bigr)2^{-jn}\exp\bigl(C(n)G_j\bigr)\}\\ &\le R(n)M < \infty. \ \square\end{aligned}$$

Let us make some remarks on Theorem 3.3. Notice that $\lambda(k)\alpha(2^{-k})^n$ estimates the total volume of the cubes or "holes" $Q_i^k(x) \in \mathcal{Q}$, $i = 1,\dots,\lambda(k)$, in the annulus $A_k(x)$, where k satisfies $\chi_k^{\mathcal{Q}}(x) = 1$. The index set I_j consists of all indices $k \le j$ for which $\chi_k^{\mathcal{Q}}(x) = 1$. Hence I_j depends on the point x. That is why the infimum is taken over all possible index sets. The condition given for I_j is implied directly by Definition 3.1.

Note the following special cases of Theorem 3.3. If we have for arbitrarily large j that

$$G_j = \inf_{I_j}\Bigl\{\sum_{k\in I_j}\frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n}\Bigr\} \ge Cj$$

with some constant $C > 0$, then it follows from Theorem 3.3 that $\dim_H(E) < n$. Note that this happens, for example, if we have the parameters $\alpha(t) = ct$ and $\lambda(k) \equiv 1$, in other words, if the set E is mean porous.

If $G_j \to \infty$ as $j \to \infty$, then $m_n(E) = 0$, and Theorem 3.3 will also give us a dimension estimate with the gauge function h. However, if G_j is bounded, i.e. there is $M \in \mathbf{R}$ such that $G_j < M$ for all j, then Theorem 3.3 does not give us a dimension estimate. Indeed, in this case the set E can have positive Lebesgue measure, see Section 7.1.

Let us also point out that, in fact, we proved more than what we claim in Theorem 3.3. Indeed, we proved inequality (6), which is a stronger condition for the set E than the claimed dimension estimate.

In the next remark we show that in certain cases the index set I_j of Theorem 3.3 can be given explicitly.

Remark 3.4. If it holds for the parameters in Theorem 3.3 that

$$p(k) := \frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n} \tag{8}$$

is increasing as a function of k, then (for even j)

$$\inf_{I_j}\Bigl\{\sum_{k\in I_j}\frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n}\Bigr\} = \sum_{k=1}^{j/2}\frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n} \ge Cj$$

with some positive constant C.

If however $p(k)$ is decreasing as a function of k, then (for even j_0)

$$\inf_{I_j}\left\{\sum_{k\in I_j}\frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n}\right\}=\sum_{k\in J_j}\frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n},$$

where

$$J_j=\left\{\frac{j_0}{2}+1,\frac{j_0}{2}+2,\dots,j_0\right\}\cup\bigl\{i\in\{j_0+1,\dots,j\}\text{ such that } i \text{ is odd}\bigr\}.$$

Moreover, for all $j>j_0$ we have that

$$\sum_{k\in J_j}\frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n}\geq\frac{1}{2}\sum_{k=j_0}^{j}\frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n}.$$

Corollary 3.5. *Let $E\subset\mathbf{R}^n$ be a weakly mean porous set with parameters (α,λ) such that $p(k)$ (defined by (8)) is a decreasing function of k and*

$$\sum_{k=j_0}^{\infty}\frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n}=\infty.$$

Then $m_n(E)=0$ and there exists a positive constant $C(n)$ such that $H^h(E)<\infty$ for each premeasure h, which satisfies

$$h(2^{-j})\leq M2^{-jn}\exp\left(C(n)\sum_{k=j_0}^{j}\frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n}\right)$$

for all $j>j_0$ with some positive constant M.

Proof. The claim follows by combining Theorem 3.3 and Remark 3.4. □

Note that this corollary is essentially sharp by an example in Section 7.1.

4. A quasihyperbolic growth condition

Let $\Omega\subset\mathbf{R}^n$ be a domain. We recall that the quasihyperbolic distance between two points $x_1,x_2\in\Omega$ is defined as

$$k_\Omega(x_1,x_2)=\inf_\gamma\int_\gamma\frac{ds}{d(x,\partial\Omega)}$$

where the infimum is taken over all rectifiable arcs joining x_1 to x_2 in Ω.

Definition 4.1. Let $\phi\colon\,]0,1] \longrightarrow]0,\infty[$ be a continuous and decreasing function. We say that a bounded domain $\Omega \subset \mathbf{R}^n$ satisfies a quasihyperbolic growth condition with a function ϕ, if there is a point $x_0 \in \Omega$ and a constant C_0 such that

$$k_\Omega(x_0, x) \le \phi\left(\frac{d(x, \partial\Omega)}{d(x_0, \partial\Omega)}\right) + C_0 \tag{9}$$

for all $x \in \Omega$.

Note that for a bounded domain we can always choose the point x_0 so that $d(x,\partial\Omega)/d(x_0,\partial\Omega) \le 1$ for all $x \in \Omega$, and hence the domain of ϕ can be assumed to be $]0,1]$. Recall that if a domain Ω satisfies condition (9) with a function $\phi(t) = C\log(1/t)$, then Ω is called a Hölder domain (see [SS1]). Thus we can say that domains defined in Definition 4.1 are *generalized Hölder domains*. It is well known that the Hausdorff dimension of the boundary of a Hölder domain is strictly smaller than n. This is shown in [JM], [SS2] and [KR]. In this section we prove a corresponding dimension estimate for domains satisfying (9) with a function ϕ that satisfies certain conditions formulated below. To indicate how fast decreasing functions ϕ allow for a generalized dimension estimate, let us already point out that, for $\phi(t) = \big(\log(1/t)\big)^s$ we will obtain a dimension bound when $s \le n/(n-1)$, whereas the boundary can have positive volume when $s > n/(n-1)$.

Definition 4.2. We say that a decreasing function $\phi\colon\,]0,1] \to]0,\infty[$ is of logarithmic type, if there exist positive constants $t_0 < 1$ and β such that ϕ satisfies the following conditions for all $t \le t_0$:

(10) $\phi(t)$ is differentiable and $-\phi'(t)t$ is a decreasing function;

$$\phi(t) \le \beta\phi\big(\sqrt{t}\,\big). \tag{11}$$

Note that, for example, a function of the form

$$\phi(t) = \begin{cases} C\left(\log\dfrac{1}{t}\right)^{s_1}\left(\log\log\dfrac{1}{t}\right)^{s_2}\cdots\left(\log^{(m)}\dfrac{1}{t}\right)^{s_m} + C, & t < a_m; \\ C, & t \ge a_m, \end{cases}$$

where $C > 0$, $m \in \mathbf{Z}^+$, $a_m = 1/\exp^{(m-1)}(e)$, $s_1 \ge 1$, $s_2, \dots, s_m \ge 0$, is of logarithmic type.

Lemma 4.3. *Let ϕ be a function of logarithmic type. Then*

$$\phi(ab) \le \beta\big(\phi(a) + \phi(b)\big)$$

for all $a, b \in]0,1[$ for which $ab \le t_0$.

Proof. Either $a \le \sqrt{ab}$ or $b \le \sqrt{ab}$, and hence we obtain $\beta\big(\phi(a)+\phi(b)\big) \ge \beta \max\{\phi(a),\phi(b)\} \ge \beta\phi\big(\sqrt{ab}\big) \ge \phi(ab)$. □

Lemma 4.4. *Let ϕ be a function of logarithmic type. Then there is $t_1 \in]0,1[$ such that the inequality*

$$\beta\phi(t^{k+1}) \le 2^{-k}\left(\frac{1}{t}\right)^k$$

holds for all $t < t_1$ and every $k \in \mathbf{Z}^+$.

Proof. Let $t_0 < 1$ be as in Definition 4.2. We show first that there is $\tilde t_1$ such that

$$\phi(t) \le \frac{1}{t} \tag{12}$$

for all $t < \tilde t_1$. Suppose that (12) is false. Then for arbitrarily large $j \in \mathbf{Z}^+$ there is $t_0^2 \le r_j \le t_0$ such that $\phi(r_j^{2^j}) > (1/r_j)^{2^j} \ge (1/t_0)^{2^j}$. By iterating condition (11), we obtain $\phi(r_j^{2^j}) \le \beta^j\phi(r_j) \le \beta^{j+1}\phi(t_0)$, and hence $(1/t_0)^{2^j} < \beta^{j+1}\phi(t_0)$. This is a contradiction with large j, and thus property (12) is proved.

Let $k \in \mathbf{Z}^+$. Applying property (11) twice, we have that

$$\beta\phi(t^{k+1}) \le \beta^3\phi(t^{(k+1)/4})$$

for all $t < t_0^2$. Then, by property (12), we obtain

$$\beta^3\phi(t^{(k+1)/4}) \le \beta^3\left(\frac{1}{t}\right)^{(k+1)/4}$$

for all $t < \tilde t_1^2$. A simple calculation yields

$$\beta^3\left(\frac{1}{t}\right)^{(k+1)/4} \le 2^{-k}\left(\frac{1}{t}\right)^k \quad \text{for all } t < \frac{1}{4\beta^6}.$$

This proves the claimed inequality for all $t < t_1 = \min\{t_0^2, \tilde t_1^2, 1/4\beta^6\}$. □

The next theorem extends a result by Smith and Stegenga in [SS1, Theorem 3] given for Hölder domains. For an intermediate result see [KOT, Lemma 4.6].

Theorem 4.5. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain that satisfies the quasihyperbolic growth condition with the function ϕ of logarithmic type. Then there is a constant $C_\Omega < \infty$ such that*

$$k_\Omega(x,x_0) \le \beta\phi\left(\frac{l\big(\gamma(x,x_1)\big)}{d(x_0,\partial\Omega)}\right) + C_\Omega \tag{13}$$

for all $x_1 \in \Omega$, where γ is a quasihyperbolic geodesic connecting x_0 to x_1, and $x \in \gamma$.

Proof. Assume that (13) is false. Then for each constant C_Ω there is a point x_1, a geodesic γ connecting x_0 to x_1, and a point $y_0 \in \gamma$ for which

$$\beta\phi\left(\frac{l\bigl(\gamma(y_0,x_1)\bigr)}{d(x_0,\partial\Omega)}\right) + C_\Omega < k_\Omega(x_0,y_0). \tag{14}$$

Let $L = l\bigl(\gamma(y_0,x_1)\bigr)$. Define points $y_k \in \gamma(y_{k-1},x_1)$ recursively so that $l\bigl(\gamma(y_{k-1},y_k)\bigr) = 2^{-k}L$ for all $k \in \mathbf{Z}^+$. Let

$$\delta_k = \sup\{d(x,\partial\Omega) : x \in \gamma(y_k,x_1)\}.$$

We can choose the constant C_Ω so large that $\delta_0/d(x_0,\partial\Omega) \le t_0$. Then, by combining (9), (14) and Lemma 4.3, we obtain the following chain of inequalities for all $x \in \gamma(y_0,x_1)$:

$$\begin{aligned}\beta\phi\left(\frac{L}{d(x_0,\partial\Omega)}\right) + C_\Omega &< k_\Omega(x_0,y_0)\\ &\le k_\Omega(x_0,x) \le \phi\left(\frac{d(x,\partial\Omega)}{d(x_0,\partial\Omega)}\right) + C_0\\ &\le \beta\phi\left(\frac{d(x,\partial\Omega)}{L}\right) + \beta\phi\left(\frac{L}{d(x_0,\partial\Omega)}\right) + C_0.\end{aligned}$$

Hence

$$C_\Omega - C_0 \le \beta\phi\left(\frac{\delta_0}{L}\right).$$

Now we can choose the constant C_Ω so large that $C_\Omega \ge C_0$ and the ratio δ_0/L is so small that, by Lemma 4.4,

$$\beta\phi\left(\left(\frac{\delta_0}{L}\right)^{k+1}\right) \le 2^{-k}\left(\frac{L}{\delta_0}\right)^k \tag{15}$$

for all $k \in \mathbf{Z}^+$.

We show by induction that $\delta_{k-1}/L \le (\delta_0/L)^k$ for all $k \in \mathbf{Z}^+$. This is trivially true if $k = 1$, so assume that it is true for some $k \ge 1$. By combining the induction assumption, Lemma 4.3 and the inequalities (14) and (15), we obtain for all $x \in \gamma(y_k,x_1)$ that

$$\begin{aligned}\beta\phi\left(\frac{L}{d(x_0,\partial\Omega)}\right) + C_\Omega + \beta\phi\left(\left(\frac{\delta_0}{L}\right)^{k+1}\right) &\le k_\Omega(x_0,y_0) + 2^{-k}\left(\frac{L}{\delta_0}\right)^k\\ &\le k_\Omega(x_0,y_0) + 2^{-k}\frac{L}{\delta_{k-1}}\\ &\le k_\Omega(x_0,y_0) + k_\Omega(y_{k-1},y_k) \le k_\Omega(x_0,x)\\ &\le \phi\left(\frac{d(x,\partial\Omega)}{d(x_0,\partial\Omega)}\right) + C_0\\ &\le \beta\phi\left(\frac{d(x,\partial\Omega)}{L}\right) + \beta\phi\left(\frac{L}{d(x_0,\partial\Omega)}\right) + C_0.\end{aligned}$$

Now we have that

$$\beta\phi\big((\delta_0/L)^{k+1}\big) + C_\Omega - C_0 \le \beta\phi\left(\frac{\delta_k}{L}\right)$$

which proves the induction hypothesis.

Since $\delta_0/L < 1$ and the inequality

$$0 < d(x_1, \partial\Omega) \le \delta_k \le L\left(\frac{\delta_0}{L}\right)^{k+1}$$

holds for all $k \in \mathbf{Z}^+$, we have a contradiction which proves the theorem. □

For the proof of the main theorem of this section we need one more lemma concerning the geometric properties of the Whitney decomposition. For the exact construction of this decomposition we refer the reader to [S].

Lemma 4.6. *Let $Q_0 \subset \mathbf{R}^n$ be a cube that has sides parallel to the coordinate planes, and let the edge length of Q_0 be 2^{-m}. Let $\widetilde{Q} \subset Q_0$ be a cube sharing a part of a face with Q_0. Let $l(\widetilde{Q}) = c2^{-m}$ with $c < 1$. Let $\mathscr{W}$ be a Whitney decomposition of Q_0. Then, there is a cube $Q \in \mathscr{W}$ for which $Q \subset \widetilde{Q}$ and $l(Q) \ge c2^{-m}/D(n)$. Moreover, there are at least $2^{i(n-1)}$ cubes $Q_j \in \mathscr{W}$ for which $Q_j \subset \widetilde{Q}$ and $l(Q_j) \ge c2^{-m-i}/D(n)$. Here $D(n) = 2 + 6\sqrt{n}$.*

Proof. Recall that each Whitney cube $Q_j^k \in \mathscr{W}$ has sides parallel to the coordinate planes and the edge length of Q_j^k is 2^{-k}. The collection $\{Q_j^k : j = 1, \dots, N_k\}$ is called the k^{th} generation of the cubes. It follows from the construction of the Whitney decomposition that the inequality

$$2^{-k}\sqrt{n} \le d(Q_j^k, \partial Q_0) < 3 \cdot 2^{-k}\sqrt{n} \tag{16}$$

holds for each cube in the k^{th} generation, see [G]. Thus we see that there must be a cube $Q \in \mathscr{W}$ such that $Q \subset \widetilde{Q}$ and

$$l(Q) \ge \frac{c2^{-m}}{2 + 6\sqrt{n}}$$

as otherwise the inequality (16) would fail for some cube near the center of $\widetilde{Q}$.

To prove the second part of the lemma let $i \in \mathbf{Z}^+$ and subdivide the cube $\widetilde{Q}$ into 2^{in} cubes with equal side lengths of at least $c2^{-m-i}$. Of these subcubes at least $2^{i(n-1)}$ cubes share a face with the cube Q_0 and, by inequality (16), from each subcube we find a cube $Q_j \in \mathscr{W}$ such that $l(Q_j) \ge c2^{-m-i}/D(n)$. □

We recall also the following property of the Whitney decomposition. Let $\mathscr{W}$ be the Whitney decomposition of a domain $\Omega \subset \mathbf{R}^n$. Pick a cube $Q_0 \in \mathscr{W}$, and set $q(Q_0) = 0$. For any two adjacent (i.e. sharing at least a part of a face) Whitney cubes, join their centers by an interval, and let $q(Q)$ be the number of intervals in the shortest chain joining the centers of Q_0 and Q. We can remove the redundant intervals so that the resulting collection of intervals is a tree. We denote the set of cubes connecting Q to Q_0 by $\operatorname{chain}(Q_0, Q)$, and the number of cubes in $\operatorname{chain}(Q_0, Q)$ by $\sharp\operatorname{chain}(Q_0, Q)$. Note that now $q(Q) + 1 = \sharp\operatorname{chain}(Q_0, Q) \leq C k_\Omega(z_0, z)$ for any $z_0 \in Q_0$ and $z \in Q$ for which $k_\Omega(z_0, z) >$ constant.

The next theorem extends the result given for Hölder domains in [KR, Theorem 5.1]. We show that the boundary of a generalized Hölder domain is weakly mean porous with appropriate parameters.

Theorem 4.7. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain that satisfies the quasihyperbolic growth condition with the function ϕ of logarithmic type. Then there is a constant $c > 0$ such that the boundary of the domain Ω is weakly mean porous with parameters $C_1(n)\alpha(t)$ and $C_2(n)\lambda(k)$, where (for small t) $\alpha(t) = c/-\phi'(t)$ and $\lambda(k)$ is the smallest integer such that $\lambda(k) \geq 2^{-k}/\alpha(2^{-k})$.*

Proof. Let $j_0 \in \mathbf{Z}^+$ such that $2^{-j_0+1} \leq d(x_0, \partial\Omega)$. Let $j > 2j_0$ and let $x \in \partial\Omega$. Choose a point

$$y \in B(x, 2^{-j-1}) \cap \Omega,$$

and let γ be a quasihyperbolic geodesic connecting y to x_0. Choose $w \in \gamma$ such that

$$l\big(\gamma(w, y)\big) = 2^{-j-1}.$$

Then $w \in B(x, 2^{-j})$. Moreover, Lemma 4.5 implies the estimate

$$k_\Omega(w, x_0) \leq \beta\phi\left(\frac{2^{-j-1}}{d(x_0, \partial\Omega)}\right) + C_\Omega. \tag{17}$$

Define for each $k \geq j_0$ a function

$$\chi_k(x) = \begin{cases} 1, & \text{if } \displaystyle\int_{A_k(x)\cap\gamma} \frac{dt}{d(t, \partial\Omega)} \leq \frac{2^{-k}}{\alpha(2^{-k})}; \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$S_j(x) = \sum_{k=j_0}^{j} \chi_k(x).$$

We prove first that

$$\frac{S_j(x)}{j} > \frac{1}{2} \tag{18}$$

for all sufficiently large j.

For those $A_k(x)$, $j_0 \le k \le j$, for which $\chi_k(x) = 0$, we have that

$$\int_{A_k(x)\cap\gamma} \frac{dt}{d(t,\partial\Omega)} > \frac{2^{-k}}{\alpha(2^{-k})}. \tag{19}$$

Suppose that the assertion $S_j(x)/j > \frac{1}{2}$ fails for some large j. Then by (19) we have that

$$k_\Omega(w,x_0) > \sum_{k=j_0}^{j} |\chi_k(x) - 1| \frac{2^{-k}}{\alpha(2^{-k})},$$

which is by property (10) at least

$$\sum_{k=j_0}^{j/2} \frac{-\phi'(2^{-k})2^{-k}}{c} \ge \frac{1}{c}\big(\phi(2^{-j/2}) - \phi(2^{-j_0})\big).$$

This number is greater than $\beta\phi\big(2^{-j-1}/d(x_0,\partial\Omega)\big) + C_\Omega$, when we choose j big enough and $c < 1/\beta^4$. Hence we have a contradiction with inequality (17), which proves (18).

Next we define a collection $\mathscr{Q}$ of disjoint cubes in Ω in the following way. Let $\mathscr{W}$ be a Whitney decomposition of the domain Ω. Then let $\mathscr{Q}$ consist of all the cubes in the Whitney decompositions of the cubes $Q \in \mathscr{W}$. We show that

$$\chi_k^{\mathscr{Q}}(x) \ge \chi_k(x) \tag{20}$$

for all $k \ge j_0$ with parameters $C_1(n)\alpha(2^{-k})$ and $C_2(n)\lambda(k)$.

Consider $k \in \mathbf{Z}^+$ such that $\chi_k(x) = 1$. Then

$$\int_{A_k(x)\cap\gamma} \frac{dt}{d(t,\partial\Omega)} \le \frac{2^{-k}}{\alpha(2^{-k})}.$$

Choose $y \in \gamma \cap S^{n-1}(x,2^{-k+1})$ and $z \in \gamma \cap S^{n-1}(x,2^{-k})$ such that $\gamma(y,z) \subset A_k(x)$. Let $Q_y, Q_z \in \mathscr{W}$ such that $y \in Q_y$ and $z \in Q_z$. Now

$$\sharp\mathrm{chain}(Q_y,Q_z) \le \frac{c_0 2^{-k}}{\alpha(2^{-k})} \tag{21}$$

with some constant c_0 depending only on n.

For each $Q_i \in \mathrm{chain}(Q_y,Q_z)$, let $\widetilde{Q}_i \subset \mathbf{R}^n$ be the largest cube such that it has sides parallel to the coordinate planes and $\widetilde{Q}_i \subset Q_i \cap \overline{A_k(x)}$. Now $\widetilde{Q}_i$ shares at least one part of a face with Q_i. Moreover

$$\sum_{i=1}^{\sharp\mathrm{chain}(Q_y,Q_z)} d(\widetilde{Q}_i) \ge 2^{-k}. \tag{22}$$

Combining (21) and (22) we have that

$$\sum_{i:d(\widetilde{Q}_i)\geq \alpha(2^{-k})/2c_0} d(\widetilde{Q}_i) \geq \frac{2^{-k}}{2}. \tag{23}$$

Applying Lemma 4.6 and inequality (23) we see that from these cubes $\widetilde{Q}_i$, for which $d(\widetilde{Q}_i) \geq \alpha(2^{-k})/2c_0$, we find at least $c_0 2^{-k}/\alpha(2^{-k})$ cubes $Q \in \mathscr{Q}$ such that $l(Q) \geq \alpha(2^{-k})/2c_0 D(n)\sqrt{n}$. Thus we have proven (20) with constants $C_1(n) = 1/2c_0 D(n)\sqrt{n}$ and $C_2(n) = c_0/2$. The claim follows immediately from (18) and (20). □

Note that property (10) for the function ϕ implies property (2) in Definition 3.1 for the function $\alpha(t) = c/-\phi'(t)$. Also note that, for a Hölder domain, $-\phi'(t)t$ is a constant, and by Theorem 4.7 the boundary of such a domain is mean porous (this result is equivalent to [KR, Theorem 5.1]).

Corollary 4.8. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain that satisfies the quasi-hyperbolic growth condition with the function ϕ of logarithmic type. Then there are positive constants M, $C(\beta,n)$ and an integer j_0 such that*

$$|\partial\Omega + 2^{-j}| \leq M \exp\biggl(-C(\beta,n)\int_{[2^{-j},2^{-j_0}]} \frac{dt}{\bigl(-\phi'(t)t\bigr)^{n-1}t}\biggr)$$

for all $j > j_0$.

Proof. By combining Theorem 4.7 and the proof of Theorem 3.3 we deduce by (6) that there are $j_0 \in \mathbf{Z}^+$ and $c = c(\beta) > 0$ such that (the summation indices follow from Remark 3.4)

$$\begin{aligned}|\partial\Omega + 2^{-j}| &\leq M \exp\biggl(-C(n)\sum_{k=j_0}^{j}\Bigl(\frac{c}{-\phi'(2^{-k})2^{-k}}\Bigr)^{n-1}\biggr)\\ &\leq M \exp\biggl(-\frac{1}{2}C(n)c^{n-1}\int_{[2^{-j},2^{-j_0}]} \frac{dt}{\bigl(-\phi'(t)t\bigr)^{n-1}t}\biggr)\end{aligned}$$

for all $j > j_0$ with a constant M depending on Ω. □

Corollary 4.9. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain that satisfies the quasi-hyperbolic growth condition with the function ϕ of logarithmic type satisfying*

$$\int_0 \frac{dt}{\bigl(-\phi'(t)t\bigr)^{n-1}t} = \infty.$$

Then $m_n(\partial\Omega) = 0$ and there is a positive constant $C(\beta,n)$ and an integer j_0 such that $H^h(\partial\Omega) < \infty$ for each premeasure h, which satisfies

$$h(2^{-j}) \leq M 2^{-jn} \exp\biggl(C(\beta,n)\int_{[2^{-j},2^{-j_0}]} \frac{dt}{\bigl(-\phi'(t)t\bigr)^{n-1}t}\biggr)$$

for all $j > j_0$ with some positive constant M.

Proof. The claim follows by Theorem 4.7, Corollary 3.5 and a similar argument as in the proof of Corollary 4.8. □

Note that Corollary 4.9 is essentially sharp by an example in Section 7.2.

Remark 4.10. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain that satisfies the quasihyperbolic growth condition with the function

$$\phi(t) = \frac{1}{\varepsilon}\left(\log\frac{1}{t}\right)^s \qquad \text{with } 1 \le s \le \frac{n}{n-1}.$$

Then, by Corollary 4.9, $m_n(\partial\Omega) = 0$ and $H^h(\partial\Omega) < \infty$ for the gauge function

$$h(t) = t^n \exp\left(\frac{C}{\big(n-(n-1)s\big)}\left(\log\frac{1}{t}\right)^{n-(n-1)s}\right)$$

when $s < n/(n-1)$, and for the gauge function

$$h(t) = t^n\left(\log\frac{1}{t}\right)^C$$

when $s = n/(n-1)$. Here the constant C depends on ε, n and s.

If $n/(n-1) < s$, then the boundary of the domain Ω can have positive Lebesgue measure, see Section 7.2.

5. Uniform continuity of quasiconformal mappings

The connection between uniform continuity of quasiconformal mappings and the concept of generalized mean porosity comes from the following observation.

Theorem 5.1. *Let $\psi\colon\,]0,1[\to]0,1[$ be an increasing bijection, and let $u(t) := \psi^{-1}(t)$. Assume that $\log\big(1/u(t)\big)$ is of logarithmic type. Let $f\colon B^n \to \Omega \subset \mathbf{R}^n$ be a K-quasiconformal map such that the inequality*

$$|f(t\omega) - f(\omega)| \le \psi(1-t) \tag{24}$$

holds for all $\omega \in S^{n-1}$ and $t_0 < t < 1$. Then there is a constant $c > 0$ such that $\partial f(B^n)$ is weakly mean porous with parameters $C_1(n)\alpha(t)$ and $C_2(n)\lambda(k)$, where (for small t) $\alpha(t) = cu(t)/u'(t)$ and $\lambda(k)$ is the smallest integer such that $\lambda(k) \ge 2^{-k}/\alpha(2^{-k})$.

Let us already remark that this theorem could be considered as a special case of Theorem 4.7. At the end of this section we discuss the connection between the uniform continuity of a quasiconformal mapping f and the quasihyperbolic growth condition in the image domain $f(B^n)$.

Proof of Theorem 5.1. Let $\omega \in \partial B^n$. Define functions χ_k and S_j as in the proof of Theorem 4.7. We prove that

$$\frac{S_j\bigl(f(\omega)\bigr)}{j} > \frac{1}{2} \tag{25}$$

for all sufficiently large $j \in \mathbf{Z}^+$.

Let $j_0 \in \mathbf{Z}^+$ such that $2^{-j_0+1} \le d\bigl(f(0), \partial f(B^n)\bigr)$ and $2^{-j_0+1} \le 1 - t_0$. Let $j > 2j_0$ and let $j_0 \le k \le j$ such that $\chi_k\bigl(f(\omega)\bigr) = 0$. The curve $\gamma = f([0,\omega])$ intersects the two boundary components of $A_k\bigl(f(\omega)\bigr)$ in two points $a = f(t_a\omega)$ and $b = f(t_b\omega)$, say. The quasihyperbolic distance $k_\Omega(a,b)$ of a and b is at least $2^{-k}/\alpha(2^{-k})$. As quasiconformal maps are quasi-isometries for large distances in the quasihyperbolic metrics (see [GO, p. 62]), the quasihyperbolic distance $k_{B^n}(t_a\omega, t_b\omega)$ is at least $C2^{-k}/\alpha(2^{-k})$, provided c is small enough. Here C depends on K and n.

Consider the largest $t < 1$ with

$$|f(t\omega) - f(\omega)| = 2^{-j}.$$

It follows from (24) that $2^{-j} \le \psi(1-t)$, and equivalently

$$\log\frac{1}{1-t} \le \log\biggl(\frac{1}{u(2^{-j})}\biggr). \tag{26}$$

On the other hand

$$\log\frac{1}{1-t} = k_{B^n}(0, t\omega) \ge \sum k_{B^n}(t_a\omega, t_b\omega) \ge \sum C\frac{2^{-k}}{\alpha(2^{-k})}, \tag{27}$$

where the summation is over all $j_0 \le k \le j$ with $\chi_k\bigl(f(\omega)\bigr) = 0$.

Suppose that the assertion $S_j\bigl(f(\omega)\bigr)/j > 1/2$ fails for some large j. Then, by combining (26) and (27), we obtain that (the summation indices follow from assumption (10))

$$\begin{aligned}\log\biggl(\frac{1}{u(2^{-j})}\biggr) &\ge C\sum_{k=j_0}^{j/2}\frac{2^{-k}u'(2^{-k})}{cu(2^{-k})}\\ &\ge C\frac{1}{c}\biggl(\log\biggl(\frac{1}{u(2^{-j/2})}\biggr) - \log\biggl(\frac{1}{u(2^{-j_0})}\biggr)\biggr).\end{aligned}$$

This contradicts property (11) when we choose j large enough and the constant $c < C/2\beta$, and thus (25) is proved. To prove the claim we can choose the collection $\mathcal{Q}$ and the constants C_1, C_2 similarly as in the proof of Theorem 4.7. □

Corollary 5.2. *Let $f\colon B^n \to \Omega \subset \mathbf{R}^n$ be a K-quasiconformal map and suppose*

$$|f(x) - f(x')| \le \psi(|x - x'|)$$

for all $x, x' \in B^n$ sufficiently close to each other. Assume that the function $\log(1/\psi^{-1})$ is of logarithmic type and that $u = \psi^{-1}$ satisfies

$$\int_0 \left(\frac{u(t)}{u'(t)}\right)^{n-1} \frac{dt}{t^n} = \infty. \tag{28}$$

Then $m_n\bigl(\partial f(B^n)\bigr) = 0$ and there is a positive constant $C = C(\beta, K, n)$ and an integer j_0 such that $H^h\bigl(\partial f(B^n)\bigr) < \infty$ for each premeasure h which satisfies

$$h(2^{-j}) \le M 2^{-jn} \exp\left(C \int_{[2^{-j}, 2^{-j_0}]} \left(\frac{u(t)}{u'(t)}\right)^{n-1} \frac{dt}{t^n}\right)$$

for all $j > j_0$ with some positive constant M.

Proof. The claim follows by combining Theorem 5.1, Corollary 3.5, and a similar argument as in the proof of Corollary 4.8. □

Remark 5.3. Consider (28) in the case $n = 2$. Assume that $u = \psi^{-1}$ satisfies the conditions of Corollary 5.2 and

$$\frac{\bigl(\log \psi(t)\bigr)'}{\log \psi(t)} t \log t \text{ is monotone for all sufficiently small } t. \tag{29}$$

Then

$$\int_0 \left(\frac{u(t)}{u'(t)}\right) \frac{dt}{t^2} = \infty \quad \text{if and only if} \quad \int_0 \left(\frac{\log \psi(t)}{\log t}\right)^2 \frac{dt}{t} = \infty.$$

The statement above seems to be known (cf. [JM, p. 453]), but for the convenience of the reader we present a proof.

Proof of Remark 5.3. By a change of variable we have for arbitrarily small $r > 0$ that

$$\begin{aligned} \int_0^r \left(\frac{u(t)}{u'(t)}\right) \frac{dt}{t^2} &= \int_0^{\psi^{-1}(r)} \left(\frac{\psi'(u)}{\psi(u)}\right)^2 u\, du \\ &= \int_0^{\psi^{-1}(r)} \bigl((\log \psi(u))'\bigr)^2 u\, du \\ &= \int_0^{\psi^{-1}(r)} \left(\frac{\bigl(\log \psi(u)\bigr)'}{(\log u)'}\right)^2 \frac{du}{u}. \end{aligned}$$

Let us write $\psi(t) = \exp\bigl(-\alpha(t)\bigr)$ and assume that $\log(1/\psi^{-1})$ is of logarithmic type (as assumed in Corollary 5.2). By (29) it suffices to show that

$$\varepsilon \le \left|\frac{\bigl(\log\psi(t)\bigr)'}{(\log t)'}\right| \Big/ \left|\frac{\log\psi(t)}{\log t}\right| = \left|\frac{\alpha'(t)}{\alpha(t)} t\log t\right| \le M$$

for some arbitrarily small $t > 0$ with some positive constants ε and M. Assume towards a contradiction that

$$\left|\frac{\alpha'(t)}{\alpha(t)} t\log t\right| < \varepsilon$$

for all small t with arbitrarily small $\varepsilon > 0$. By Gronwall's lemma ([W, p. 436]) we deduce that $\alpha(t) \le C\bigl(\log(1/t)\bigr)^{\varepsilon}$ for small t and hence

$$\psi(t) \ge \exp\biggl(-C\biggl(\log\frac{1}{t}\biggr)^{\varepsilon}\biggr).$$

This implies that

$$\psi^{-1}(t) \le \exp\biggl(-\biggl(\frac{1}{C}\log\frac{1}{t}\biggr)^{1/\varepsilon}\biggr)$$

or equivalently

$$\log\frac{1}{\psi^{-1}(t)} \ge \biggl(\frac{1}{C}\log\frac{1}{t}\biggr)^{1/\varepsilon}$$

for small t. But this is a contradiction with (11) when ε is chosen small enough. Let us then assume that $|\alpha'(t)/\alpha(t)t\log t| > M$ for all small t with arbitrarily large M. Again by Gronwall we deduce that $\alpha(t) \ge C\bigl(\log(1/t)\bigr)^{M}$ and hence $\psi(t) \le \exp\bigl(-C\bigl(\log(1/t)\bigr)^{M}\bigr)$. This implies that

$$\psi^{-1}(r) \ge \exp\biggl(-\biggl(\frac{1}{C}\log\frac{1}{t}\biggr)^{1/M}\biggr)$$

or equivalently

$$\biggl(\frac{1}{C}\log\frac{1}{t}\biggr)^{1/M} \ge \log\biggl(\frac{1}{\psi^{-1}(t)}\biggr).$$

This, however, contradicts property (10) when M is chosen large enough. □

By Remark 5.3 we see that, in the case $n = 2$, condition (28) is essentially equivalent with the assumption of [JM, Theorem C.1]. Jones and Makarov proved in this paper that this condition, implying $m_2\bigl(\partial f(B^2)\bigr) = 0$, is sharp. We will show in Section 7.2 that even the dimension estimate of Corollary 5.2 is essentially sharp.

Remark 5.4. Let $f\colon B^n \to \Omega \subset \mathbf{R}^n$ be a K-quasiconformal map and suppose

$$|f(x) - f(x')| \leq \psi(|x - x'|)$$

for all $x, x' \in B^n$ with the function

$$\psi(t) = \exp\biggl(-\Bigl(\varepsilon \log \frac{1}{t}\Bigr)^{1/s}\biggr)$$

where $1 \leq s \leq n/(n-1)$. Then, by Corollary 5.2, $m_n\bigl(\partial f(B^n)\bigr) = 0$ and $H^h\bigl(\partial f(B^n)\bigr) < \infty$ for the gauge function

$$h(t) = t^n \exp\biggl(\frac{C}{\bigl(n-(n-1)s\bigr)}\Bigl(\log \frac{1}{t}\Bigr)^{n-(n-1)s}\biggr)$$

when $s < n/(n-1)$, and for the gauge function

$$h(t) = t^n \Bigl(\log \frac{1}{t}\Bigr)^{C}$$

when $s = n/(n-1)$. Here the constant C depends on K, ε, n and s.

If $n/(n-1) < s$, then the boundary of the domain $f(B^n)$ can have positive Lebesgue measure, see Section 7.2.

Note that the previous example is roughly equivalent with Remark 4.10. Indeed, by using the fact that quasiconformal mappings are quasi-isometries for large distances in the quasihyperbolic metrics, we see the following connection between uniform continuity of quasiconformal mappings and the quasihyperbolic growth condition. If $f\colon B^n \to \Omega \subset \mathbf{R}^n$ is a K-quasiconformal mapping from the unit ball onto a bounded domain Ω, and

$$|f(x) - f(x')| \leq \psi(|x - x'|)$$

for all $x, x' \in B^n$ sufficiently close to each other, where ψ satisfies the conditions of Theorem 5.1, then the image domain $f(B^n)$ satisfies the quasihyperbolic growth condition with the function $\phi(t) = C \log\bigl(1/\psi^{-1}(t)\bigr)$. Moreover, the dimension estimates implied by Corollaries 4.9 and 5.2 for the boundary of $f(B^n)$ are equivalent (except perhaps with different constants).

6. John domains

Definition 6.1. Let $\varphi\colon [0,\infty[\to [0,\infty[$ be a continuous function such that $\varphi(t)/t$ is an increasing function. We say that a domain Ω is a φ-John domain, if there is a John center $x_0 \in \Omega$ such that for all $x \in \Omega$ there is a curve $\gamma\colon [0,l] \to \Omega$, parametrized by arclength and with $\gamma(0) = x$, $\gamma(l) = x_0$, and $d(\gamma(t), \partial\Omega) \geq \varphi(t)$ for all $0 < t < l$.

Note that, when $\varphi(t) = ct$ with some $c < 1$, this definition is equivalent to the definition of a usual c-John domain. The Hausdorff dimension of the boundary of a usual c-John domain is known to be strictly less than n, see e.g. [KR, p. 599]. The question arises, whether one could establish a dimension bound for the boundary of a φ-John domain $\Omega \subset \mathbf{R}^n$ with $\varphi(t) = ct^s$ for some $s > 1$. This cannot be done, however. In Section 7.3 we show that, for any $s > 1$, the boundary $\partial\Omega$ can have positive Lebesgue measure. However, with a proper function φ, a dimension estimate for the boundary can be established by applying generalized mean porosity. It is indeed immediate that the boundary of a φ-John domain is weakly mean porous with parameters $\alpha(t) = C\varphi(t)$ (for small t) and $\lambda(k) \geq 2^{-k}/2\alpha(2^{-k})$ (take $\mathcal{Q}$ to be the collection of the Whitney decompositions of all the cubes in the Whitney decomposition of $\mathbf{R}^n \setminus \partial\Omega$). By applying Corollary 3.5 we obtain the following result.

Corollary 6.2. *Let $\Omega \subset \mathbf{R}^n$ be a φ-John domain. Assume that*

$$\int_0 \frac{\varphi(t)^{n-1}dt}{t^n} = \infty.$$

Then $m_n(\partial\Omega) = 0$ and there is a positive constant $C(n)$ and an integer j_0 such that $H^h(\partial\Omega) < \infty$ for each premeasure h which satisfies

$$h(2^{-j}) \leq M2^{-jn} \exp\biggl(C(n) \int_{[2^{-j},2^{-j_0}]} \frac{\varphi(t)^{n-1}\,dt}{t^n}\biggr)$$

for all $j > j_0$ with some positive constant M.

Note that this corollary is essentially sharp by an example given in Section 7.3. Therefore, if

$$\varphi(t) = \frac{t}{\bigl(\log(1/t)\bigr)^s}$$

for small t, then we obtain a dimension bound when $s \leq 1/(n-1)$, whereas the boundary can have positive volume when $s > 1/(n-1)$.

Remark 6.3. Let $\Omega \subset \mathbf{R}^n$ be a φ-John domain with

$$\varphi(t) = t\biggl(\log\frac{1}{t}\biggr)^{-1/(n-1)}$$

for small t. Then Corollary 6.2 implies that $m_n(\partial\Omega)=0$ and, moreover, $H^h(\partial\Omega) < \infty$ for the gauge function

$$h(t) = t^n \left(\log \frac{1}{t}\right)^{C(n)}.$$

7. Sharpness of the results

Recall the well-known Frostman lemma. We denote by $\mathscr{M}(A)$ the set of Radon measures μ such that $\operatorname{spt}(\mu) \subset A$ and $\mu(\mathbf{R}^n)=1$.

Lemma 7.1. *Let A be a Borel set in $\mathbf{R}^n$ and suppose there exists $\mu \in \mathscr{M}(A)$ such that $\mu\big(B(x,r)\big) \le h(r)$ for $x \in \mathbf{R}^n$ and $r>0$. Then $H^h(A)>0$.*

Proof. Take any collection of balls $B_i = B(x_i,r_i)$, $x_i \in \mathbf{R}^n$, $r_i < r$, such that $A \subset \bigcup_i B_i$. Then $1=\mu(A) \le \mu(\bigcup_i B_i) \le \sum_i \mu(B_i) \le \sum_i h(r_i)$. □

7.1. Sharpness of Corollary 3.5. We show the sharpness of Corollary 3.5 by constructing an example of a set E which is weakly mean porous with parameters α and λ and for which $H^h(E)>0$ with a premeasure h satisfying for all $j \in \mathbf{Z}^+$

$$h(2^{-j}) \le 2^{-jn} \exp\left(\widetilde{C}(n) \sum_{k=1}^{j} \frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n}\right)$$

with some constant $\widetilde{C}(n)$.

Let α and λ be as in Corollary 3.5. Then

$$p(k) = \frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n} \quad \text{is a decreasing function of } k. \tag{30}$$

Let

$$Q_0 = \left\{x \in \mathbf{R}^n : |x_i| \le \frac{1}{4\sqrt{n}} \text{ for all } i=1,2,\dots,n\right\}.$$

We define for each $k \in \mathbf{Z}^+$ a collection $\mathscr{E}_k$ of closed sets $F \subset \mathbf{R}^n$ in the following way. $\mathscr{E}_0$ consists of Q_0 alone. To obtain $\mathscr{E}_k$ from $\mathscr{E}_{k-1}$, subdivide Q_0 into 2^{nk} closed dyadic cubes Q_k^i of diameter 2^{-k-1} in the natural way. For each $i=1,\dots,2^{nk}$, let $\widetilde{Q}_k^i \subset Q_k^i$ be the smallest open cube in the center of Q_k^i such that it contains $2^n\lambda(k)$ disjoint open cubes of side length $\alpha(2^{-k})$. We may assume that the diameter of $\widetilde{Q}_k^i$ is dyadic and, by condition (30),

$$\frac{d(\widetilde{Q}_k^i)}{d(Q_k^i)} \le \frac{1}{4} \quad \text{for all } i \text{ and } k. \tag{31}$$

Let

$$\mathscr{E}_k = \{F \cap Q_k^i \setminus \widetilde{Q}_k^i : F \in \mathscr{E}_{k-1},\ i=1,\dots,2^{nk}\}.$$

We show that the set

$$E = \bigcap_{k=1}^{\infty} \bigcup_{F \in \mathscr{E}_k} F$$

is weakly mean porous with parameters α and λ.

Define for every $k \in \mathbf{Z}^+$ a collection $\widetilde{\mathscr{Q}}_k$ of disjoint cubes by subdividing each $\widetilde{Q}_k^i$, $i = 1, \ldots, 2^{nk}$, into $2^n \lambda(k)$ subcubes of side length $\alpha(2^{-k})$. Then let

$$\mathscr{Q}_k = \Big\{ Q \in \widetilde{\mathscr{Q}}_k : Q \subset \bigcup_{i=1}^{2^{nk}} \widetilde{Q}_k^i \setminus \bigcup_{j=0}^{k-1} \bigcup_{l=1}^{2^{nj}} \widetilde{Q}_j^l \Big\},$$

and let

$$\mathscr{Q} = \{ Q \in \mathscr{Q}_k : k = 1, 2, \ldots \}.$$

To see that $\mathscr{Q}$ indeed satisfies the conditions of Definition 3.1, notice the following geometric facts. Fix $x \in E$ and take any Q_k (a dyadic subcube of Q_0 with diameter 2^{-k-1}) such that $x \in Q_k$. Now, there exists a dyadic subcube $Q_{k+3} \subset Q_k$ such that $Q_{k+3} \subset A_{k+3}(x)$. Moreover, by inequality (31), we can choose the cube Q_{k+3} so that it is not completely covered by the set $\bigcup_{j=0}^{k+2} \bigcup_{l=1}^{2^{nj}} \widetilde{Q}_j^l$ (here $\widetilde{Q}_j^l$ is defined as above). Now, by inequality (31), there is at most one dyadic subcube Q_j such that $j \le k+2$ and $\widetilde{Q}_j \cap \widetilde{Q}_k \neq \emptyset$. Since the diameter of the cube $\widetilde{Q}_j$ is dyadic, we see that the set $\widetilde{Q}_{k+3} \setminus \bigcup_{j=0}^{k+2} \bigcup_{l=1}^{2^{nj}} \widetilde{Q}_j^l$ contains at least $\lambda(k+3)$ cubes $Q \in \mathscr{Q}$ such that $l(Q) \ge \alpha(2^{-k-3})$. Thus we have that

$$\chi_k^{\mathscr{Q}}(x) = 1$$

for every k and hence E is weakly mean porous.

Next we estimate the dimension of E by using Frostman's lemma. Define the density

$$\Delta_{k,F} = \frac{\left| \bigcup_{G \in \mathscr{E}_{k+1}} G \cap F \right|}{|F|} = \sum_{G \in \mathscr{E}_{k+1}} \frac{|G \cap F|}{|F|}$$

for each (nonempty) set $F \in \mathscr{E}_k$. The construction above implies

$$\Delta_{k,F} \ge \left(1 - \frac{C(n)\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n} \right) =: \Delta_k. \tag{32}$$

For each set $F \in \mathscr{E}_k$ let $F_i \in \mathscr{E}_i$, $i = 0, 1, \ldots, k-1$, be the (unique) sets for which $F \subset F_i$. We define a sequence of Radon measures μ_k, $k = 0, 1, 2, \ldots$, so that

$$\mu_0(A) = \frac{1}{|Q_0|} |Q_0 \cap A|$$

and

$$\mu_k(A) = \frac{1}{|Q_0|} \sum_{F \in \mathscr{E}_k} \frac{|F \cap A|}{\Delta_{k-1,F_{k-1}} \times \cdots \times \Delta_{0,F_0}}$$

for all measurable $A \subset \mathbf{R}^n$. Then $\operatorname{spt}(\mu_k) \subset \bigcup_{F \in \mathscr{E}_k} F$ and

$$\begin{aligned}\mu_{k+1}(F) &= \frac{1}{|Q_0|} \sum_{G \in \mathscr{E}_{k+1}} \frac{|G \cap F|}{\Delta_{k,F} \times \Delta_{k-1,F_{k-1}} \times \cdots \times \Delta_{0,F_0}} \\ &= \frac{1}{|Q_0|} \frac{|F|}{\Delta_{k-1,F_{k-1}} \times \cdots \times \Delta_{0,F_0}} = \mu_k(F)\end{aligned}$$

for all $F \in \mathscr{E}_k$. Hence $\mu_l(F) = \mu_k(F)$ for all $l \geq k$. In particular, for all k

$$\mu_k(\mathbf{R}^n) = \mu_k(Q_0) = \mu_0(Q_0) = 1$$

and thus $\mu_k \in \mathscr{M}(\mathbf{R}^n)$ for all k.

Recall that $\mathscr{M}(\mathbf{R}^n)$ is a compact metric space with an appropriate metric d (see [M, pp. 52–55]). Hence there is a subsequence μ_{k_i} converging to a measure $\mu \in \mathscr{M}(\mathbf{R}^n)$ in the metric d. Note that $\operatorname{spt}(\mu) \subset E$.

Let $j \in \mathbf{Z}^+$ and let $x \in \mathbf{R}^n$. By (32) we have for each $G \in \mathscr{E}_{j+1}$ that

$$\mu(G) \leq \frac{|G|}{|Q_0| \prod_{k=1}^{j} \Delta_k}$$

and, by the construction, the ball $B(x, 2^{-j})$ intersects at most $P(n)$ sets $G^i \in \mathscr{E}_{j+1}$, where $P(n)$ is a constant depending only on n. Hence

$$\mu\big(B(x, 2^{-j})\big) \leq \mu\left(\bigcup_{i=1}^{P(n)} G^i\right) \leq \frac{\sum_{i=i}^{P(n)} |G^i|}{|Q_0| \prod_{k=1}^{j} \Delta_k} \leq \frac{P(n) 2^{-jn}}{|Q_0| \prod_{k=1}^{j} \Delta_k}.$$

We choose the constant $Q(n) \geq P(n)/|Q_0|$ and the function

$$\begin{aligned}h(2^{-j}) &= \frac{Q(n) 2^{-jn}}{\prod_{k=1}^{j} \Delta_k} = Q(n) 2^{-jn} \exp\left(-\log\left(\prod_{k=1}^{j} \Delta_k\right)\right) \\ &= Q(n) 2^{-jn} \exp\left(-\sum_{k=1}^{j} \log \Delta_k\right) \\ &= Q(n) 2^{-jn} \exp\left(-\sum_{k=1}^{j} \log\left(1 - \frac{C(n)\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n}\right)\right).\end{aligned}$$

Clearly we have that

$$h(2^{-j}) \leq 2^{-jn} \exp\left(\widetilde{C}\,(n) \sum_{k=1}^{j} \frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n}\right),$$

when we choose the constant $\widetilde{C}(n)$ large enough. Therefore

$$\mu\big(B(x,2^{-j})\big) \le h(2^{-j}) \le 2^{-jn} \exp\biggl(\tilde{C}(n) \sum_{k=1}^{j} \frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n}\biggr)$$

and the claim follows by Frostman's lemma.

Note especially that if

$$\sum_{k=1}^{\infty} \frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n} < \infty,$$

then Frostman's lemma implies that $m_n(E) > 0$.

7.2. Sharpness of Corollary 4.9. We show the sharpness of Corollary 4.9 in the case $n = 2$ by constructing a domain $\Omega \subset \mathbf{R}^2$ such that it satisfies the quasihyperbolic growth condition with a function ϕ, and $H^h(\partial\Omega) > 0$ with a premeasure h satisfying

$$h(2^{-j}) \le 2^{-2j} \exp\biggl(\widetilde{C} \int_{[2^{-j},2^{-j_0}]} \frac{dt}{-\phi'(t)t^2}\biggr)$$

for all large $j \in \mathbf{Z}^+$, where the constant $\widetilde{C}$ depends only on β.

Let ϕ: $]0,1] \to]0,\infty[$ be a function satisfying the conditions of Corollary 4.9. By condition (10) we can take j_0 to be the smallest integer such that

$$\frac{4\beta^2}{-\phi'(2^{-j_0})} \le \frac{2^{-j_0}}{16}.$$

Then define a function α so that it satisfies

$$\alpha(2^{-k}) = \frac{4\beta^2}{-\phi'(2^{-k})}$$

for all $k \ge j_0$. Thus $\alpha(2^{-k})/2^{-k} \le 1/16$ for all $k \ge j_0$. We may assume that $\alpha(2^{-k})$ is dyadic for all $k \ge j_0$.

Let

$$Q_{j_0} = \bigl\{(x,y) \in \mathbf{R}^2 : |x| \le 2^{-j_0-1} \text{ and } |y| \le 2^{-j_0-1}\bigr\}.$$

Let $\Omega^x_{j_0}$ be the $\alpha(2^{-j_0})$ neighborhood of the x-coordinate axis in the square Q_{j_0}. Let $\Omega^y_{j_0}$ be the $\alpha(2^{-j_0})$ neighborhood of the y-coordinate axis in the square Q_{j_0}. Let $\Omega_{j_0} = \Omega^x_{j_0} \cup \Omega^y_{j_0}$. For each $k > j_0$ define Ω_k by subdividing Q_{j_0} into dyadic squares Q^i_k, $i = 1, 2, \ldots, 2^{2(k-j_0)}$, of side length 2^{-k}. Let $\widetilde{\Omega}^x_k$ be the union of the $\alpha(2^{-k})$ neighborhoods of the centered x-coordinate axes in the squares Q^i_k and let

$\Omega_k^x = \widetilde{\Omega}_k^x \setminus \bigcup_{i=j_0}^{k-1} \overline{\Omega}_i$. Similarly let $\widetilde{\Omega}_k^y$ be the union of the $\alpha(2^{-k})$ neighborhoods of the centered y-coordinate axes in the squares Q_k^i and let $\Omega_k^x = \widetilde{\Omega}_k^x \setminus \bigcup_{i=j_0}^{k-1} \overline{\Omega}_i$. Let $\Omega_k = \Omega_k^x \cup \Omega_k^y$.

Define the domain Ω by

$$\Omega = \bigcup_{k=j_0}^{\infty} \Omega_k \cup \bigcup_{k=j_0}^{\infty} \operatorname{int}_1(\overline{\Omega}_k^x \cap \overline{\Omega}_{k+1}^y),$$

where we denote by $\operatorname{int}_1(\,\cdot\,)$ the one-dimensional interior of the set.

Notice that the domain Ω satisfies the quasihyperbolic growth condition with the function ϕ: Let $x_0 = 0$ and let $x_j \in \Omega$ such that $\alpha(2^{-j-1}) \le d(x_j, \partial\Omega) \le \alpha(2^{-j})$ for some $j \ge 2j_0$. Now we have that

$$\begin{aligned}
k_\Omega(x_0, x_j) &\le 2 \sum_{k=j_0}^{j+1} \int_0^{2^{-k}} \frac{dt}{\alpha(2^{-k})} \le 2 \sum_{k=j_0}^{j+1} \frac{2^{-k}}{\alpha(2^{-k})} \\
&= \frac{1}{2\beta^2} \sum_{k=j_0}^{j+1} -\phi'(2^{-k}) 2^{-k} \le \frac{1}{\beta^2} \int_{2^{-j-1}}^{2^{-j_0+1}} -\phi'(t)\,dt \\
&\le \frac{1}{\beta^2} \phi(2^{-j-1}) \le \phi(2^{-j/2}) \le \phi\left(\frac{2^{-j}}{2^{-j_0}}\right) \\
&\le \phi\left(\frac{\alpha(2^{-j})}{\alpha(2^{-j_0})}\right) \le \phi\left(\frac{d(x_j, \partial\Omega)}{d(x_0, \partial\Omega)}\right).
\end{aligned}$$

Hence there is a constant C_0 such that the inequality

$$k(x, x_0) \le \phi\left(\frac{d(x, \partial\Omega)}{d(x_0, \partial\Omega)}\right) + C_0$$

holds for all $x \in \Omega$.

We obtain the desired dimension estimate for $\partial\Omega$ similarly to Section 7.1. Let $\mathscr{E}_{j_0} = \{Q_{j_0} \setminus \Omega_{j_0}\}$ and define for all $k > j_0$

$$\mathscr{E}_k = \bigl\{ F \cap Q_k^i \setminus \Omega_k : F \in \mathscr{E}_{k-1},\ i = 1, 2, \dots, 2^{2(k-j_0)} \bigr\}.$$

Now

$$\partial\Omega = E = \bigcap_{k=j_0}^{\infty} \bigcup_{F \in \mathscr{E}_k} F,$$

and for the density $\Delta_{k,F}$ we have the estimate

$$\Delta_{k,F} \ge \left(1 - \frac{C\alpha(2^{-k})}{2^{-k}}\right)$$

for each $F \in \mathscr{E}_k$. Hence

$$\Delta_{k,F} \geq \left(1 - \frac{C}{-\phi'(2^{-k})2^{-k}}\right),$$

where the constant C depends only on β. The claim follows as in Section 7.1 by using Frostman's lemma. Note especially that if the sum $\sum 1/-\phi'(2^{-k})2^{-k}$ converges, or equivalently, if

$$\int_0 \frac{dt}{-\phi'(t)t^2} < \infty,$$

then $m_n(\partial\Omega) > 0$.

Note that this example also shows the essential sharpness of Corollary 5.2. Indeed, the domain Ω is simply connected and hence it is the image of the disk B^2 for some quasiconformal mapping $f\colon B^2 \to \mathbf{R}^2$. Since Ω satisfies the quasi-hyperbolic growth condition with the function ϕ which satisfies the conditions of Corollary 4.9, the mapping f is uniformly continuous with a corresponding modulus of continuity $\psi(t) = C\phi^{-1}\big(C\log(1/t)\big)$; see [HK] for details. In this case the dimension estimates of the Corollaries 4.9 and 5.2 are essentially equivalent (except perhaps with different constants).

To prove the sharpness of the dimension estimate in $\mathbf{R}^n$ with $n > 2$, a similar construction can be carried out. We sketch an outline for the case $n = 3$. Let

$$Q_{j_0} = \big\{x \in \mathbf{R}^3 : |x_i| \leq 2^{-j_0-1} \text{ for all } i = 1,2,3\big\}.$$

Define Ω_{j_0} now by removing the $\alpha(2^{-j_0})$ neighborhoods of the coordinate axes in Q_{j_0} and of the lines $(t, \pm 2^{j_0-2}, 0)$ in Q_{j_0}. Define Ω_j accordingly for $j > j_0$ in the dyadic subcubes of Q_{j_0} and, finally, define the domain Ω by attaching the x_3-components of Ω_{j+1} to the x_1-components of Ω_j. By [V] one can deduce that with this construction Ω is a quasiconformal ball.

7.3. Sharpness of Corollary 6.2. We show the sharpness of Corollary 6.2 in the case $n = 2$ by constructing a φ-John domain $\Omega \subset \mathbf{R}^2$ for which $H^h(\partial\Omega) > 0$ with a premeasure h satisfying

$$h(2^{-j}) \leq 2^{-2j} \exp\left(\widetilde{C} \int_{[2^{-j},2^{-j_0}]} \frac{\varphi(t)dt}{t^2}\right)$$

for all large j with some constant $\widetilde{C}$.

Let $\varphi\colon]0,1[\to]0,1[$ be a continuous function such that $\varphi(t)/t$ is an increasing function. Choose $\alpha(t) = \varphi(t)$ and construct a domain Ω similarly as in Section 7.2. Notice that Ω is a φ-John domain: Let $x_0 = 0$ be the John center, and for any

$x \in \Omega$ let $\gamma(x_0, x)$ be the quasihyperbolic geodesic joining x to x_0. Now the length of γ is at most 2^{-j_0} and $d\bigl(\gamma(t), \partial\Omega\bigr) \geq \varphi(t)$ for all $0 < t < l \leq 2^{-j_0}$.

We obtain the desired dimension estimate similarly as in Section 7.2. Now, for the density $\Delta_{k,F}$, we have the following estimate. For each $F \in \mathscr{E}_k$

$$\Delta_{k,F} \geq \left(1 - \frac{C\varphi(2^{-k})}{2^{-k}}\right).$$

The claim follows as in Section 7.1 by using Frostman's lemma. To prove the sharpness of the dimension estimate in $\mathbf{R}^n$ with $n > 2$, a similar construction can be carried out. The case $n = 3$ is outlined at the end of Section 7.2.

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