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ON HECKE L-FUNCTIONS ASSOCIATED WITH CUSP FORMS II: ON THE SIGN CHANGES OF $S_f(T)$

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In honour of Professor T. N. Shorey on his sixtieth birthday

Abstract. We study the number of sign changes of $S_f(t)$ (related to Hecke L-functions attached to holomorphic cusp forms of even positive integral weight with respect to the full modular group) over shorter intervals.

1. Introduction

Let

$$S(t) = \pi^{-1} \arg \zeta\big(\tfrac{1}{2} + it\big),$$

where the argument is obtained by continuous variation along the straight lines joining 2, $2+it$ and $\frac{1}{2}+it$, starting with the value zero. When t is equal to the imaginary part of any zero of $\zeta(s)$, we put

$$S(t) = \lim_{\varepsilon \to 0} \tfrac{1}{2}\big\{S(t+\varepsilon) + S(t-\varepsilon)\big\}.$$

As for Atle Selberg's comment on a deep result of Littlewood on $S(t)$, A. Ghosh established that (see Theorem 1 of [5] and also the paper of Selberg [16]) $S(t)$ changes its sign at least

$$T(\log T)\exp\big(-A(\delta)(\log\log T)(\log\log\log T)^{-(1/2)+\delta}\big)$$

times in the interval $(T, 2T)$. Here δ is any arbitrarily small positive constant, and $A(\delta) > 0$ depending only on δ. In fact, he proved this result over shorter intervals.

Let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ be a holomorphic cusp form of even integral weight $k > 0$ with respect to the full modular group $\Gamma = SL(2, \mathbf{Z})$. We define the associated Hecke L-function

$$L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{1.1}$$

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for $\operatorname{Re} s > (k+1)/2$. Throughout this paper, we assume that $f(z)$ is a Hecke eigenform with $a_1 = 1$. It is known (see [7]) that $L_f(s)$ admits analytic continuation to $\mathbf{C}$ as an entire function and it satisfies the functional equation

$$(2\pi)^{-s}\Gamma(s)L_f(s) = (-1)^{k/2}(2\pi)^{-(k-s)}\Gamma(k-s)L_f(k-s). \tag{1.2}$$

$L_f(s)$ has an Euler-product representation (for $\operatorname{Re} s > (k+1)/2$)

$$L_f(s) = \prod_p \bigl(1 - a_p p^{-s} + p^{k-1}p^{-2s}\bigr)^{-1}. \tag{1.3}$$

The non-trivial zeros of $L_f(s)$ lie within the critical strip $(k-1)/2 < \operatorname{Re} s < (k+1)/2$. These zeros are located symmetrically to the real axis and they are also symmetrical about the line $\operatorname{Re} s = k/2$. The Riemann hypothesis in this situation asserts that all the non-trivial zeros are on the critical line $\operatorname{Re} s = k/2$. From Deligne's proof of Ramanujan–Petersson's conjecture (see [1] and [2]), we have the bound for the coefficients

$$|a_n| \le d(n)n^{(k-1)/2}. \tag{1.4}$$

Several interesting deep results about the Hecke L-functions have been established lately. As a sample, a certain average growth of these L-functions in the weight aspect on the critical line has been investigated in the papers of Peter Sarnak (see [15]) and of Matti Jutila and Yoichi Motohashi (see [9]).

Let $N_f(T)$ denote the number of zeros $\beta + i\gamma$ of $L_f(s)$ for which $0 < \gamma < T$. If T is equal to the ordinate of any zero, then we define

$$N_f(T) := \lim_{\varepsilon \to 0} \tfrac{1}{2}\{N_f(T+\varepsilon) + N_f(T-\varepsilon)\}. \tag{1.5}$$

Now, one can show that (following Theorem 9.3 of [18])

$$N_f(T) = \frac{T}{\pi}\log\frac{T}{\pi} - \frac{T}{\pi} + 1 + S_f(T) + O\left(\frac{1}{T}\right), \tag{1.6}$$

where

$$S_f(t) = \frac{1}{\pi}\arg\ L_f\left(\frac{k}{2} + it\right). \tag{1.7}$$

The argument is obtained by a continuous variation along the straight lines joining the points $\frac{1}{2}k+1$, $\frac{1}{2}k+1+it$ and $\frac{1}{2}k+it$, starting with the value $\frac{1}{2}(k-1)$. Hence the variation of $S_f(t)$ is closely connected with the distribution of the imaginary parts of the zeros of $L_f(s)$.

We now define, for $\sigma \ge k/2$, $T \ge 1$ and $H \le T$,

$$N_f(\sigma, T, T+H) = \#\bigl\{\beta + i\gamma : L_f(\beta + i\gamma) = 0,\ \beta \ge \sigma,\ T \le \gamma \le T+H\bigr\}. \tag{1.8}$$

In [14], we proved the following two theorems:

Theorem A. *For $t \geq 2$, $2 \leq x \leq t^2$, we have*

$$S_f(t) = -\frac{1}{\pi} \sum_{n<x^3} \frac{\Lambda_{x,f}(n)\sin(t\log n)}{n^{\sigma_{x,t}}\log n} + O\biggl((\sigma_{x,t} - k/2)\biggl|\sum_{n<x^3} \frac{\Lambda_{x,f}(n)}{n^{\sigma_{x,t}+it}}\biggr|\biggr) + O\bigl((\sigma_{x,t} - k/2)\log t\bigr),$$

where

$$\sigma_{x,t} = k/2 + 2\max(\beta - k/2, 2/\log x),$$

$\varrho = \beta + i\gamma$ running over those zeros for which

$$|t-\gamma| \leq x^{3|\beta-k/2|}(\log x)^{-1},$$

and $\Lambda_{x,f}(n)$ is as in (2.6).

As corollaries we obtained (by choosing $x = \sqrt{\log t}$)

$$S_f(t) = O(\log t)$$

unconditionally, and assuming the Riemann hypothesis for $L_f(s)$, we got

$$S_f(t) = O\biggl(\frac{\log t}{\log\log t}\biggr).$$

Theorem A$'$. *Let B be any fixed small positive constant. Let*

$$B' = \frac{19}{20} + \frac{13.505}{5}B$$

and $B' < \alpha \leq 1$. Then for $T^\alpha \leq H \leq T$, we have

$$N_f(\sigma, T, T+H) \ll H\biggl(\frac{H}{T^{B'}}\biggr)^{-(B/(1-B'))(\sigma-k/2)} \log T$$

uniformly for $k/2 \leq \sigma \leq (k+1)/2$.

As an application to the above Theorems A and A$'$, the object of this paper is now to prove

Main theorem. *Let B' be the constant as in Theorem A$'$. Let $B' < \alpha \leq 1$. If $(T+1)^\alpha \leq H \leq T$ and $\delta > 0$ is an arbitrarily small real number, there is an $A = A(\alpha,\delta) > 0$ and a $T_0 = T_0(\alpha,\delta) > 0$ such that when $T > T_0$, $S_f(t)$ changes its sign at least*

$$H(\log T)\exp\bigl(-A(\log\log T)(\log\log\log T)^{-(1/2)+\delta}\bigr)$$

times in the interval $(T, T+H)$.

Remark 1. This main theorem is an analogous result of the theorem in the case of $S(t)$ related to the ordinary Riemann zeta-function, which was established by A. Ghosh (see Theorem 1 of [5]). In the case of $S(t)$, B' can be replaced by $\frac{1}{2}$ (or even by a better positive constant).

Remark 2. If we assume the Riemann hypothesis for $L_f(s)$, then the main theorem is true with $0 < \alpha \leq 1$.

The proof requires asymptotic formulae for integrals of the type

$$\int_T^{T+H} |S_f(t)|^{2l}\, dt$$

and

$$\int_T^{T+H} |S_{1,f}(t+h) - S_{1,f}(t)|^{2l}\, dt,$$

where

$$S_{1,f}(t) := \int_0^t S_f(u)\, du$$

with the error terms uniform in integers $l \geq 1$ and $h > 0$ with a suitable value of h. It should be mentioned that the asymptotic formulae for higher moments of $S(t)$ over shorter intervals have been extensively studied earlier in [3], [4], [5] and [6].

In fact, first we establish the following theorems from which the main theorem follows. The constants B and B' occurring in the sequel are as in Theorem A$'$, which we do not mention hereafter.

Theorem 1. *Let $B' < \alpha \leq 1$. If $T^\alpha \leq H \leq T$, then there is an absolute positive constant $A_1 = A_1(\alpha)$ such that for any integer l satisfying*

$$1 \leq l \ll (\log\log T)^{1/3},$$

we have

$$\int_T^{T+H} |S_f(t)|^{2l}\, dt = \frac{(2l)!}{l!}\left(\frac{1}{2\pi}\right)^{2l} H(\log\log T)^l + O\big(A_1^l l^{l-(1/2)} H(\log\log T)^{l-(1/2)}\big),$$

where the implied constants depend at most on α.

Theorem 2. *Let $B' < \alpha \leq 1$. If $(T+h)^\alpha \leq H \leq T$, then there is an absolute positive constant $A_2 = A_2(\alpha)$ such that for any integer l, with*

$$1 \leq l \ll (\log\log T)^{1/3},$$

and any h satisfying

$$(\log T)^{1/2} < h^{-1} < \frac{1}{10l}\log T,$$

we have

$$\int_T^{T+H} |S_{1,f}(t+h) - S_{1,f}(t)|^{2l}\,dt = \frac{(2l)!}{l!}\left(\frac{h}{2\pi}\right)^{2l} H(\log h^{-1})^l + O\big(A_2^l l^{l-(1/2)} H h^{2l} (\log\log T)^{l-(1/2)}\big).$$

Remark 3. Theorems 1 and 2 are analogous results of Theorems 2 and 3 of [5]. However, here the range of l as well as the error terms have been improved. In fact, Theorems 2 and 3 of [5] hold with this range of l as well as with this error term, which can be easily noticed from our arguments.

As a consequence of Theorems 1 and 2, we obtain

Theorem 3. *Let $B' < \alpha \le 1$. If $T^\alpha \le H \le T$, then for any given $\delta > 0$, we have*

$$\int_T^{T+H} |S_f(t)|\,dt = \frac{2}{\sqrt{\pi}}\frac{H}{2\pi}(\log\log T)^{(1/2)} + O_\delta\Big(H\big((\log\log T)(\log\log\log T)^{-(1/2)+\delta}\big)^{(1/2)}\Big),$$

where the implied constants depend on α and δ.

Theorem 4. *Let $B' < \alpha \le 1$. If $(T+h)^\alpha \le H \le T$, then for any given $\delta > 0$ and any h satisfying*

$$(\log T)^{1/2} < h^{-1} < \varepsilon_1 \frac{\log T}{\log\log T},$$

for some suitable constant $\varepsilon_1 = \varepsilon_1(\alpha) > 0$, we have

$$\int_T^{T+H} |S_{1,f}(t+h) - S_{1,f}(t)|\,dt = \frac{2}{\sqrt{\pi}}\frac{Hh}{2\pi}(\log h^{-1})^{1/2} + O\Big(Hh\big((\log\log T)(\log\log\log T)^{-(1/2)+\delta}\big)^{1/2}\Big),$$

where the implied constants depend on α and δ.

Remark 4. We prove Theorems 1 and 2 in detail adapting the approach of [5] to our situation. However, we need an asymptotic estimate for the quantity $\sum_{p\le x} a_p^2 \log p/p^{k-1}$ which is proved in Section 4 using Shimura's split of the Rankin–Selberg L-function into the ordinary Riemann zeta-function and the symmetric square L-function associated to a Hecke eigenform f for the full modular group. Apart from this, Theorem A' plays a crucial role (on the whole) particularly in proving the main theorem over shorter intervals.

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2. Notation and preliminaries

Throughout the paper, the implied constants A are effective absolute positive constants and they need not be the same at each occurrence. When k is even, it is known that a_n s are real. In fact, they are totally real algebraic numbers. Hence a_p is real from (1.1) and (1.3). By Deligne's estimate, we also have $|a_p| \leq 2p^{(k-1)/2}$. We define a real number A'_p such that $a_p = 2A'_p p^{(k-1)/2}$, and hence, $|A'_p| \leq 1$. Let α'_p and $\overline{\alpha'_p}$ be the roots of the equation $x^2 - 2A'_p x + 1 = 0$ and we note that $|\alpha'_p| = 1$. Therefore, from the Euler product of $L_f(s)$, we can write

$$L_f(s) = \prod_p (1 - \alpha_p p^{-s})^{-1}(1 - \overline{\alpha_p} p^{-s})^{-1} \tag{2.1}$$

with $|\alpha_p| = p^{(k-1)/2}$ and $a_p = \alpha_p + \overline{\alpha_p}$. Taking the logarithm and differentiating both sides of (2.1) with respect to s, we find that

$$-\frac{L'_f(s)}{L_f(s)} = \sum_{m\geq 1,p} (\alpha_p^m + \overline{\alpha_p}^m) p^{-ms}(\log p). \tag{2.2}$$

Now we define

$$\Lambda_f(n) = (\alpha_p^m + \overline{\alpha_p}^m)(\log p) \quad \text{if } n = p^m;\ 0 \text{ otherwise.} \tag{2.3}$$

Hence we obtain

$$-\frac{L'_f(s)}{L_f(s)} = \sum_{n=2}^{\infty} \Lambda_f(n) n^{-s} \quad (\text{in } \operatorname{Re} s > (k+1)/2). \tag{2.4}$$

Note that

$$\Lambda_f(n) \leq 2(\log n) n^{(k-1)/2}. \tag{2.5}$$

For $x > 1$, we define

$$\Lambda_{x,f}(n) = \begin{cases} \Lambda_f(n), & \text{if } 1 \leq n \leq x, \\ \Lambda_f(n) \dfrac{\left\{\left(\log\left(\dfrac{x^3}{n}\right)\right)^2 - 2\left(\log\left(\dfrac{x^2}{n}\right)\right)^2\right\}}{2(\log x)^2}, & \text{if } x \leq n \leq x^2, \\ \Lambda_f(n) \dfrac{\left(\log\left(\dfrac{x^3}{n}\right)\right)^2}{2(\log x)^2} & \text{for } x^2 \leq n \leq x^3. \end{cases} \tag{2.6}$$

3. Some lemmas

Lemma 3.1. *Let τ be a real positive number and suppose that $\delta(n)$ are complex numbers satisfying*

$$|\delta(n)| \leq C$$

for some fixed constant $C > 0$. Then, for any integer $l \geq 1$, we have

$$\begin{aligned} S_1 &:= \sum_{\substack{p_1,\cdots,p_l<y,\\ q_1,\cdots,q_l<y,\\ p_1\cdots p_l=q_1\cdots q_l}} \frac{\delta(p_1)\cdots\delta(p_l)\delta(q_1)\cdots\delta(q_l)}{(p_1\cdots p_l q_1\cdots q_l)^\tau} \\ &= l!\Big(\sum_{p<y}\frac{\delta^2(p)}{p^{2\tau}}\Big)^l + O\Big(C^{2l}l!\Big(\sum_{p<y}p^{-2\tau}\Big)^{l-2}\Big(\sum_{p<y}p^{-4\tau}\Big)\Big). \end{aligned}$$

Proof. See, for example, Lemma 1 of [5]. □

For $x \geq 2$, $t > 0$, we define the number $\sigma_{x,t}$ by

$$\sigma_{x,t} = k/2 + 2\max\big(\beta - k/2, 2/\log x\big),$$

where $\varrho = \beta + i\gamma$ runs over all zeros of $L_f(s)$ for which

$$|t-\gamma| \leq x^{3|\beta-k/2|}(\log x)^{-1}.$$

Lemma 3.2. *Suppose that $T^\alpha \leq H \leq T$, where $B' < \alpha \leq 1$ and $x \geq 2$, $1 \leq \xi \leq x^{8l}$, $x^3\xi^2 \leq (H/T^{B'})^{1/4}$. Then, for $0 \leq \nu \leq 8l$, we have*

$$\begin{aligned} I_1 &:= \int_T^{T+H}\Big(\sigma_{x,t}-\frac{k}{2}\Big)^\nu \xi^{\sigma_{x,t}-(k/2)}\,dt \ll A^l\frac{H}{(\log x)^\nu} \\ &+ A^l\Bigg(H\log T\bigg((\nu)!\frac{\log T}{\log x}\Big(\frac{4}{\log\big(H/T^{B'}\big)}\Big)^{\nu+1} \\ &+ (\nu)!\frac{1}{\log x}\Big(\frac{4}{\log\big(H/T^{B'}\big)}\Big)^{\nu}\bigg)\Bigg). \end{aligned}$$

Proof. The proof follows using Theorem A′ at the appropriate place of the proof of Lemma 12 of [16]. □

Lemma 3.3. *Let $H > 1$, $l \geq 1$ and $1 < y \leq H^{1/l}$. Suppose that β_p are complex numbers satisfying*

$$|\beta_p| < B_1\frac{\log p}{\log y} \quad \textit{for } p < y. \tag{3.3.1}$$

Then, we have

$$\int_0^H \Big|\sum_{p<y} \beta_p p^{-(1/2)-it}\Big|^{2l} dt \ll (AB_1^2 l)^l H, \tag{3.3.2}$$

and if $|\beta_p| < B_1$, *then we have*

$$\int_0^H \Big|\sum_{p<y} \beta_p p^{-1-2it}\Big|^{2l} dt \ll (AB_1^2 l)^l H. \tag{3.3.3}$$

Proof. See, for example, Lemma 3 of [5]. ▫

Remark. It should be mentioned here that a general mean-value theorem for the Dirichlet polynomial with a better error term is also available, for which we refer to [10].

Lemma 3.4. *Let* $B' < \alpha \le 1$, $T^\alpha \le H \le T$ *and* $x = T^{(\alpha - B')/(60l)}$. *Then, for* $T \le t \le T+H$, *we have*

$$\begin{aligned} S_f(t) &+ \frac{1}{\pi}\sum_{p<x^3} \frac{(\alpha_p + \overline{\alpha_p})\sin(t\log p)}{p^{k/2}} \\ &= O\bigg(\Big|\sum_{p<x^3} \frac{\Lambda_f(p) - \Lambda_{x,f}(p)}{p^{k/2}\log p} p^{-it}\Big|\bigg) \\ &\quad + O\bigg(\Big|\sum_{p<x^{3/2}} \frac{\Lambda_{x,f}(p^2)}{p^k \log p} p^{-2it}\Big|\bigg) + O\bigg(\Big(\sigma_{x,t} - \frac{k}{2}\Big)\log T\bigg) \\ &\quad + O\bigg(\Big(\sigma_{x,t} - \frac{k}{2}\Big) x^{(\sigma_{x,t} - (k/2))} \int_{k/2}^\infty x^{(k/2)-\sigma} \Big|\sum_{p<x^3} \frac{\Lambda_{x,f}(p)\log(xp)}{p^{\sigma+it}}\Big| d\sigma\bigg). \end{aligned}$$

Proof. From Theorem A (stated in the introduction), we obtain

$$\begin{aligned} S_f(t) = &-\frac{1}{\pi}\sum_{p<x^3} \frac{\Lambda_{x,f}(p)\sin(t\log p)}{p^{\sigma_{x,t}}\log p} - \frac{1}{\pi}\sum_{p^2<x^3} \frac{\Lambda_{x,f}(p^2)\sin(t\log p^2)}{p^{2\sigma_{x,t}}(\log p^2)} \\ &+ O\bigg(\Big(\sigma_{x,t} - \frac{k}{2}\Big)\Big|\sum_{p<x^3} \frac{\Lambda_{x,f}(p)}{p^{\sigma_{x,t}+it}}\Big|\bigg) \\ &+ O\bigg(\Big(\sigma_{x,t} - \frac{k}{2}\Big)\Big|\sum_{p^2<x^3} \frac{\Lambda_{x,f}(p^2)}{p^{2\sigma_{x,t}+2it}}\Big|\bigg) \\ &+ O\bigg(\Big|\sum_{\substack{p^r<x^3\\ r>2}} \frac{\Lambda_{x,f}(p^r)\sin(t\log p^r)}{p^{r\sigma_{x,t}}(\log p^r)}\Big|\bigg) \\ &+ O\bigg(\Big(\sigma_{x,t} - \frac{k}{2}\Big)\Big|\sum_{\substack{p^r<x^3\\ r>2}} \frac{\Lambda_{x,f}(p^r)}{p^{r\sigma_{x,t}+rit}}\Big|\bigg) + O\bigg(\Big(\sigma_{x,t} - \frac{k}{2}\Big)\log t\bigg). \end{aligned} \tag{3.4.1}$$

Note that $\sigma_{x,t} \geq \frac{1}{2}k$ and

$$|\Lambda_{x,f}(n)| \leq |\Lambda_f(n)| \leq 2(\log n)n^{(k-1)/2}.$$

Now, it is easy to see that

$$\sum_{\substack{p^r<x^3\\ r>2}} \frac{\Lambda_{x,f}(p^r)\sin(t\log p^r)}{p^{r\sigma_{x,t}}(\log p^r)} = O(1) = O\bigg(\Big(\sigma_{x,t}-\frac{k}{2}\Big)\log T\bigg), \tag{3.4.2}$$

$$\begin{aligned}\Big(\sigma_{x,t}-\frac{k}{2}\Big)\bigg|\sum_{\substack{p^r<x^3\\ r>2}} \frac{\Lambda_{x,f}(p^r)}{p^{r\sigma_{x,t}+rit}}\bigg| &= O\bigg(\Big(\sigma_{x,t}-\frac{k}{2}\Big)\bigg)\\ &= O\bigg(\Big(\sigma_{x,t}-\frac{k}{2}\Big)\log T\bigg)\end{aligned} \tag{3.4.3}$$

and

$$\begin{aligned}\Big(\sigma_{x,t}-\frac{k}{2}\Big)\bigg|\sum_{p^2<x^3} \frac{\Lambda_{x,f}(p^2)}{p^{2\sigma_{x,t}+2it}}\bigg| &= O\bigg(\Big(\sigma_{x,t}-\frac{k}{2}\Big)\log x\bigg)\\ &= O\bigg(\Big(\sigma_{x,t}-\frac{k}{2}\Big)\log T\bigg).\end{aligned} \tag{3.4.4}$$

Now, we write the first four terms on the right-hand side of (3.4.1) in the following manner, namely,

$$\begin{aligned} S_f(t) &+ \frac{1}{\pi}\sum_{p<x^3}\frac{(\alpha_p+\overline{\alpha_p})\sin(t\log p)}{p^{k/2}}\\ &= O\bigg(\bigg|\sum_{p<x^3}\frac{\Lambda_f(p)-\Lambda_{x,f}(p)}{p^{k/2}\log p}p^{-it}\bigg|\bigg)\\ &\quad + O\bigg(\bigg|\sum_{p<x^3}\frac{\Lambda_{x,f}(p)}{p^{k/2}\log p}(1-p^{(k/2)-\sigma_{x,t}})p^{-it}\bigg|\bigg)\\ &\quad + O\bigg(\Big(\sigma_{x,t}-\frac{k}{2}\Big)\bigg|\sum_{p<x^3}\frac{\Lambda_{x,f}(p)}{p^{\sigma_{x,t}+it}}\bigg|\bigg)\\ &\quad + O\bigg(\bigg|\sum_{p<x^{3/2}}\frac{\Lambda_{x,f}(p^2)}{p^{k}\log p}p^{-2it}\bigg|\bigg)\\ &\quad + O\bigg(\bigg|\sum_{p<x^{3/2}}\frac{\Lambda_{x,f}(p^2)}{p^{k}\log p}(1-p^{k-2\sigma_{x,t}})p^{-2it}\bigg|\bigg)\\ &\quad + O\bigg(\Big(\sigma_{x,t}-\frac{k}{2}\Big)\log T\bigg).\end{aligned} \tag{3.4.5}$$

We note that

$$
\begin{aligned}
Q_1 &:= \Big| \sum_{p<x^{3/2}} \frac{\Lambda_{x,f}(p^2)}{p^k \log p}(1-p^{k-2\sigma_{x,t}})p^{-2it}\Big| \\
&< \sum_{p<x^{3/2}} \frac{2(\log p)p^{k-1}}{p^k \log p}(1-p^{k-2\sigma_{x,t}}) \qquad (3.4.6)\\
&< \sum_{p<x^{3/2}} \frac{4\big(\sigma_{x,t}-\frac{1}{2}k\big)\log p}{p} = O\big(\big(\sigma_{x,t}-\tfrac{1}{2}k\big)\log T\big),
\end{aligned}
$$

since

$$\sigma_{x,t} \geq \frac{k}{2} + \frac{4}{\log x}$$

and $1-e^{-x}<x$. Further, we have

$$
\begin{aligned}
Q_2 &:= \Big| \sum_{p<x^3} \frac{\Lambda_{x,f}(p)}{p^{k/2}\log p}\big(1-p^{(k/2)-\sigma_{x,t}}\big)p^{-it}\Big| \qquad (3.4.7)\\
&= \Big| \int_{k/2}^{\sigma_{x,t}} \sum_{p<x^3} \frac{\Lambda_{x,f}(p)}{p^{\sigma'+it}}\, d\sigma' \Big| \leq \int_{k/2}^{\sigma_{x,t}} \Big| \sum_{p<x^3} \frac{\Lambda_{x,f}(p)}{p^{\sigma'+it}} \Big| \, d\sigma'.
\end{aligned}
$$

If $\frac{1}{2}k \leq \sigma' \leq \sigma_{x,t}$, then

$$
\begin{aligned}
\Big| \sum_{p<x^3} \frac{\Lambda_{x,f}(p)}{p^{\sigma'+it}} \Big| &= \Big| x^{\sigma'-(k/2)} \int_{\sigma'}^{\infty} x^{(k/2)-\sigma} \sum_{p<x^3} \frac{\Lambda_{x,f}(p)(\log xp)}{p^{\sigma+it}}\, d\sigma \Big| \qquad (3.4.8)\\
&\leq x^{\sigma_{x,t}-(k/2)} \int_{k/2}^{\infty} x^{(k/2)-\sigma} \Big| \sum_{p<x^3} \frac{\Lambda_{x,f}(p)(\log xp)}{p^{\sigma+it}} \Big| \, d\sigma,
\end{aligned}
$$

and therefore, from (3.4.7) and (3.4.8), we get

$$Q_2 \leq \left(\sigma_{x,t} - \frac{k}{2}\right) x^{\sigma_{x,t}-(k/2)} \int_{k/2}^{\infty} x^{(k/2)-\sigma} \Big| \sum_{p<x^3} \frac{\Lambda_{x,f}(p)(\log xp)}{p^{\sigma+it}} \Big| \, d\sigma. \qquad (3.4.9)$$

Now, the lemma follows from (3.4.5), (3.4.6) and (3.4.9). □

Lemma 3.5. *Let $B' < \alpha \leq 1$ and suppose that $T^\alpha \leq H \leq T$. Put $x = T^{(\alpha-B')/(60l)}$. Then, for $l \ll \log T$, we have*

$$\int_T^{T+H} \Big| S_f(t) + \frac{1}{\pi} \sum_{p<x^3} \frac{(\alpha_p + \overline{\alpha_p})\sin(t\log p)}{p^{k/2}} \Big|^{2l} dt \ll A^l l^{2l} H.$$

Proof. Let

$$\sum_1(t) := \sum_{p<x^3} \frac{(\alpha_p + \overline{\alpha_p})\sin(t\log p)}{p^{k/2}}, \tag{3.5.1}$$

$$E_1(t) := \sum_{p<x^3} \frac{\Lambda_f(p) - \Lambda_{x,f}(p)}{p^{k/2}\log p} p^{-it}, \tag{3.5.2}$$

$$E_2(t) := \sum_{p<x^{3/2}} \frac{\Lambda_{x,f}(p^2)}{p^k \log p} p^{-2it}, \tag{3.5.3}$$

$$E_3(t) := \left(\sigma_{x,t} - \tfrac{1}{2}k\right)\log T, \tag{3.5.4}$$

and

$$E_4(t) := \left(\sigma_{x,t} - \tfrac{1}{2}k\right) x^{(\sigma_{x,t}-(k/2))} \int_{k/2}^{\infty} x^{(k/2)-\sigma} \left| \sum_{p<x^3} \frac{\Lambda_{x,f}(p)\log(xp)}{p^{\sigma+it}} \right| d\sigma. \tag{3.5.5}$$

Now, clearly from Lemma 3.4, we have

$$\left| S_f(t) + \pi^{-1} \sum_1(t) \right|^{2l} \ll A^l \left(|E_1(t)|^{2l} + |E_2(t)|^{2l} + |E_3(t)|^{2l} + |E_4(t)|^{2l} \right). \tag{3.5.6}$$

If we take

$$\beta_p = \frac{\Lambda_f(p) - \Lambda_{x,f}(p)}{p^{(k-1)/2}\log p},$$

then from the definition of $\Lambda_f(n)$ and $\Lambda_{x,f}(n)$, we easily find that

$$\beta_p = 0 \quad \text{for } 2 \le p \le x,$$
$$|\beta_p| \le 2\left(\frac{\log p}{\log x} - 1\right)^2 \le 2\frac{\log p}{\log x} \quad \text{for } x \le p \le x^2,$$

and

$$|\beta_p| \le 6\frac{\log p}{\log x} \quad \text{for } x^2 \le p \le x^3.$$

Therefore,

$$|\beta_p| \le B_1 \frac{\log p}{\log x} \quad \text{for } p \le x^3$$

with some absolute positive constant B_1. Similarly, if we take

$$\beta'_p = \frac{\Lambda_{x,f}(p^2)}{p^{k-1}\log p},$$

then from the definition of $\Lambda_{x,f}(n)$, we find that

$$\Lambda_{x,f}(p^2) \le 9p^{k-1}(\log p),$$

and so we get $|\beta'_p| < B_2$ with some absolute positive constant B_2. Therefore, from Lemma 3.3, ((3.3.2), (3.3.3), respectively), we obtain

$$\int_T^{T+H} |E_1(t)|^{2l}\,dt \ll (Al)^l H \tag{3.5.7}$$

and

$$\int_T^{T+H} |E_2(t)|^{2l}\,dt \ll (Al)^l H. \tag{3.5.8}$$

Note that we have fixed $x = T^{(\alpha-B')/(60l)}$. From Lemma 3.2, with $\xi = 1$ and $\nu = 2l$, we get

$$\int_T^{T+H} |E_3(t)|^{2l}\,dt \ll A^l\big(l(2l)! + l^{2l}\big)H \ll A^l l(2l)^{2l-1}H \ll A^l l^{2l} H, \tag{3.5.9}$$

since,

$$(2l)! \le (2l)^{2l-1} \quad \text{for } l \ge 1, \tag{3.5.10A}$$

(3.5.10B)

$$S_2 := H\log T\left((\nu)!\frac{\log T}{\log x}\left(\frac{4}{\log\big(H/T^{B'}\big)}\right)^{\nu+1} + (\nu)!\frac{1}{\log x}\left(\frac{4}{\log\big(H/T^{B'}\big)}\right)^{\nu}\right)$$
$$\ll \nu!\frac{H}{(\log x)(\log T)^{\nu-1}}$$

and

$$\frac{H}{(\log x)^\nu} \ll A^l l^{2l} H. \tag{3.5.10C}$$

Now, we notice that

$$\int_T^{T+H} |E_4(t)|^{2l}\,dt \le Q_3 Q_4 \tag{3.5.11}$$

where

$$Q_3 := \left(\int_T^{T+H} \big(\sigma_{x,t} - \tfrac{1}{2}k\big)^{4l} x^{4l(\sigma_{x,t}-(k/2))}\,dt\right)^{1/2}$$

and

$$Q_4 := \left(\int_T^{T+H}\left(\int_{k/2}^{\infty} x^{(k/2)-\sigma}\left|\sum_{p<x^3}\frac{\Lambda_{x,f}(p)(\log xp)}{p^{\sigma+it}}\right| d\sigma\right)^{4l} dt\right)^{1/2}.$$

From Lemma 3.2, (with $\xi = x^{4l}$, $\nu = 4l$), we obtain

$$Q_3 \ll \left(A^l(l^{4l} + l(4l)!)H(\log T)^{-4l}\right)^{1/2} \ll A^l l^{2l} H^{1/2}(\log T)^{-2l}. \tag{3.5.12}$$

By Hölder's inequality, we get

$$\begin{aligned} Q_4^2 &\le \int_T^{T+H}\left(\int_{k/2}^{\infty} x^{(k/2)-\sigma}\, d\sigma\right)^{4l-1} \\ &\quad\times \left(\int_{k/2}^{\infty} x^{(k/2)-\sigma}\left|\sum_{p<x^3}\frac{\Lambda_{x,f}(p)(\log xp)}{p^{\sigma+it}}\right|^{4l} d\sigma\right) dt \\ &\le (\log x)^{1-4l}\left(\int_{k/2}^{\infty} x^{(k/2)-\sigma}\right. \\ &\quad\times \left.\left(\int_T^{T+H}\left|\sum_{p<x^3}\frac{\Lambda_{x,f}(p)(\log xp)}{p^{\sigma+it}}\right|^{4l} dt\right) d\sigma\right). \end{aligned} \tag{3.5.13}$$

By taking

$$\beta_p = \frac{\Lambda_{x,f}(p)(\log xp)}{p^{(k-1)/2}(\log x)^2},$$

we observe that $|\beta_p| \le 10\log p/\log x$. Now, by (3.3.2), we obtain

$$\int_T^{T+H}\left|\sum_{p<x^3}\frac{\Lambda_{x,f}(p)(\log xp)}{p^{\sigma+it}}\right|^{4l} dt \ll (AB_1^2 l)^{2l} H(\log x)^{8l}. \tag{3.5.14}$$

Therefore, we get from (3.5.13) and (3.5.14)

$$Q_4^2 \ll (AB_1^2 l)^{2l} H(\log x)^{4l}. \tag{3.5.15}$$

From (3.5.11), (3.5.12) and (3.5.15), with our choice of x, we get

$$\begin{aligned} \int_T^{T+H} |E_4(t)|^{2l}\, dt &\ll A^l l^{2l} H^{1/2}(\log T)^{-2l}(AB_1^2 l)^l H^{1/2}(\log x)^{2l} \\ &\ll A^l l^l H. \end{aligned} \tag{3.5.16}$$

This proves the lemma. □

Lemma 3.6. *Let $B' < \alpha \le 1$ and $T^\alpha \le H \le T$. Then, if $l \ge 1$ is an integer and*

$$x^3 = T^{(\alpha - B')/(20l)} \le z \le H^{1/l},$$

we have

$$Q_5 := \int_T^{T+H} \left| S_f(t) + \frac{1}{\pi} \sum_{p<z} \frac{(\alpha_p + \overline{\alpha_p}) \sin(t \log p)}{p^{k/2}} \right|^{2l} dt \ll A^l l^{2l} H. \tag{3.6.1}$$

Proof. We clearly have

$$\begin{aligned} Q_5 \ll 4^l \int_T^{T+H} \left| S_f(t) + \frac{1}{\pi} \sum_{p<x^3} \frac{(\alpha_p + \overline{\alpha_p}) \sin(t \log p)}{p^{k/2}} \right|^{2l} dt \\ + 4^l \int_T^{T+H} \left| \sum_{x^3 \le p < z} p^{-(1/2)-it} \right|^{2l} dt. \end{aligned} \tag{3.6.2}$$

From Lemma 3.5, we observe that

$$\int_T^{T+H} \left| S_f(t) + \frac{1}{\pi} \sum_{p<x^3} \frac{(\alpha_p + \overline{\alpha_p}) \sin(t \log p)}{p^{k/2}} \right|^{2l} dt \ll A^l l^{2l} H. \tag{3.6.3}$$

From (3.3.2) (with $B_1 = O(1)$), we have

$$\int_T^{T+H} \left| \sum_{x^3 \le p < z} p^{-(1/2)-it} \right|^{2l} dt \ll A^l l^l H. \tag{3.6.4}$$

In the notation of Lemma 3.3,

$$\beta_p = 1 = \frac{\log p}{\log z} \frac{\log z}{\log p} \ll \frac{\log p}{\log z}$$

so that (3.3.1) is satisfied with z in place of y. This proves the lemma. □

4. Prime number theorem related to the Dirichlet series $\sum_{n=1}^\infty a_n^2/n^s$

We know that

$$L_f(s) = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\overline{\alpha_p}}{p^s}\right)^{-1} = \sum_{n=1}^\infty \frac{a_n}{n^s} \tag{4.1}$$

is an entire function, $|\alpha_p| = p^{(k-1)/2}$, $\alpha_p\overline{\alpha_p} = p^{k-1}$ and $a_p = \alpha_p + \overline{\alpha_p}$. Now, let

$$L_{f^2}(s) := \sum_{n=1}^{\infty} \frac{a_n^2}{n^s} \tag{4.2}$$

and

$$L_{f\otimes f}(s) = \prod_p \left(1 - \frac{\alpha_p^2}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p\overline{\alpha_p}}{p^s}\right)^{-1} \left(1 - \frac{\overline{\alpha_p}^2}{p^s}\right)^{-1}, \tag{4.3}$$

where the symbol $\otimes$ in (4.3) denotes the Rankin–Selberg convolution. The important relation between (4.2) and (4.3) is given by (see [12], [11], [17] and [13])

$$\zeta(s-k+1)L_{f\otimes f}(s) = \zeta(2s-2k+2)L_{f^2}(s), \tag{4.4}$$

where $\zeta(s)$ is the ordinary Riemann zeta-function. It has been proved by Rankin (see [12]) that $L_{f^2}(s)$ has a simple pole at $s = k$ with residue $k\alpha$ (α is a certain constant). Therefore, the series $-\big(L'_{f^2}(k-1+s)\big)/\big(L_{f^2}(k-1+s)\big)$ has a simple pole at $s = 1$ with residue 1.

We define

$$\Lambda^*(n) = \begin{cases} \dfrac{\big(\alpha_p^{2m} + \overline{\alpha_p}^{2m} + \big(\alpha_p\overline{\alpha_p}\big)^m + (-1)^{m+1}(\alpha_p\overline{\alpha_p})^m\big)\log p}{p^{m(k-1)}}, & \text{if } n = p^m, \\ 0, & \text{otherwise.} \end{cases} \tag{4.5}$$

We have the usual von Mangoldt's function, namely,

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m, \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

We also define $\Psi^*_{f^2}(x)$ and $\Psi_{f^2}(x)$ by

$$\Psi^*_{f^2}(x) = \sum_{n\le x} \Lambda^*(n)(x-n) \tag{4.7}$$

and

$$\Psi^*_{f^2}(x) = \int_0^x \Psi_{f^2}(u)\,du = \int_1^x \Psi_{f^2}(u)\,du. \tag{4.8}$$

It is obvious that

$$\Psi_{f^2}(x) = \sum_{n\le x} \Lambda^*(n). \tag{4.9}$$

The aim of this section is to prove:

Theorem 4.1. *For $x \geq x_0$, we have*

$$\Psi_{f^2}(x) = x + O\big(xe^{-C\sqrt{\log x}}\big).$$

To prove this theorem, we need the following lemmas.

Lemma 4.1. *There exists a positive constant C (>0) such that*

$$L_{f^2}(k-1+s) \neq 0 \text{ in } \sigma > 1 - \frac{C}{\log(|t|+2)}.$$

Proof. See, for example, [8]. □

Lemma 4.2. *Suppose that $L_{f^2}(s)$ has no zeros in the domain*

$$\sigma > 1 - \eta(|t|),$$

where $\eta(t)$, for $t \geq 0$, a decreasing function, has a continuous derivative $\eta'(t)$ and satisfies

(i) $0 < \eta(t) < \frac{1}{2}$,
(ii) $\eta'(t) \to 0$ *as* $t \to \infty$,
(iii) $\dfrac{1}{\eta(t)} = O(\log t)$ *as* $t \to \infty$.

Let α_1' be a fixed number satisfying $0 < \alpha_1' < 1$. Then,

$$-\frac{L'_{f^2}(s)}{L_{f^2}(s)} = O\big(\log^2(|t|)\big)$$

uniformly in the region $\sigma \geq 1 - \alpha_1'\eta(|t|)$ as $t \to \pm\infty$.

Proof. Since we have an Euler-product representation for $L_{f^2}(s)$ from (4.3) and (4.4), the proof of this lemma follows in a similar fashion to that of Theorem 20 of [8]. □

Lemma 4.3. *Under the conditions of Lemma 4.2, we have*

$$\Psi^*_{f^2}(x) = \tfrac{1}{2}x^2 + O\big(x^2 e^{-\alpha_1'\omega(x)}\big)$$

as $x \to \infty$, where $\omega(x)$ is the minimum of $\eta(t)\log x + \log t$ for $t \geq 1$.

Proof. First of all, we note that (for $C>1$)

(4.3.1)

$$\begin{aligned}\Psi^*_{f^2}(x) &= \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{x^{s+1}}{s(s+1)}\Big(-\frac{L'_{f^2}(k-1+s)}{L_{f^2}(k-1+s)}\Big)\,ds\\ &= \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{x^{s+1}}{s(s+1)}\Big(-\frac{L'_{f^2}(k-1+s)}{L_{f^2}(k-1+s)}-\frac{\zeta'(2s)}{\zeta(2s)}+\frac{\zeta'(2s)}{\zeta(2s)}\Big)\,ds\\ &= \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{x^{s+1}}{s(s+1)}\Big(-\frac{L'_{f\otimes f}(k-1+s)}{L_{f\otimes f}(k-1+s)}-\frac{\zeta'(s)}{\zeta(s)}+\frac{\zeta'(2s)}{\zeta(2s)}\Big)\,ds\\ &= \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{x^{s+1}}{s(s+1)}\Big(-\frac{L'_{f\otimes f}(k-1+s)}{L_{f\otimes f}(k-1+s)}-\frac{\zeta'(s)}{\zeta(s)}\Big)\,ds+O\big(x^{7/4}\big),\end{aligned}$$

since

$$\begin{aligned}\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{x^{s+1}}{s(s+1)}\Big(-\frac{\zeta'(2s)}{\zeta(2s)}\Big)\,ds &= \frac{1}{2\pi i}\int_{C-iT}^{C+iT}\frac{x^{s+1}}{s(s+1)}\Big(-\frac{\zeta'(2s)}{\zeta(2s)}\Big)\,ds\\ &\quad+O\Big(\frac{x^{C+1}}{T}\Big).\end{aligned}$$

Now, by moving the line of integration to $\sigma=\frac{3}{4}$, we see that the horizontal portions contribute an error which is in the absolute value at most $O(x^{C+1}/T)$, and the vertical portion contributes at most $O(x^{7/4})$. We can choose $C=1+\varepsilon$ (ε is a small positive constant) and $T=x^{1/2}$. From (4.3.1), we get

(4.3.2)

$$\frac{\Psi^*_{f^2}(x)}{x^2}=\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{x^{s-1}}{s(s+1)}\Big(-\frac{L'_{f\otimes f}(k-1+s)}{L_{f\otimes f}(k-1+s)}-\frac{\zeta'(s)}{\zeta(s)}\Big)\,ds+O(x^{-1/4}).$$

Now, we move the line of integration of the integral appearing on the right-hand side of (4.3.2) to $\sigma=1-\alpha'_1\eta(|t|)$. Therefore, this lemma follows when applying Lemmas 4.1 and 4.2.

Now, from Lemma 4.3, Theorem 4.1 follows by standard arguments (see, for example, [8]). ◻

5. Proof of Theorem 1

We fix $z=T^{\alpha/(5l)}$. Notice that $\alpha_p+\overline{\alpha_p}=a_p$. Let us write

$$\Delta_z(t):=\Delta(t):=S_f(t)+\frac{1}{\pi}\sum_{p<z}\frac{a_p\sin(t\log p)}{p^{k/2}}. \tag{5.1}$$

Then, from the binomial theorem, we have

$$
\begin{aligned}
\big(S_f(t)\big)^{2l} &= \Big(\frac{1}{\pi}\sum_{p<z}\frac{a_p\sin(t\log p)}{p^{k/2}}\Big)^{2l}\\
&\qquad + \sum_{j=1}^{2l}\binom{2l}{j}\Delta^j(t)\Big(-\frac{1}{\pi}\sum_{p<z}\frac{a_p\sin(t\log p)}{p^{k/2}}\Big)^{2l-j}\\
&= Q_6 + Q_7, \quad \text{say.}
\end{aligned}
\tag{5.2}
$$

We observe that

$$
Q_7 \ll 4^l\, l\,|\Delta(t)|\Big(|\Delta(t)|^{2l-1} + \Big|\sum_{p<z}\frac{a_p\sin(t\log p)}{p^{k/2}}\Big|^{2l-1}\Big).
$$

Therefore, we obtain (using Hölder's inequality)

$$
\begin{aligned}
Q_8 &:= \int_T^{T+H}|S_f(t)|^{2l}\,dt - \frac{1}{\pi^{2l}}\int_T^{T+H}\Big|\sum_{p<z}\frac{a_p\sin(t\log p)}{p^{k/2}}\Big|^{2l}\,dt\\
&\ll A^l\int_T^{T+H}|\Delta(t)|^{2l}\,dt + A^l\int_T^{T+H}|\Delta(t)|\,\Big|\sum_{p<z}\frac{a_p\sin(t\log p)}{p^{k/2}}\Big|^{2l-1}\,dt\\
&\ll A^l\int_T^{T+H}|\Delta(t)|^{2l}\,dt\\
&\quad + A^l\Big(\int_T^{T+H}|\Delta(t)|^{2l}\,dt\Big)^{1/2l}\Big(\int_T^{T+H}\Big|\sum_{p<z}\frac{a_p\sin(t\log p)}{p^{k/2}}\Big|^{2l}\,dt\Big)^{1-(1/2l)}.
\end{aligned}
\tag{5.3}
$$

Let $\eta_1 := \eta_1(t) := \sum_{p<z} a_p p^{-(k/2)-it}$, and hence,

$$
\sum_{p<z} a_p p^{-k/2}\sin(t\log p) = \frac{i}{2}\big(\eta_1 - \overline{\eta_1}\big).
\tag{5.4}
$$

Therefore, from the binomial expansion, we obtain

$$
\begin{aligned}
Q_9 &:= \int_T^{T+H}\Big|\sum_{p<z}\frac{a_p\sin(t\log p)}{p^{k/2}}\Big|^{2l}\,dt\\
&= \Big(\frac{1}{2}\Big)^{2l}\sum_{j=0}^{2l}(-1)^j\binom{2l}{j}\int_T^{T+H}\eta_1^j\,\overline{\eta_1}^{(2l-j)}\,dt\\
&= 2^{-2l}\frac{(2l)!}{(l!)^2}\int_T^{T+H}|\eta_1(t)|^{2l}\,dt\\
&\quad + O\Big(4^{-l}\sum_{\substack{j=0,1,\ldots,2l\\ j\neq l}}\binom{2l}{j}\Big|\int_T^{T+H}\eta_1^j\,\overline{\eta_1}^{(2l-j)}\,dt\Big|\Big).
\end{aligned}
\tag{5.5}
$$

We note that the integral in the error term of (5.5) is

$$\ll \sum_{\substack{p_1,\cdots,p_j<z\\ q_1,\cdots,q_{(2l-j)}<z}} \frac{a_{p_1}\cdots a_{p_j}a_{q_1}\cdots a_{q_{(2l-j)}}}{(p_1\cdots p_j q_1\cdots q_{(2l-j)})^{k/2}}\left|\log\left(\frac{p_1\cdots p_j}{q_1\cdots q_{(2l-j)}}\right)\right|^{-1}. \tag{5.6}$$

We note that $|a_p|\le 2p^{(k-1)/2}$ and $z=T^{\alpha/(5l)}$. Since

$$\min\left(\frac{1}{a},\frac{1}{b}\right)\le\left|\log\left(\frac{a}{b}\right)\right| \tag{5.7}$$

for any two distinct positive integers a and b, from (5.6) and (5.7) (for $j\neq l$), we get,

$$\int_T^{T+H}\eta_1^j\overline{\eta_1}^{(2l-j)}\,dt\ll z^{2l}\left(\sum_{p<z}|a_p|p^{-k/2}\right)^{2l}\ll A^l z^{3l}\ll A^l H. \tag{5.8}$$

Therefore, the error term in (5.5) is

$$\ll A^l H. \tag{5.9}$$

Now,

$$\begin{aligned} I_2 &:= \int_T^{T+H}|\eta_1(t)|^{2l}\,dt\\ &= H\sum_{\substack{p_1,\dots,p_l<z\\ q_1,\dots,q_l<z\\ p_1\cdots p_l=q_1\cdots q_l}}\frac{a_{p_1}\cdots a_{p_l}a_{q_1}\cdots a_{q_l}}{(p_1\cdots p_l q_1\cdots q_l)^{k/2}}\\ &\quad+O\left(\sum_{\substack{p_1,\dots,p_l<z\\ q_1,\dots,q_l<z\\ p_1\cdots p_l\neq q_1\cdots q_l}}\frac{a_{p_1}\cdots a_{p_l}a_{q_1}\cdots a_{q_l}}{(p_1\cdots p_l q_1\cdots q_l)^{k/2}}\left|\log\left(\frac{p_1\cdots p_l}{q_1\cdots q_l}\right)\right|^{-1}\right). \end{aligned} \tag{5.10}$$

Arguments similar to (5.6) yield the error term in (5.10) as

$$\ll A^l H. \tag{5.11}$$

Since $|a_p|\le 2p^{(k-1)/2}$, we have $|\delta(p)|:=|a_p/p^{(k-1)/2}|\le 2$. Therefore, choosing $C=2$ and $\tau=\frac{1}{2}$ in Lemma 3.1, we obtain the first term on the right-hand side of (5.10) as

$$= Hl!\left(\sum_{p<z}\frac{a_p^2}{p^k}\right)^l+O\left(H2^{2l}l!\left(\sum_{p<z}p^{-1}\right)^{l-2}\left(\sum_{p<z}p^{-2}\right)\right). \tag{5.12}$$

We note that (from Theorem 4.1),

$$\Psi_{f^2}(x) = \sum_{n\le x} \Lambda^*(n) = \sum_{p\le x} \frac{a_p^2 \log p}{p^{k-1}} + O\big(x^{1/2}\log x\big) = x + O\big(xe^{-C\sqrt{\log x}}\big), \tag{5.13}$$

and hence, using Abel's identity, we obtain

$$\sum_{p\le z} \frac{a_p^2}{p^k} = \log\log z + O(1) = \log\log T - \log(5l) + O(1). \tag{5.14}$$

Hence, from (5.10), (5.11), (5.12) and (5.14), we get

$$\int_T^{T+H} |\eta_1(t)|^{2l}\, dt = l!H(\log\log T)^l + O\big(A^l l!(\log l)H(\log\log T)^{l-1}\big). \tag{5.15}$$

Therefore, from (5.5), (5.9) and (5.15), we find that

$$\begin{aligned}\int_T^{T+H} \Bigg|\sum_{p<z} \frac{a_p \sin(t\log p)}{p^{k/2}}\Bigg|^{2l} dt &= \frac{(2l)!}{l!} 4^{-l} H(\log\log T)^l \\ &\quad + O\big(A^l l!(\log l)H(\log\log T)^{l-1}\big) \\ &\ll A^l l! H(\log\log T)^l,\end{aligned} \tag{5.16}$$

since $1\le l \ll (\log\log T)^{1/3}$. Note that we have used

$$\frac{(2l)!}{(l!)^2} = \binom{2l}{l} \le 2^{2l}.$$

From Lemma 3.6 and (5.16), we see that the right-hand side of (5.3) is

$$\ll (Al)^{2l}H + A^l l H^{1/2l}\big(A^l l^{l-1} H(\log\log T)^l\big)^{1-(1/2l)}, \tag{5.17}$$

since (for $l \ge 1$) we have

$$l! \le l^{l-1}. \tag{5.18}$$

Therefore, the right-hand side of (5.17) becomes the total error, which is

$$\ll (Al)^{2l}H + A^l l^{l-(1/2)} H(\log\log T)^{l-(1/2)}. \tag{5.19}$$

Note that

$$l^{2l} \ll l^{l-(1/2)}(\log\log T)^{l-(1/2)} \quad \text{provided } l \ll (\log\log T)^{(l-(1/2))/(l+(1/2))},$$

and

$$\min_{l\ge1}\left(\frac{l-\frac12}{l+\frac12}\right) = \min_{l\ge1}\left(1 - \frac{1}{l+\frac12}\right) = \frac13.$$

Hence, Theorem 1 holds with this error term

$$O\big(A^l l^{l-(1/2)} H(\log\log T)^{l-(1/2)}\big),$$

provided $1 \le l \ll (\log\log T)^{1/3}$. This proves Theorem 1.

6. Proof of Theorem 2

First, we write

$$\Delta_z(t) := \Delta(t) := S_f(t) + \pi^{-1} \sum_{p<z} \frac{a_p \sin(t \log p)}{p^{k/2}} := S_f(t) + \pi^{-1} \sum_2 (t).$$

Then,

$$S_{1,f}(t+h) - S_{1,f}(t) = \int_t^{t+h} S_f(u)\,du = -\pi^{-1} \int_t^{t+h} \sum_2 (u)\,du + \int_t^{t+h} \Delta(u)\,du.$$

Therefore,

$$\begin{aligned} \left|\int_t^{t+h} S_f(u)\,du\right|^{2l} &= \frac{1}{\pi^{2l}} \left|\int_t^{t+h} \sum_2 (u)\,du\right|^{2l} \\ &\quad + O\Big(A^l \Big|\int_t^{t+h} \Delta(u)\,du\Big|^{2l}\Big) \\ &\quad + O\Big(A^l \Big|\int_t^{t+h} \Delta(u)\,du\Big| \Big|\int_t^{t+h} \sum_2 (u)\,du\Big|^{2l-1}\Big) \end{aligned} \tag{6.1}$$

exactly as in (5.3). We notice that

$$\left|\int_t^{t+h} \Delta(u)\,du\right|^{2l} \le h^{2l-1} \int_t^{t+h} |\Delta(u)|^{2l}\,du,$$

and hence, by Hölder's inequality, we get

$$\begin{aligned} Q_{10} &:= \int_T^{T+H} \left|\int_t^{t+h} S_f(u)\,du\right|^{2l} dt \\ &= \frac{1}{\pi^{2l}} \int_T^{T+H} \left|\int_t^{t+h} \sum_2 (u)\,du\right|^{2l} dt \\ &\quad + O\Big(A^l h^{2l-1} \int_T^{T+H} \int_t^{t+h} |\Delta(u)|^{2l}\,du\Big) \\ &\quad + O\bigg(A^l \Big(h^{2l-1} \int_T^{T+H} \int_t^{t+h} |\Delta(u)|^{2l}\,du\,dt\Big)^{1/2l} \\ &\qquad \times \Big(\int_T^{T+H} \Big|\int_t^{t+h} \sum_2 (u)\,du\Big|^{2l} dt\Big)^{1-(1/2l)}\bigg). \end{aligned} \tag{6.2}$$

We notice that

$$\int_T^{T+H}\int_t^{t+h}|\Delta(u)|^{2l}\,du\,dt=\int_0^h du\int_{T+u}^{T+u+H}|\Delta(t)|^{2l}\,dt, \tag{6.3}$$

and hence, by Lemma 3.6, with $(T+h)^{\alpha}\le H\le T$, $B'<\alpha\le 1$ and

$$(T+h)^{(\alpha-B')/(20l)}\le z\le H^{1/l},$$

we have

$$\int_T^{T+H}|\Delta(t)|^{2l}\,dt\ll (Al)^{2l}H. \tag{6.4}$$

With these restrictions, we have

$$\begin{aligned} Q_{10}:=&\int_T^{T+H}\left|\int_t^{t+h}S_f(u)\,du\right|^{2l}dt\\ =&\frac{1}{\pi^{2l}}\int_T^{T+H}\left|\int_t^{t+h}\sum_2(u)\,du\right|^{2l}dt\\ &+O\biggl((Al)^{2l}h^{2l}H+A^l lH^{1/2l}h\Bigl(\int_T^{T+H}\left|\int_t^{t+h}\sum_2(u)\,du\right|^{2l}dt\Bigr)^{1-(1/2l)}\biggr). \end{aligned} \tag{6.5}$$

Now, the main term on the right-hand side of (6.5) (apart from the constant π^{-2l}) is

$$\int_T^{T+H}\left|\sum_{p<z}\frac{a_p\bigl(\cos\bigl((t+h)\log p\bigr)-\cos(t\log p)\bigr)}{p^{k/2}\log p}\right|^{2l}dt. \tag{6.6}$$

We put

$$\eta_2=\eta_2(t)=\sum_{p<z}a_p p^{-(k/2)-it}(\log p)^{-1}(p^{-ih}-1), \tag{6.7}$$

so that

$$\sum_{p<z}\frac{a_p\bigl(\cos\bigl((t+h)\log p\bigr)-\cos(t\log p)\bigr)}{p^{k/2}\log p}=\frac{\eta_2+\overline{\eta_2}}{2}. \tag{6.8}$$

The integral in (6.6) becomes equal to

$$2^{-2l}\frac{(2l)!}{(l!)^2}\int_T^{T+H}|\eta_2(t)|^{2l}\,dt+O\biggl(4^{-l}\sum_{\substack{j=0,1,\dots,2l\\ j\ne l}}\binom{2l}{j}\left|\int_T^{T+H}\eta_2^j\overline{\eta_2}^{(2l-j)}\,dt\right|\biggr). \tag{6.9}$$

Now, (for $j \neq l$)

$$Q_{11} := \int_T^{T+H} \eta_2^j \overline{\eta_2}^{(2l-j)}\, dt$$

(6.10)
$$\ll \sum_{\substack{p_1,\ldots,p_j<z \\ q_1,\ldots,q_{(2l-j)}<z}} \frac{a_{p_1}\cdots a_{p_j} a_{q_1}\cdots a_{q_{(2l-j)}}}{(p_1\cdots p_j q_1\cdots q_{(2l-j)})^{k/2}} \times \prod_{m=1}^{j} \frac{|p_m^{ih}-1|}{(\log p_m)} \times \prod_{n=1}^{2l-j} \frac{|q_n^{ih}-1|}{(\log q_n)} \times \left|\log\left(\frac{p_1\cdots p_j}{q_1\cdots q_{(2l-j)}}\right)\right|^{-1}$$
$$\ll A^{2l} z^{2l} h^{2l} \left(\sum_{p<z} p^{-1/2}\right)^{2l},$$

since $|a_p| \leq 2p^{(k-1)/2}$ and

$$|p^{ih}-1| = 2\left|\sin\left(\frac{h\log p}{2}\right)\right| \leq h\log p.$$

Hence, the error term in (6.9) is

(6.11)
$$\ll A^l h^{2l} H,$$

by taking $z = T^{\alpha/(5l)}$.

Now, we have

(6.12)
$$Q_{12} := \int_T^{T+H} |\eta_2(t)|^{2l}\, dt = H \sum_{\substack{p_1,\ldots,p_l<z \\ q_1,\ldots,q_l<z \\ p_1\cdots p_l = q_1\cdots q_l}} \frac{a_{p_1}\cdots a_{p_l} a_{q_1}\cdots a_{q_l}}{(p_1\cdots p_l q_1\cdots q_l)^{k/2}} \times \prod_{j=1}^{l} \frac{(p_j^{ih}-1)(q_j^{-ih}-1)}{(\log p_j)(\log q_j)} + O(A^l h^{2l} H),$$

in the exact way as we obtained (5.10) and (5.11). Now, by Lemma 3.1, with $\tau = \frac{1}{2}$,

$$\delta(p_j) = \begin{cases} \dfrac{a_{p_j}(p_j^{ih}-1)}{p_j^{(k-1)/2}(\log p_j)} & \text{for } 1 \leq j \leq l, \\ \dfrac{a_{p_j}(p_j^{-ih}-1)}{p_j^{(k-1)/2}(\log p_j)} & \text{for } l+1 \leq j \leq 2l, \end{cases}$$

and $C=2h$, the main term in (6.12) becomes equal to

$$
\begin{aligned}
Q_{13} &:= l!\Big(\sum_{p<z}\frac{a_p^2|p^{ih}-1|^2}{p^k(\log p)^2}\Big)^l H + O\Big(2^{2l}h^{2l}l!\Big(\sum_{p<z}p^{-1}\Big)^{l-2}H\Big) \\
&= l!H\Big(4\sum_{p<z}\frac{a_p^2}{p^k}\Big(\frac{\sin\big(\frac{1}{2}h\log p\big)}{\log p}\Big)^2\Big)^l \\
&\quad + O\big(A^l l^l h^{2l} H(\log\log T)^{l-2}\big).
\end{aligned}
\tag{6.13}
$$

Now, assuming that $1<h^{-1}<\log T/(10l)<\log z$, we write the sum on the right-hand side of (6.13) as

$$
\Big(\sum_{p<e^{1/h}} + \sum_{e^{1/h}\le p<z}\Big)\frac{a_p^2}{p^k}\Big(\frac{\sin\big(\frac{1}{2}h\log p\big)}{\log p}\Big)^2. \tag{6.14}
$$

The first sum in (6.14) is

$$
\frac{h^2}{4}\sum_{p<e^{1/h}}\frac{a_p^2}{p^k} + O\Big(h^2\sum_{p<e^{1/h}}\frac{a_p^2}{p^k}h^2(\log p)^2\Big) = \frac{h^2}{4}\log h^{-1} + O(h^2), \tag{6.15}
$$

since

$$
\sum_{p\le x}\frac{a_p^2}{p^k} = \log\log x + O(1) \tag{6.16}
$$

as in (5.14), and in the error term, we have used $a_p^2 \le 4p^{k-1}$ and

$$
\sum_{p\le x}\frac{(\log p)^2}{p} = O\big((\log x)^2\big). \tag{6.17}
$$

The second sum in (6.14) is

$$
\ll \sum_{e^{1/h}\le p<z}\frac{\log p}{p(\log p)^3} \ll \int_{1/h}^{\infty} v^{-3}\,dv \ll h^2. \tag{6.18}
$$

Hence, we obtain from (6.14), (6.15) and (6.18)

$$
\sum_{p<z}\frac{a_p^2}{p^k}\Big(\frac{\sin\big(\frac{1}{2}h\log p\big)}{\log p}\Big)^2 = \frac{h^2}{4}\log h^{-1} + O(h^2). \tag{6.19}
$$

Therefore, from (6.12), (6.13) and (6.19), we get

$$\int_T^{T+H} |\eta_2(t)|^{2l}\,dt = l!Hh^{2l}\bigl(\log h^{-1}+O(1)\bigr)^l + O\bigl(A^l l^l h^{2l} H(\log\log T)^{l-2}\bigr). \tag{6.20}$$

Substituting (6.11) and (6.20) in (6.5) and using the inequality $l! \le l^{l-1}$ for $l \ge 1$, we arrive at

$$\begin{aligned} Q_{10} &:= \int_T^{T+H} \left| \int_t^{t+h} S_f(u)\,du \right|^{2l} dt \\ &= \frac{(2l)!}{l!}\left(\frac{h}{2\pi}\right)^{2l} H(\log h^{-1})^l + O(A^l l^{2l} h^{2l} H) \\ &\quad + O\bigl(A^l H h^{2l} l^{l-(1/2)} (\log\log T)^{l-(1/2)}\bigr) \end{aligned} \tag{6.21}$$

from which Theorem 2 follows.

7. Completion of the proof of the main theorem

The proofs of Theorems 3 and 4 are verbatim the same as in [5] (see Section 5 and 6 of [5]). Hence, the proof of the main theorem follows from the arguments similar to those in Section 7 of [5].

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