Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 31, 2006, 265–286

ON ENTIRE FUNCTIONS THAT SHARE A VALUE WITH THEIR DERIVATIVES

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Abstract. Let f be a nonconstant entire function, a a finite complex number, k and m two distinct positive integers, and (k,m) the greatest common divisor of k and m. If f, $f^{(k)}$ and $f^{(m)}$ share a CM, then

$$f(z)=\frac{c-1}{c}a+\sum_{j=1}^{q}C_je^{\lambda_j z},$$

where q is a positive integer with $q\le(k,m)$, c and C_j for $1\le j\le q$ are nonzero constants, and λ_j for $1\le j\le q$, are distinct nonzero constants satisfying $(\lambda_j)^k=(\lambda_j)^m=c$, for $a\ne0$, and $(\lambda_j)^k=c$, $(\lambda_j)^m=d$, for $a=0$, where d is a nonzero constant. This answers a question of Yang and Yi [14] for entire functions, and extends a result of Csillag [2].

1. Introduction and main results

Let f and g be two nonconstant meromorphic functions in the complex plane, and let a be a finite complex number. If $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities, then we say that f and g share a CM.

In 1986, Jank–Mues–Volkmann [8] proved

Theorem A. *Let f be a nonconstant meromorphic function and a a nonzero finite complex number. If f, f' and f'' share a CM, then $f\equiv f'$.*

By Theorem A, the following question was posed.

Question 1 (see [6], [7], [13], [14]). *Let f be a nonconstant meromorphic function, a a nonzero finite complex number, and k, m two distinct positive integers. Suppose that f, $f^{(k)}$ and $f^{(m)}$ share a CM. Can we get $f\equiv f^{(k)}$?*

The following example [12] shows that the answer to Question 1 is, in general, negative.

2000 Mathematics Subject Classification: Primary 30D35.

Supported by the NNSF of China (Grant No. 10471065), the NSF of Education Department of Jiangsu Province (Grant No. 04KJD110001) and the Presidential Foundation of South China Agricultural University.

Example 1. Let k, m be positive integers satisfying $m > k+1$, b a nonzero constant such that $b^k = b^m \neq 1$ and $a = b^k$. Set

$$f(z) = e^{bz} + a - 1.$$

Then f, $f^{(k)}$ and $f^{(m)}$ share a CM. But $f \not\equiv f^{(k)}$.

In Example 1, f is an entire function, and f, $f^{(k)}$ and $f^{(m)}$ share a CM. Although $f \not\equiv f^{(k)}$, we have $f^{(k)} \equiv f^{(m)}$.

Naturally, we pose the following question.

Question 2. *Let f be a nonconstant meromorphic function, a a nonzero finite complex number and k, m two distinct positive integers. Suppose that f, $f^{(k)}$ and $f^{(m)}$ share a CM. Can we get $f^{(k)} \equiv f^{(m)}$?*

In this paper, we give an affirmative answer to Question 2 for entire functions. In fact, we have proved the following more general result.

Theorem 1. *Let f be a nonconstant entire function, a a finite complex number, k and m two distinct positive integers, and (k,m) the greatest common divisor of k and m. If f, $f^{(k)}$ and $f^{(m)}$ share a CM, then*

$$f(z) = \left(1 - \frac{1}{c}\right)a + \sum_{j=1}^{q} C_j e^{\lambda_j z}, \tag{1.1}$$

where q is a positive integer with $q \leq (k,m)$, c and C_j, $1 \leq j \leq q$, are nonzero constants, and λ_j, $1 \leq j \leq q$, are distinct nonzero constants satisfying

$$(\lambda_j)^k = (\lambda_j)^m = c, \quad \text{for } a \neq 0; \tag{1.2}$$

and

$$(\lambda_j)^k = c, \quad (\lambda_j)^m = d, \quad \text{for } a = 0, \tag{1.3}$$

where d is a nonzero constant.

By Theorem 1, we can easily obtain the following results.

Corollary 2. *Let f be a nonconstant entire function, a a nonzero finite complex number, and k, m two distinct positive integers. Suppose that f, $f^{(k)}$ and $f^{(m)}$ share a CM. Then $f^{(k)} \equiv f^{(m)}$.*

Corollary 2 gives an affirmative answer to Question 2 for entire functions.

Corollary 3 ([10, Theorem 1]). *Let f be a nonconstant entire function, a a nonzero finite complex number and k a positive integer. If f, $f^{(k)}$ and $f^{(k+1)}$ share a CM, then $f \equiv f'$.*

Corollary 4 ([10, Theorem 2]). *Let f be a nonconstant entire function, a a nonzero finite complex number and $k \geq 2$ a positive integer. If f, f' and $f^{(k)}$ share a CM, then*

$$f(z) = \left(1 - \frac{1}{c}\right)a + Ce^{cz}, \tag{1.4}$$

where C and c are nonzero constants with $c^{k-1} = 1$.

Corollary 5 (Csillag [2], cf. [4, p. 67]). *Let f be a nonconstant entire function, and k and m two distinct positive integers. If $ff^{(k)}f^{(m)} \neq 0$, then $f = e^{Az+B}$, where A $(\neq 0)$ and B are constants.*

Let f be a nonconstant meromorphic function in the complex plane. Throughout this paper, we use the basic results and notations of Nevanlinna theory (cf. [3], [4], [11], [14]). In particular, $S(r,f)$ denotes any function satisfying

$$S(r,f) = o\{T(r,f)\},$$

as $r \to +\infty$, possibly outside of a set of finite linear measure, where $T(r,f)$ is Nevanlinna's characteristic function.

As usual, the order $\varrho(f)$ of f is defined as

$$\varrho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

2. Some lemmas

We will use $P_d[f]$ to denote a differential polynomial in f of degree $\leq d$ with constant coefficients which may be different at different occurrence. We denote the set of differential polynomials in f with constant coefficients by $\mathscr{P}[f]$.

Lemma 1 (Clunie [1], cf. [4, p. 68]). *Let f be a nonconstant meromorphic function, n be a positive integer, $P[f]$ and $Q[f]$ two differential polynomials in f with constant coefficients, and $P[f] \not\equiv 0$. If the degree of $P[f]$ is at most n and*

$$f^n Q[f] = P[f],$$

then

$$m(r, Q[f]) = S(r,f).$$

Lemma 2 (cf. [9, p. 29–34]). *Let f be a nonconstant entire function, n be a positive integer and a_j, $0 \leq j \leq n$, meromorphic functions with $a_n \not\equiv 0$. Suppose that*

$$a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 \equiv 0. \tag{2.1}$$

Then

$$T(r,f) \leq O\left(1 + \sum_{j=0}^{n} T(r, a_n)\right). \tag{2.2}$$

The following result is an instant corollary of Lemma 2.

Lemma 3. *Let f be a nonconstant entire function, n a positive integer and a_j, $0 \le j \le n$, meromorphic functions satisfying $T(r, a_j) = S(r, f)$. If*

$$a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 \equiv 0,$$

then $a_j \equiv 0$ for $j = 0, 1, \ldots, n$.

Lemma 4 ([3, Lemma 3.12]). *Let $f_j(z)$ $(\not\equiv 0)$, $j = 1, 2, \ldots, n$, be n meromorphic functions which are linearly independent such that*

$$f_1(z) + f_2(z) + \cdots + f_n(z) \equiv 1. \tag{2.3}$$

Then for every j, $1 \le j \le n$,

$$T(r, f_j) \le \sum_{k=1}^{n} N\left(r, \frac{1}{f_k}\right) + N(r, f_j) + N(r, D) + S(r), \tag{2.4}$$

where $D = W(f_1, f_2, \ldots, f_n)$ is the Wronskian, and $S(r)$ is a function which satisfies

$$S(r) = o\Big(\max_{1 \le k \le n} T(r, f_k)\Big)$$

as $r \to \infty$, possibly outside a set of finite linear measure.

Lemma 5 ([3, Lemma 5.1]). *Let $a_j(z)$, $j = 0, 1, \ldots, n$, be entire and of finite order $\le \varrho$ $(< \infty)$. Let $g_j(z)$, $j = 1, \ldots, n$, be also entire such that each of the functions $g_i - g_j$, $i \ne j$, is a transcendental function or a polynomial of degree greater than ϱ. If*

$$\sum_{j=1}^{n} a_j(z) e^{g_j(z)} \equiv a_0(z), \tag{2.5}$$

then

$$a_j(z) \equiv 0, \quad j = 0, 1, \ldots, n. \tag{2.6}$$

Lemma 6. *Let f and α be nonconstant entire functions, a a finite complex number and k a positive integer. Suppose that*

$$f^{(k)} = a + e^{\alpha} f. \tag{2.7}$$

Then for any positive integer j, $1 \le j \le k-1$, we have

$$f^{(k+j)} = \gamma_{0,j} f + \gamma_{1,j} f' + \cdots + \gamma_{j,j} f^{(j)}, \tag{2.8}$$

and $\gamma_{i,j}$ *are entire functions satisfying*

$$\begin{pmatrix} \gamma_{0,j} \\ \vdots \\ \gamma_{j,j} \end{pmatrix} = \begin{pmatrix} A_{0,1,j}e^{\alpha} \\ \vdots \\ A_{j,1,j}e^{\alpha} \end{pmatrix} \tag{2.9}$$

where

$$\begin{aligned} A_{i,1,j} &= \frac{j!}{i!(j-i)!} e^{-\alpha}(e^{\alpha})^{(j-i)} \\ &= \frac{j!}{i!(j-i)!}\big((\alpha')^{j-i} + P_{j-i-1}[\alpha']\big), \quad 0 \le i \le j, \end{aligned} \tag{2.10}$$

are differential polynomials in α' *with constant coefficients. In particular,* $A_{j,1,j} \equiv 1$ *for* $1 \le j \le k-1$. *Here* $P_d[\alpha'] \equiv 0$ *for* $d \le 0$.

Proof. We prove this lemma by mathematical induction on j. By (2.7), we have $f^{(k+1)} = \alpha' e^{\alpha} f + e^{\alpha} f'$, so that (2.8)–(2.10) are true for $j = 1$. Now suppose that (2.8)–(2.10) are true for $j \le k-2$. Thus by (2.8), we get

$$\begin{aligned} f^{(k+j+1)} &= \gamma'_{0,j} f + \gamma'_{1,j} f' + \cdots + \gamma'_{j,j} f^{(j)} \\ &\quad + \gamma_{0,j} f' + \cdots + \gamma_{j-1,j} f^{(j)} + \gamma_{j,j} f^{(j+1)} \\ &= \gamma_{0,j+1} f + \gamma_{1,j+1} f' + \cdots + \gamma_{j,j+1} f^{(j)} + \gamma_{j+1,j+1} f^{(j+1)}, \end{aligned} \tag{2.11}$$

where

$$\gamma_{0,j+1} = \gamma'_{0,j}, \tag{2.12}$$

$$\begin{aligned} \gamma_{1,j+1} &= \gamma'_{1,j} + \gamma_{0,j}, \\ &\vdots \\ \gamma_{i,j+1} &= \gamma'_{i,j} + \gamma_{i-1,j}, \\ &\vdots \\ \gamma_{j,j+1} &= \gamma'_{j,j} + \gamma_{j-1,j}, \end{aligned} \tag{2.13}$$

$$\gamma_{j+1,j+1} = \gamma_{j,j}. \tag{2.14}$$

By (2.11)–(2.14), we know that (2.8)–(2.10) are true for $j+1$. Thus (2.8)–(2.10) are true for $j = 1, 2, \ldots, k-1$. Lemma 6 is proved.

Lemma 7. *Let* f *and* α *be nonconstant entire functions,* a *a finite complex number and* k *a positive integer. Suppose that*

$$f^{(k)} = a + e^{\alpha} f. \tag{2.15}$$

Then for any positive integer j $(\geq k)$ $j = sk + l$, $s \geq 1$, $0 \leq l \leq k-1$, *we have*

$$f^{(k+j)} = \gamma_{-1,j} + \gamma_{0,j} f + \gamma_{1,j} f' + \cdots + \gamma_{k-1,j} f^{(k-1)}, \tag{2.16}$$

and $\gamma_{i,j}$ *are entire functions satisfying*

$$\begin{pmatrix} \gamma_{-1,j} \\ \gamma_{0,j} \\ \vdots \\ \gamma_{l,j} \\ \gamma_{l+1,j} \\ \vdots \\ \gamma_{k-1,j} \end{pmatrix} = \begin{pmatrix} aA_{-1,1,j}e^{\alpha} + a\sum_{t=2}^{s-1} A_{-1,t,j}(e^{\alpha})^t + aA_{-1,s,j}(e^{\alpha})^s \\ A_{0,1,j}e^{\alpha} + \sum_{t=2}^{s} A_{0,t,j}(e^{\alpha})^t + A_{0,s+1,j}(e^{\alpha})^{s+1} \\ \vdots \\ A_{l,1,j}e^{\alpha} + \sum_{t=2}^{s} A_{l,t,j}(e^{\alpha})^t + A_{l,s+1,j}(e^{\alpha})^{s+1} \\ A_{l+1,1,j}e^{\alpha} + \sum_{t=2}^{s-1} A_{l+1,t,j}(e^{\alpha})^t + A_{l+1,s,j}(e^{\alpha})^s \\ \vdots \\ A_{k-1,1,j}e^{\alpha} + \sum_{t=2}^{s-1} A_{k-1,t,j}(e^{\alpha})^t + A_{k-1,s,j}(e^{\alpha})^s \end{pmatrix}, \tag{2.17}$$

where $A_{i,t,j}$ $(\in \mathscr{P}[\alpha'])$ *satisfy*

$$\begin{pmatrix} A_{-1,s,j} \\ A_{0,s+1,j} \\ \vdots \\ A_{l-1,s+1,j} \\ A_{l,s+1,j} \\ A_{l+1,s,j} \\ \vdots \\ A_{k-1,s,j} \end{pmatrix} = \begin{pmatrix} C_{-1,s,j}(\alpha')^l + P_{l-1}[\alpha'] \\ C_{0,s+1,j}(\alpha')^l + P_{l-1}[\alpha'] \\ \vdots \\ C_{l-1,s+1,j}\alpha' + P_0[\alpha'] \\ 1 \\ C_{l+1,s,j}(\alpha')^{k-1} + P_{k-2}[\alpha'] \\ \vdots \\ C_{k-1,s,j}(\alpha')^{l+1} + P_l[\alpha'] \end{pmatrix}, \tag{2.18}$$

and $C_{i,s+1,j}$, $-1 \leq i \leq l-1$, *and* $C_{i,s,j}$, $l+1 \leq i \leq k-1$, *are positive integers, and*

$$A_{i,1,j} = \frac{j!}{i!(j-i)!} e^{-\alpha}(e^{\alpha})^{(j-i)} = \frac{j!}{i!(j-i)!}\big((\alpha')^{j-i} + P_{j-i-1}[\alpha']\big). \tag{2.19}$$

Here $P_d[\alpha'] \equiv 0$ *for* $d \leq 0$.

Proof. We prove this lemma by mathematical induction on j. First we prove that (2.16)–(2.19) are true for $j = k$. By Lemma 6, we have

$$f^{(2k-1)} = \gamma_{0,k-1} f + \gamma_{1,k-1} f' + \cdots + \gamma_{k-1,k-1} f^{(k-1)}. \tag{2.20}$$

This together with (2.15) yields

$$\begin{aligned} f^{(2k)} &= (f^{(2k-1)})' \\ &= \gamma'_{0,k-1} f + \gamma'_{1,k-1} f' + \cdots + \gamma'_{k-1,k-1} f^{(k-1)} \\ &\quad + \gamma_{0,k-1} f' + \cdots + \gamma_{k-2,k-1} f^{(k-1)} + a\gamma_{k-1,k-1} + e^{\alpha}\gamma_{k-1,k-1} f \\ &= \gamma_{-1,k} + \gamma_{0,k} f + \gamma_{1,k} f' + \cdots + \gamma_{k-1,k} f^{(k-1)}. \end{aligned} \tag{2.21}$$

By Lemma 6, we get

$$
\begin{aligned}
\gamma_{-1,k} &= a\gamma_{k-1,k-1} = ae^{\alpha}, \qquad (2.22)\\
\gamma_{0,k} &= \gamma'_{0,k-1} + e^{\alpha}\gamma_{k-1,k-1}\\
&= (e^{\alpha})^{(k)} + (e^{\alpha})^2, \qquad (2.23)\\
\gamma_{i,k} &= \gamma'_{i,k-1} + \gamma_{i-1,k-1}\\
&= \frac{(k-1)!}{i!(k-1-i)!}(e^{\alpha})^{(k-i)} + \frac{(k-1)!}{(i-1)!(k-i)!}(e^{\alpha})^{(k-i)}\\
&= \frac{k!}{i!(k-i)!}(e^{\alpha})^{(k-i)}, \quad i = 1,\dots,k-1. \qquad (2.24)
\end{aligned}
$$

Thus (2.16)–(2.19) are true for $j = k$.

Now we assume that this lemma is true for a given $j = sk+l$ with $s \ge 1$ and $0 \le l \le k-1$. Next we show that this lemma is true for $j+1$. First by (2.15) and (2.16), we get

$$
\begin{aligned}
f^{(k+j+1)} &= \gamma'_{-1,j} + \gamma'_{0,j}f + \gamma'_{1,j}f' + \cdots + \gamma'_{k-1,j}f^{(k-1)}\\
&\quad + \gamma_{0,j}f' + \cdots + \gamma_{k-2,j}f^{(k-1)} + a\gamma_{k-1,j} + e^{\alpha}\gamma_{k-1,j}f.
\end{aligned}
$$

It follows that (2.16) is true for $j+1$ with

$$
\begin{aligned}
\gamma_{-1,j+1} &= \gamma'_{-1,j} + a\gamma_{k-1,j}, \qquad (2.25)\\
\gamma_{0,j+1} &= \gamma'_{0,j} + e^{\alpha}\gamma_{k-1,j}, \qquad (2.26)\\
\gamma_{i,j+1} &= \gamma'_{i,j} + \gamma_{i-1,j}, \quad i = 1,2,\dots,k-1. \qquad (2.27)
\end{aligned}
$$

Thus for $l \le k-2$, by the assumptions,

$$
\begin{aligned}
\gamma_{-1,j+1} &= \left(aA_{-1,1,j}e^{\alpha} + a\sum_{t=2}^{s-1}A_{-1,t,j}(e^{\alpha})^t + aA_{-1,s,j}(e^{\alpha})^s\right)'\\
&\quad + a\left(A_{k-1,1,j}e^{\alpha} + \sum_{t=2}^{s-1}A_{k-1,t,j}(e^{\alpha})^t + A_{k-1,s,j}(e^{\alpha})^s\right) \qquad (2.28)\\
&= a(A'_{-1,1,j} + \alpha' A_{-1,1,j} + A_{k-1,1,j})e^{\alpha}\\
&\quad + a\sum_{t=2}^{s-1}(A'_{-1,t,j} + t\alpha' A_{-1,t,j} + A_{k-1,t,j})(e^{\alpha})^t + aA_{-1,s,j+1}(e^{\alpha})^s,
\end{aligned}
$$

where $A_{-1,s,j+1} = A'_{-1,s,j} + s\alpha' A_{-1,s,j} + A_{k-1,s,j}$,

$$\begin{aligned}\gamma_{0,j+1} &= \Big(A_{0,1,j}e^{\alpha} + \sum_{t=2}^{s} A_{0,t,j}(e^{\alpha})^t + A_{0,s+1,j}(e^{\alpha})^{s+1}\Big)' \\ &\quad + e^{\alpha}\Big(A_{k-1,1,j}e^{\alpha} + \sum_{t=2}^{s-1} A_{k-1,t,j}(e^{\alpha})^t + A_{k-1,s,j}(e^{\alpha})^s\Big) \\ &= A_{0,1,j+1}e^{\alpha} + \sum_{t=2}^{s}(A'_{0,t,j} + t\alpha' A_{0,t,j} + A_{k-1,t-1,j})(e^{\alpha})^t \\ &\quad + A_{0,s+1,j+1}(e^{\alpha})^{s+1},\end{aligned} \tag{2.29}$$

where $A_{0,1,j+1} = A'_{0,1,j} + A_{0,1,j}\alpha'$, $A_{0,s+1,j+1} = A'_{0,s+1,j} + (s+1)\alpha' A_{0,s+1,j} + A_{k-1,s,j}$, and for $1 \le i \le l$,

$$\begin{aligned}\gamma_{i,j+1} &= \gamma'_{i,j} + \gamma_{i-1,j} \\ &= \Big(A_{i,1,j}e^{\alpha} + \sum_{t=2}^{s} A_{i,t,j}(e^{\alpha})^t + A_{i,s+1,j}(e^{\alpha})^{s+1}\Big)' \\ &\quad + A_{i-1,1,j}e^{\alpha} + \sum_{t=2}^{s} A_{i-1,t,j}(e^{\alpha})^t + A_{i-1,s+1,j}(e^{\alpha})^{s+1} \\ &= A_{i,1,j+1}e^{\alpha} + \sum_{t=2}^{s}(A'_{i,t,j} + t\alpha' A_{i,t,j} + A_{i-1,t,j})(e^{\alpha})^t \\ &\quad + A_{i,s+1,j+1}(e^{\alpha})^{s+1},\end{aligned} \tag{2.30}$$

where $A_{i,1,j+1} = A'_{i,1,j} + \alpha' A_{i,1,j} + A_{i-1,1,j}$, $A_{i,s+1,j+1} = A'_{i,s+1,j} + (s+1)\alpha' A_{i,s+1,j} + A_{i-1,s+1,j}$, and for $i = l+1$,

$$\begin{aligned}\gamma_{l+1,j+1} &= \gamma'_{l+1,j} + \gamma_{l,j} \\ &= \Big(A_{l+1,1,j}e^{\alpha} + \sum_{t=2}^{s-1} A_{l+1,t,j}(e^{\alpha})^t + A_{l+1,s,j}(e^{\alpha})^s\Big)' \\ &\quad + A_{l,1,j}e^{\alpha} + \sum_{t=2}^{s} A_{l,t,j}(e^{\alpha})^t + A_{l,s+1,j}(e^{\alpha})^{s+1} \\ &= A_{l+1,1,j+1}e^{\alpha} + \sum_{t=2}^{s}(A'_{l+1,t,j} + t\alpha' A_{l+1,t,j} + A_{l,t,j})(e^{\alpha})^t \\ &\quad + A_{l+1,s+1,j+1}(e^{\alpha})^{s+1},\end{aligned} \tag{2.31}$$

where $A_{l+1,1,j+1} = A'_{l+1,1,j} + \alpha' A_{l+1,1,j} + A_{l,1,j}$, $A_{l+1,s+1,j+1} = A_{l,s+1,j}$, and for $l+2 \le i \le k-1$,

$$\gamma_{i,j+1} = \gamma'_{i,j} + \gamma_{i-1,j}$$

$$
\begin{aligned}
&= \Big(A_{i,1,j} e^{\alpha} + \sum_{t=2}^{s-1} A_{i,t,j} (e^{\alpha})^t + A_{i,s,j} (e^{\alpha})^s \Big)' \\
&\quad + A_{i-1,1,j} e^{\alpha} + \sum_{t=2}^{s-1} A_{i-1,t,j} (e^{\alpha})^t + A_{i-1,s,j} (e^{\alpha})^s \\
&= A_{i,1,j+1} e^{\alpha} + \sum_{t=2}^{s-1} (A'_{i,t,j} + t\alpha' A_{i,t,j} + A_{i-1,t,j})(e^{\alpha})^t + A_{i,s,j+1} (e^{\alpha})^s,
\end{aligned} \tag{2.32}
$$

where $A_{i,1,j+1} = A'_{i,1,j} + \alpha' A_{i,1,j} + A_{i-1,1,j}$, $A_{i,s,j+1} = A'_{i,s,j} + s\alpha' A_{i,s,j} + A_{i-1,s,j}$.

By (2.28)–(2.32), we know that (2.16)–(2.19) are true for $j+1$ when $j = sk+l$ with $0 \le l \le k-2$. Similarly, we can prove (2.16)–(2.19) are true for $j+1$ when $j = sk+k-1$. We omit the details here. Thus Lemma 7 is proved.

Lemma 8. *Let*

$$
\Delta_j = \begin{pmatrix} \gamma_{0,j} & \gamma_{0,j+1} & \cdots & \gamma_{0,j+k-1} \\ \gamma_{1,j} & \gamma_{1,j+1} & \cdots & \gamma_{1,j+k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{k-1,j} & \gamma_{k-1,j+1} & \cdots & \gamma_{k-1,j+k-1} \end{pmatrix}, \tag{2.33}
$$

where $\gamma_{i,j}$ are entire functions defined in Lemmas 6–7 (for $1 \le j \le k-1$ and $i > j$ set $\gamma_{i,j} = 0$). Denote the determinant of Δ_j by $\det(\Delta_j)$. Then for $j = sk + l$ (≥ 1) with $s \ge 0$, $0 \le l \le k-1$, we have

$$
\begin{aligned}
\det(\Delta_j) &= \big((\alpha')^{kj} + P_{kj-1}[\alpha']\big)(e^{\alpha})^k \\
&\quad + \sum_{t=k+1}^{(s+1)k+l-1} A_{t,j} (e^{\alpha})^t + (-1)^{l(k-l)} (e^{\alpha})^{(s+1)k+l},
\end{aligned} \tag{2.34}
$$

where $A_{t,j} \in \mathscr{P}[\alpha']$.

Proof. Obviously, by Lemmas 6–7, we have

$$
\det(\Delta_j) = \sum_{t=k}^{\nu} A_{t,j} (e^{\alpha})^t \tag{2.35}
$$

with $\nu \ge k$ and $A_{t,j} \in \mathscr{P}[\alpha']$. Thus we need only to show that

$$
\nu = (s+1)k + l, \quad A_{\nu,j} = (-1)^{l(k-l)}, \tag{2.36}
$$

and

$$
A_{k,j} = (\alpha')^{kj} + P_{kj-1}[\alpha']. \tag{2.37}
$$

First we prove (2.36). By Lemmas 6–7, we have

$$M_1=\begin{pmatrix}\gamma_{0,j} & \gamma_{0,j+1} & \cdots & \gamma_{0,j+k-1-l}\\ \gamma_{1,j} & \gamma_{1,j+1} & \cdots & \gamma_{1,j+k-1-l}\\ \vdots & \vdots & \ddots & \vdots\\ \gamma_{l-1,j} & \gamma_{l-1,j+1} & \cdots & \gamma_{l-1,j+k-1-l}\end{pmatrix}_{l\times(k-l)}$$
$$=(\text{polynomials in } e^{\alpha} \text{ of degrees } \le s+1)_{l\times(k-l)},$$

$$M_2=\begin{pmatrix}\gamma_{0,j+k-l} & \gamma_{0,j+k-l+1} & \cdots & \gamma_{0,j+k-1}\\ \gamma_{1,j+k-l} & \gamma_{1,j+k-l+1} & \cdots & \gamma_{1,j+k-1}\\ \vdots & \vdots & \ddots & \vdots\\ \gamma_{l-1,j+k-l} & \gamma_{l-1,j+k-l+1} & \cdots & \gamma_{l-1,j+k-1}\end{pmatrix}$$
$$=\begin{pmatrix}1 & A_{0,s+2,j+k-l+1} & \cdots & A_{0,s+2,j+k-1}\\ 0 & 1 & \cdots & A_{1,s+2,j+k-1}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix}_{l\times l}(e^{\alpha})^{s+2}$$
$$+(\text{polynomials in } e^{\alpha} \text{ of degrees } \le s+1)_{l\times l}$$
$$=A_{l\times l}(e^{\alpha})^{s+2}+(\text{polynomials in } e^{\alpha} \text{ of degrees } \le s+1)_{l\times l},$$

$$M_3=\begin{pmatrix}\gamma_{l,j} & \gamma_{l,j+1} & \cdots & \gamma_{l,j+k-1-l}\\ \gamma_{l+1,j} & \gamma_{l+1,j+1} & \cdots & \gamma_{l+1,j+k-1-l}\\ \vdots & \vdots & \ddots & \vdots\\ \gamma_{k-1,j} & \gamma_{k-1,j+1} & \cdots & \gamma_{k-1,j+k-1-l}\end{pmatrix}_{(k-l)\times(k-l)}$$
$$=\begin{pmatrix}1 & A_{l,s+1,j+1} & \cdots & A_{l,s+1,j+k-1-l}\\ 0 & 1 & \cdots & A_{l+1,s+1,j+k-1-l}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix}_{(k-l)\times(k-l)}(e^{\alpha})^{s+1}$$
$$+(\text{polynomials in } e^{\alpha} \text{ of degrees } \le s)_{(k-l)\times(k-l)}$$
$$=B_{(k-l)\times(k-l)}(e^{\alpha})^{s+1}+(\text{polynomials in } e^{\alpha} \text{ of degrees } \le s)_{(k-l)\times(k-l)},$$

$$M_4=\begin{pmatrix}\gamma_{l,j+k-l} & \gamma_{l,j+k-l+1} & \cdots & \gamma_{l,j+k-1}\\ \gamma_{l+1,j+k-l} & \gamma_{l+1,j+k-l+1} & \cdots & \gamma_{l+1,j+k-1}\\ \vdots & \vdots & \ddots & \vdots\\ \gamma_{k-1,j+k-l} & \gamma_{k-1,j+k-l+1} & \cdots & \gamma_{k-1,j+k-1}\end{pmatrix}_{(k-l)\times l}$$
$$=\begin{pmatrix}A_{l,s+1,j+k-l} & A_{l,s+1,j+k-l+1} & \cdots & A_{l,s+1,j+k-1}\\ A_{l+1,s+1,j+k-l} & A_{l+1,s+1,j+k-l+1} & \cdots & A_{l+1,s+1,j+k-1}\\ \vdots & \vdots & \ddots & \vdots\\ A_{k-1,s+1,j+k-l} & A_{k-1,s+1,j+k-l+1} & \cdots & A_{k-1,s+1,j+k-1}\end{pmatrix}_{(k-l)\times l}(e^{\alpha})^{s+1}$$

$$+ (\text{polynomials in } e^{\alpha} \text{ of degrees } \leq s)_{(k-l)\times l}$$
$$= C_{(k-l)\times l}(e^{\alpha})^{s+1} + (\text{polynomials in } e^{\alpha} \text{ of degrees } \leq s)_{(k-l)\times l},$$

where $A_{l\times l}, B_{(k-l)\times(k-l)}, C_{(k-l)\times l}$ are matrices whose elements are differential polynomials in α'. In particular, $A_{l\times l}$ and $B_{(k-l)\times(k-l)}$ are upper triangular matrices whose principal diagonal elements equal 1. Thus by (2.33) we get

$$\det(\Delta_j) = \begin{vmatrix} M_1 & M_2 \\ M_3 & M_4 \end{vmatrix}$$
$$= \begin{vmatrix} 0 & A \\ B & C \end{vmatrix} (e^{\alpha})^{(s+1)k+l} + (\text{terms of degree} \leq (s+1)k+l-1)$$
$$= (-1)^{l(k-l)} \det(A)\det(B)(e^{\alpha})^{(s+1)k+l} + (\text{terms of degree} \leq (s+1)k+l-1)$$
$$= (-1)^{l(k-l)}(e^{\alpha})^{(s+1)k+l} + (\text{terms of degree} \leq (s+1)k+l-1),$$

where $0 = 0_{l\times(k-l)}$ is the zero matrix, $A = A_{l\times l}$, $B = B_{(k-l)\times(k-l)}$, $C = C_{(k-l)\times l}$. This proves (2.36).

Next we prove (2.37). By Lemmas 6–7, we have

$$(2.38) \qquad A_{k,j} = \begin{vmatrix} A_{0,1,j} & A_{0,1,j+1} & \cdots & A_{0,1,j+k-1} \\ A_{1,1,j} & A_{1,1,j+1} & \cdots & A_{1,1,j+k-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k-1,1,j} & A_{k-1,1,j+1} & \cdots & A_{k-1,1,j+k-1} \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \binom{j}{1} & \binom{j+1}{1} & \cdots & \binom{j+k-1}{1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{j}{i} & \binom{j+1}{i} & \cdots & \binom{j+k-1}{i} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{j}{k-1} & \binom{j+1}{k-1} & \cdots & \binom{j+k-1}{k-1} \end{vmatrix} (\alpha')^{kj} + P_{kj-1}[\alpha'],$$

where

$$\binom{j}{i} = \frac{j!}{i!(j-i)!}$$

are the binomial coefficients. Since

$$\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}$$

is a polynomial in x of degree i, by the calculating properties of determinant and the well-known Vandermonde's determinant, we see that $A_{k,j} = C(\alpha')^{kj} + P_{kj-1}[\alpha']$, where C is a nonzero constant which is equal to

$$\prod_{s=1}^{k-1} \frac{1}{s!} \prod_{1\leq i<t\leq k} (t-i) = 1.$$

This proves (2.37). Thus Lemma 8 is proved.

3. Proof of Theorem 1

By the assumptions, there exist two entire functions $\alpha(z)$ and $\beta(z)$ such that

$$\frac{f^{(k)}(z)-a}{f(z)-a}=e^{\alpha(z)}, \tag{3.1}$$

$$\frac{f^{(m)}(z)-a}{f(z)-a}=e^{\beta(z)}. \tag{3.2}$$

Next we consider two cases.

Case 1. Either α or β is a constant. Without loss of generality, we assume that α is a constant. Set $e^{\alpha}=c$. Then by (3.1), we get

$$f^{(k)}-cf=(1-c)a. \tag{3.3}$$

Solving (3.3), we get

$$f(z)=\left(1-\frac{1}{c}\right)a+\sum_{j=1}^{q}C_je^{\lambda_j z}, \tag{3.4}$$

where q $(\leq k)$ is a positive integer, and C_j, λ_j are nonzero constants satisfying $(\lambda_j)^k=c$ and $\lambda_i\neq\lambda_j$, $i\neq j$. By (3.4) and (3.2), it follows that $\varrho(e^{\beta})\leq 1$, where ϱ (e^{β}) is the order of e^{β}, so that $e^{\beta}=de^{\mu z}$, where d $(\neq 0)$ and μ are constants. Thus by (3.2) and (3.4), we get

$$-a+\sum_{j=1}^{q}(\lambda_j)^mC_je^{\lambda_j z}=-\frac{da}{c}e^{\mu z}+\sum_{j=1}^{q}C_jde^{(\lambda_j+\mu)z}. \tag{3.5}$$

Applying Lemma 5 to (3.5), we deduce that $\mu=0$ and $(\lambda_j)^m=d$. Further, if $a\neq 0$, then $c=d$.

By $(\lambda_j)^k=c$, $(\lambda_j)^m=d$ and the fact that λ_j, $1\leq j\leq q$, are distinct, we know that $q\leq(k,m)$, where (k,m) is the greatest common divisor of k and m. In fact, by Euclidean division algorithm, there exist integers k_0 and m_0 such that $(k,m)=k_0k+m_0m$. Thus $(\lambda_j)^{(k,m)}=\left[(\lambda_j)^k\right]^{k_0}\left[(\lambda_j)^m\right]^{m_0}=c^{k_0}d^{m_0}$. Hence by the fact that λ_j, $1\leq j\leq q$, are distinct, it follows that $q\leq(k,m)$.

Case 2. Both α and β are not constants.

We will prove that this case cannot occur. Without loss of generality, we assume $k<m$. Let

$$F(z)=f(z)-a. \tag{3.6}$$

Then by (3.1) and (3.2), we have

$$F^{(k)} = a + e^{\alpha} F, \tag{3.7}$$

$$F^{(m)} = a + e^{\beta} F. \tag{3.8}$$

Set

$$\phi = \frac{F^{(m)} - F^{(k)}}{F}. \tag{3.9}$$

Then by (3.7) and (3.8), we get

$$\phi = e^{\beta} - e^{\alpha}. \tag{3.10}$$

Next we consider two subcases.

Case 2.1: $\phi \equiv 0$. Then by (3.10), we get

$$e^{\beta} = e^{\alpha}. \tag{3.11}$$

Thus by (3.1), (3.2) and (3.11), we get

$$f^{(m)} - f^{(k)} = 0. \tag{3.12}$$

Solving (3.12), we get

$$f(z) = b(z) + \sum_{j=1}^{s} C_j e^{\lambda_j z}, \tag{3.13}$$

where b is a polynomial with $\deg b \le k-1$, $s \le m-k$ is a positive integer, and C_j, λ_j are nonzero constants with $(\lambda_j)^{m-k} = 1$ and $\lambda_i \ne \lambda_j$, $i \ne j$. By (3.1) and (3.13), we know that $\varrho(e^{\alpha}) \le 1$. This together with that α is nonconstant yields that $e^{\alpha} = Ce^{cz}$, where C and c are nonzero constants. Thus by (3.1) and (3.13), we get

$$-a + \sum_{j=1}^{s} C_j (\lambda_j)^k e^{\lambda_j z} = C[b(z) - a]e^{cz} + \sum_{j=1}^{s} CC_j e^{(\lambda_j + c)z}. \tag{3.14}$$

Applying Lemma 5 to (3.14), we get that $c = 0$, a contradiction.

Case 2.2: $\phi \not\equiv 0$. Then by the logarithmic derivative lemma, it follows from (3.9) that

$$m(r, \phi) = S(r, F). \tag{3.15}$$

By (3.10), ϕ is an entire function. Thus by (3.15), we get

$$T(r,\phi)=S(r,F). \tag{3.16}$$

Since $\phi\not\equiv 0$, by (3.10), we get

$$\frac{e^{\beta}}{\phi}=1+\frac{e^{\alpha}}{\phi}. \tag{3.17}$$

Thus by (3.16), (3.17) and the second fundamental theorem we deduce that

$$\begin{aligned} T\left(r,\frac{e^{\beta}}{\phi}\right) &\le \overline{N}\left(r,\frac{e^{\beta}}{\phi}\right)+\overline{N}\left(r,\frac{\phi}{e^{\beta}}\right)+\overline{N}\left(r,\frac{1}{\frac{e^{\beta}}{\phi}-1}\right)+S\left(r,\frac{e^{\beta}}{\phi}\right)\\ &\le \overline{N}\left(r,\frac{e^{\beta}}{\phi}\right)+\overline{N}\left(r,\frac{\phi}{e^{\beta}}\right)+\overline{N}\left(r,\frac{1}{\frac{e^{\alpha}}{\phi}}\right)+S\left(r,\frac{e^{\beta}}{\phi}\right)\\ &\le S(r,F)+S\left(r,\frac{e^{\beta}}{\phi}\right). \end{aligned} \tag{3.18}$$

This together with (3.16) yields that $T(r,e^{\beta})=S(r,F)$. It follows from (3.10) and (3.16) that $T(r,e^{\alpha})=T(r,e^{\beta}-\phi)=S(r,F)$. Thus we get

$$T(r,e^{\alpha})+T(r,e^{\beta})=S(r,F). \tag{3.19}$$

Now, for $0\le j\le k-1$, set

$$p_{i,j}=\gamma_{i,m-k+j},\quad i=-1,0,1,\dots,k-1, \tag{3.20}$$

where $\gamma_{i,j}$ are defined as in Lemmas 6–8. Then by Lemmas 6–7, we have

$$\begin{aligned} F^{(m+j)} &= F^{(k+m-k+j)}\\ &= p_{-1,j}+p_{0,j}F+p_{1,j}F'+\cdots+p_{k-1,j}F^{(k-1)},\quad j=0,1,\dots,k-1. \end{aligned} \tag{3.21}$$

On the other hand, by (3.8) and Lemma 6, for $1\le j\le k-1$, we have

$$F^{(m+j)}=q_{0,j}e^{\beta}F+q_{1,j}e^{\beta}F'+\cdots+q_{j,j}e^{\beta}F^{(j)}, \tag{3.22}$$

where $q_{i,j}$, $i\le j$, are differential polynomials in β' with constant coefficients. In particular, $q_{j,j}\equiv 1$ for $j=1,2,\dots,k-1$. Thus by (3.8), (3.21) and (3.22), we get

$$(\,F,F',\dots,F^{(k-1)}\,)\,(e^{\beta}Q-P)=\Gamma, \tag{3.23}$$

where

$$P = \begin{pmatrix} p_{0,0} & p_{0,1} & \cdots & p_{0,k-1} \\ p_{1,0} & p_{1,1} & \cdots & p_{1,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{k-1,0} & p_{k-1,1} & \cdots & p_{k-1,k-1} \end{pmatrix}, \tag{3.24}$$

$$Q = \begin{pmatrix} 1 & q_{0,1} & \cdots & q_{0,k-1} \\ 0 & 1 & \cdots & q_{1,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{3.25}$$

$$\Gamma = (\, p_{-1,0} - a, p_{-1,1}, \ldots, p_{-1,k-1}\,). \tag{3.26}$$

By (3.23) and the theory of linear equations, we get

$$\det(e^{\beta}Q - P)F = \det(T), \tag{3.27}$$

where T is a matrix whose first line is Γ and the other lines are the same as those of $e^{\beta}Q - P$.

Thus by (3.19) and (3.27), we know that

$$\det(e^{\beta}Q - P) = 0. \tag{3.28}$$

This yields that

$$\det(e^{\beta}I - R) = 0, \tag{3.29}$$

where $I = I_{k\times k}$ is the kth unit matrix, $R = Q^{-1}P$ and Q^{-1} is the inverse matrix of Q. Obviously, the matrix Q^{-1} is also an upper triangular matrix whose elements are differential polynomial in β'. By (3.29), we get

$$(e^{\beta})^k - a_1(e^{\beta})^{k-1} + \cdots + (-1)^t a_t (e^{\beta})^{k-t} + \cdots + (-1)^k a_k = 0, \tag{3.30}$$

where a_t is the sum of all the principle minors of order t of R. In particular, $a_k = \det(R) = \det(P)$. Here, for a matrix

$$A = (a_{i,j}) = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix},$$

and t integers $1 \leq i_1 < i_2 < \cdots < i_t \leq n$, we call

$$\begin{vmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \cdots & a_{i_1,i_t} \\ a_{i_2,i_1} & a_{i_2,i_2} & \cdots & a_{i_2,i_t} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_t,i_1} & a_{i_t,i_1} & \cdots & a_{i_t,i_t} \end{vmatrix}$$

a principle minor of order t of A.

Obviously, by (3.24), (3.25) and the definition of a_t, a_t, $1 \leq t \leq k$, are polynomials in e^α whose coefficients are differential polynomials in α' and β' with constant coefficients.

Next we consider the degrees of these polynomials a_t. Since $m > k$, there exist integers $s \geq 1$ and $0 \leq l \leq k-1$ such that

$$m = sk + l. \tag{3.31}$$

It is obvious that if $l = 0$ then $s > 1$. We claim that for $l \geq 1$,

$$\deg(a_t) \leq ts + l - 1, \quad t = 1, 2, \ldots, k-1, \tag{3.32}$$

and for $l = 0$,

$$\deg(a_t) \leq ts, \quad t = 1, 2, \ldots, k-1. \tag{3.33}$$

and

$$\deg(a_k) = m = ks + l. \tag{3.34}$$

In order to prove (3.32)–(3.34), we first consider the degree of the elements of $R = (r_{i,j})$ which are polynomials in e^α. By (3.20), we see that for $0 \leq j \leq k-1-l$, $p_{i,j} = \gamma_{i,(s-1)k+j+l}$, while for $k-l \leq j \leq k-1$, $p_{i,j} = \gamma_{i,sk+j+l-k}$. Thus by Lemmas 6–7, for $0 \leq i, j \leq k-1$,

$$\deg(p_{i,j}) \leq \begin{cases} s & \text{if } 0 \leq j \leq k-1-l,\ 0 \leq i \leq j+l, \\ s-1 & \text{if } 0 \leq j \leq k-1-l,\ j+l+1 \leq i \leq k-1, \\ s+1 & \text{if } k-l \leq j \leq k-1,\ 0 \leq i \leq j+l-k, \\ s & \text{if } k-l \leq j \leq k-1,\ j+l-k+1 \leq i \leq k-1. \end{cases} \tag{3.35}$$

By (3.25), we may assume that

$$Q^{-1} = \begin{pmatrix} 1 & q^*_{0,1} & \cdots & q^*_{0,k-1} \\ 0 & 1 & \cdots & q^*_{1,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{3.36}$$

where $q^*_{i,j}$, $0 \le i < j \le k-1$, are differential polynomials in β' with constant coefficients. Thus by $(r_{i,j}) = R = Q^{-1}P$, we get

$$r_{i,j} = p_{i,j} + q^*_{i,i+1}p_{i+1,j} + q^*_{i,i+2}p_{i+2,j} + \cdots + q^*_{i,k-1}p_{k-1,j}. \tag{3.37}$$

Thus by (3.35) and (3.37), we see that for $0 \le i, j \le k-1$,

$$\deg(r_{i,j}) \le \begin{cases} s & \text{if } 0 \le j \le k-1-l,\ 0 \le i \le j+l, \\ s-1 & \text{if } 0 \le j \le k-1-l,\ j+l+1 \le i \le k-1, \\ s+1 & \text{if } k-l \le j \le k-1,\ 0 \le i \le j+l-k, \\ s & \text{if } k-l \le j \le k-1,\ j+l-k+1 \le i \le k-1. \end{cases} \tag{3.38}$$

Now let

$$L_{i_1,i_2,\ldots,i_t} = \begin{vmatrix} r_{i_1,i_1} & r_{i_1,i_2} & \cdots & r_{i_1,i_t} \\ r_{i_2,i_1} & r_{i_2,i_2} & \cdots & r_{i_2,i_t} \\ \vdots & \vdots & \ddots & \vdots \\ r_{i_t,i_1} & r_{i_t,i_2} & \cdots & r_{i_t,i_t} \end{vmatrix}$$

be a principle minor of order $t \le k-1$ of R, where $0 \le i_1 < i_2 < \cdots < i_t \le k-1$.

By (3.38), for the case of $l = 0$, the degrees of all $r_{i,j}$ are at most s, so that the degree of $L_{i_1,i_2,\ldots,i_t}$ is at most ts. It follows that the degree of a_t is at most ts. This proves (3.33).

Next we consider the case of $1 \le l \le k-1$. By the definition of determinant, we have

$$L_{i_1,i_2,\ldots,i_t} = \sum \delta_{j_1,j_2,\ldots,j_t} r_{i_1,j_1} r_{i_2,j_2} \cdots r_{i_t,j_t},$$

where the sum takes over all the permutations of $(i_1, i_2, \ldots, i_t)$, and $\delta_{j_1,j_2,\ldots,j_t} = \pm 1$ according to the permutation $(j_1, j_2, \ldots, j_t)$ of $(i_1, i_2, \ldots, i_t)$ is even or odd. Let

$$L_t = r_{i_1,j_1} r_{i_2,j_2} \cdots r_{i_t,j_t}.$$

For $t \le l-1$, by (3.38), the degree of L_t is at most $t(s+1) \le ts + l - 1$.

For $t \ge l$, if there exist $x \le l-1$ polynomials in $r_{i_1,j_1}, r_{i_2,j_2}, \ldots, r_{i_t,j_t}$ with degree $s+1$, then by (3.38), the degree of L_t is at most $x(s+1)+(t-x)s = ts+x \le ts+l-1$. If there exist l polynomials in $r_{i_1,j_1}, r_{i_2,j_2}, \ldots, r_{i_t,j_t}$ with degree $s+1$, then by (3.38), $\{0, 1, \ldots, l-1\} \subset \{i_1, i_2, \ldots, i_t\}$. It follows that there exists at least one of $r_{i_1,j_1}, r_{i_2,j_2}, \ldots, r_{i_t,j_t}$ whose degree is $s-1$ (for otherwise, we must have $\{l, \ldots, k-1\} \subset \{i_1, i_2, \ldots, i_t\}$. This together with $\{0, 1, \ldots, l-1\} \subset \{i_1, i_2, \ldots, i_t\}$ yields that $t \ge k$, which contradicts $t \le k-1$). Hence the degree of L_t is at most $l(s+1)+(s-1)+(t-l-1)s = ts+l-1$. It follows that $\deg(L_{i_1,i_2,\ldots,i_t}) \le ts+l-1$. Thus (3.32) is proved.

Next we prove (3.34). In fact, it can be seen from (3.20), (3.24) and Lemma 8 that

$$\begin{aligned} a_k &= \det(P) = \det(\Delta_{m-k}) \\ &= \big((\alpha')^{k(m-k)} + P_{k(m-k)-1}[\alpha']\big)(e^\alpha)^k \\ &\quad + \sum_{t=k+1}^{m-1} A_{t,m-k}(e^\alpha)^t + (-1)^{l(k-l)}(e^\alpha)^m. \end{aligned} \tag{3.39}$$

Thus we get (3.34).

By (3.32)–(3.34), we see that for $1 \le t \le k-1$, $\deg(a_t) < \deg(a_k)$. Thus by (3.30) and Lemma 2, it follows that

$$T(r,e^{\beta}) = O\big(T(r,e^{\alpha})\big) + S(r,e^{\beta}),$$
$$T(r,e^{\alpha}) = O\big(T(r,e^{\beta})\big) + S(r,e^{\alpha}).$$

Hence we get

$$S(r,e^{\alpha}) = S(r,e^{\beta}) = S(r) \quad \text{(say)}. \tag{3.40}$$

Next we prove the following claims.

Claim I. For any rational number $\theta = \nu/\mu$ with $\nu \in \mathbf{Z}$ and $\mu \in \mathbf{N}$,

$$T(r,e^{\beta-\theta\alpha}) \ne S(r). \tag{3.41}$$

Suppose on the contrary that there exists a rational number $\theta = \nu/\mu$ such that

$$T(r,e^{\beta-\theta\alpha}) = S(r). \tag{3.42}$$

Let

$$b(z) = e^{\beta-\theta\alpha}. \tag{3.43}$$

Then $b(z) \ne 0$ is entire and $T(r,b) = S(r)$. By (3.43),

$$e^{\beta} = b(z)e^{\theta\alpha} = b(z)(e^{\alpha/\mu})^{\nu}. \tag{3.44}$$

On the other hand, by (3.20), (3.24) and Lemmas 6–7, we have

$$P = (e^{\alpha})^{s+1}P_0 + (e^{\alpha})^{s}P_1 + \cdots + (e^{\alpha})P_s, \tag{3.45}$$

where P_j are $k \times k$ matrices whose elements are differential polynomials in α'. In particular, $\det(P_s) \equiv A_{k,m-k}$, where $A_{k,m-k}$ is defined by (2.38). By (3.28), (3.44) and (3.45), we get

$$\det\big(b(e^{\alpha/\mu})^{\nu}Q - (e^{\alpha})^{s+1}P_0 - (e^{\alpha})^{s}P_1 - \cdots - (e^{\alpha})P_s\big) = 0. \tag{3.46}$$

If $\nu > \mu$, then by (3.46), we get

$$\det\big(b(e^{\alpha/\mu})^{\nu-\mu}Q - (e^{\alpha/\mu})^{s\mu}P_0 - \cdots - (e^{\alpha/\mu})^{\mu}P_{s-1} - P_s\big) = 0. \tag{3.47}$$

Since the left side of (3.47) is a polynomial in $e^{\alpha/\mu}$ whose "constant" term is $\det(-P_s) = (-1)^k A_{k,m-k}$, by Lemma 3, we get $A_{k,m-k} = 0$. Thus by (2.10), (2.19) and the fact that α is nonconstant, α' is nonconstant. For otherwise, let $\alpha' = c$. Then $c \neq 0$, and by (2.10), (2.19), we have

$$A_{i,1,j} = \binom{j}{i}(c)^{j-i},$$

so that by (2.38), it follows that $A_{k,m-k} = (c)^{k(m-k)} \neq 0$, which contradicts $A_{k,m-k} = 0$. Hence α' is nonconstant. Thus by (2.37) and Lemma 1, we deduce that $T(r,\alpha') = m(r,\alpha') = S(r,\alpha')$, a contradiction.

If $\nu < \mu$, then by (3.46), we get

$$\det\big(bQ - (e^{\alpha/\mu})^{(s+1)\mu-\nu}P_0 - (e^{\alpha/\mu})^{s\mu-\nu}P_1 - \cdots - (e^{\alpha/\mu})^{\mu-\nu}P_s\big) = 0.$$

Using the same argument as that in case $\nu > \mu$, we deduce that $\det(bQ) = 0$. Thus by $\det(Q) = 1$, we get that $b = 0$, a contradiction.

If $\nu = \mu$, then $e^{\beta} = b(z)e^{\alpha}$. Thus by (3.32)–(3.34), we see that the left side of (3.30) is a polynomial in e^{α} whose leading term is $\varepsilon(e^{\alpha})^m$, where $\varepsilon = \pm 1$ is a constant. Thus applying Lemma 2 to (3.30), we get a contradiction: $T(r,e^{\alpha}) = S(r)$.

Hence Claim I is proved.

Claim II. We have

$$H = \sum_{t=1}^{k-1}(-1)^t a_t (e^{\beta})^{k-t} \equiv 0. \tag{3.48}$$

Suppose that $H \not\equiv 0$. Then by the fact that a_t are polynomials in e^{α}, we can rewrite H as

$$H = \sum_{(t,i)\in T\times I} a_{t,i}e^{(k-t)\beta + i\alpha}, \tag{3.49}$$

where $T \subset \{1,\dots,k-1\}$ and I are finite index sets, $a_{t,i} \not\equiv 0$ are differential polynomials in α' and β' such that all the functions $a_{t,i}e^{(k-t)\beta+i\alpha}$, $(t,i) \in T\times I$ are linearly independent.

By (3.39), we rewrite a_k as

$$(-1)^k a_k = \sum_{i\in J} a_{k,i}e^{i\alpha}, \tag{3.50}$$

where $J \supset \{m\}$ is a finite index set, and $a_{k,i}$ ($\not\equiv 0$), $i \in J$, are differential polynomials in α'.

Hence by (3.30), (3.48)–(3.50), we get

$$e^{k\beta} + \sum_{(t,i)\in T\times I} a_{t,i}e^{(k-t)\beta+i\alpha} + \sum_{i\in J} a_{k,i}e^{i\alpha} = 0. \tag{3.51}$$

By (3.51), we get

$$\sum_{(t,i)\in T\times I}(-a_{t,i})e^{-t\beta+i\alpha}+\sum_{i\in J}(-a_{k,i})e^{-k\beta+i\alpha}=1. \tag{3.52}$$

If the functions $(-a_{t,i})e^{-t\beta+i\alpha}$, $(t,i)\in T\times I$ and $(-a_{k,i})e^{-k\beta+i\alpha}$, $i\in J$ are linearly independent, then by Lemma 4 and the fact that $m\in J$, we get

$$T\big(r,(-a_{k,m})e^{-k\beta+m\alpha}\big)=S(r),$$

so that

$$T(r,e^{-k\beta+m\alpha})=S(r),$$

which contradicts Claim I.

Hence the functions $(-a_{t,i})e^{-t\beta+i\alpha}$, $(t,i)\in T\times I$ and $(-a_{k,i})e^{-k\beta+i\alpha}$, $i\in J$ are linearly dependent. That is, there exist constants $C_{t,i},(t,i)\in T\times I$ and $C_{k,i}$, $i\in J$, at least one of them is not equal to 0, such that

$$\sum_{(t,i)\in T\times I}C_{t,i}a_{t,i}e^{-t\beta+i\alpha}+\sum_{i\in J}C_{k,i}a_{k,i}e^{-k\beta+i\alpha}=0,$$

so that

$$\sum_{(t,i)\in T\times I}C_{t,i}a_{t,i}e^{(k-t)\beta+i\alpha}+\sum_{i\in J}C_{k,i}a_{k,i}e^{i\alpha}=0. \tag{3.53}$$

By Lemma 3, at least one of $C_{t,i},(t,i)\in T\times I$ is not equal to 0. Set $T_1\times I_1=\big\{(t,i)\in T\times I: C_{t,i}\neq 0\big\}$. Then $T_1\times I_1\neq\emptyset$ (empty set). By the assumption that $a_{t,i}e^{(k-t)\beta+i\alpha}$, $(t,i)\in T\times I$ are linearly independent, at least one of $C_{k,i},i\in J$ is not equal to 0. Set $J_1=\big\{i\in J: C_{k,i}\neq 0\big\}$. Then $J_1\neq\emptyset$. Let $i_1\in J_1$. Then by (3.53), we get

$$\sum_{(t,i)\in T_1\times I_1}\frac{-C_{t,i}a_{t,i}}{C_{k,i_1}a_{k,i_1}}e^{(k-t)\beta+(i-i_1)\alpha}+\sum_{i\in J_1\setminus\{i_1\}}\frac{-C_{k,i}a_{k,i}}{C_{k,i_1}a_{k,i_1}}e^{(i-i_1)\alpha}=1.$$

If the functions

$$\begin{aligned}&\frac{-C_{t,i}a_{t,i}}{C_{k,i_1}a_{k,i_1}}e^{(k-t)\beta+(i-i_1)\alpha},(t,i)\in T_1\times I_1\quad\text{and}\\&\frac{-C_{k,i}a_{k,i}}{C_{k,i_1}a_{k,i_1}}e^{(i-i_1)\alpha},i\in J_1\setminus\{i_1\}\end{aligned} \tag{3.54}$$

are linearly independent, then by Lemma 4, we get for $(t_0,i_0)\in T_1\times I_1$,

$$T\left(r,\frac{C_{t_0,i_0}a_{t_0,i_0}}{C_{k,i_1}a_{k,i_1}}e^{(k-t_0)\beta+(i_0-i_1)\alpha}\right)=S(r),$$

so that

$$T\big(r, e^{(k-t_0)\beta+(i_0-i_1)\alpha}\big) = S(r),$$

which again contradicts Claim I. Thus the functions showed in (3.54) are linearly dependent. Thus there exist constants $D_{t,i}, (t,i) \in T_1 \times I_1$ and $D_{k,i}, i \in J_1 \setminus \{i_1\}$, at least one of them is not equal to 0, such that

$$\sum_{(t,i)\in T_1\times I_1} \frac{D_{t,i}a_{t,i}}{C_{k,i_1}a_{k,i_1}} e^{(k-t)\beta+(i-i_1)\alpha} + \sum_{i\in J_1\setminus\{i_1\}} \frac{D_{k,i}a_{k,i}}{C_{k,i_1}a_{k,i_1}} e^{(i-i_1)\alpha} = 0,$$

so that

$$\sum_{(t,i)\in T_1\times I_1} D_{t,i}a_{t,i}e^{(k-t)\beta+i\alpha} + \sum_{i\in J_1\setminus\{i_1\}} D_{k,i}a_{k,i}e^{i\alpha} = 0. \tag{3.55}$$

By Lemma 3, we see that at least one of $D_{t,i}, (t,i) \in T \times I$ is not equal to 0, so that $T_2 \times I_2 = \big\{(t,i) \in T_1 \times I_1 : D_{t,i} \neq 0\big\} \neq \emptyset$. By the assumption that $a_{t,i}e^{(k-t)\beta+i\alpha}$, $(t,i) \in T \times I$ are linearly independent, at least one of $D_{k,i}, i \in J\setminus\{i_1\}$ is not equal to 0, so that $J_2 = \big\{i \in J_1 \setminus \{i_1\} : D_{k,i} \neq 0\big\} \neq \emptyset$. Let $i_2 \in J_2$. Then using an argument similar to that in the above step, there exist constants $E_{t,i}, (t,i) \in T_2 \times I_2$ and $E_{k,i}, i \in J_2 \setminus \{i_2\}$, at least one of them is not equal to 0, such that

$$\sum_{(t,i)\in T_2\times I_2} E_{t,i}a_{t,i}e^{(k-t)\beta+i\alpha} + \sum_{i\in J_2\setminus\{i_2\}} E_{k,i}a_{k,i}e^{i\alpha} = 0.$$

Step by step, it follows that J is an infinite set. It is impossible. Hence we have proved Claim II.

Next we continue to prove Theorem 1. By (3.30), (3.50) and Claim II, we get

$$e^{k\beta} + \sum_{i\in J} a_{k,i}e^{i\alpha} = 0,$$

so that

$$\sum_{i\in J}(-a_{k,i})e^{i\alpha-k\beta} = 1. \tag{3.56}$$

If the functions $(-a_{k,i})e^{i\alpha-k\beta}$, $i \in J$, are linearly independent, then by Lemma 3, we get

$$T(r, a_{k,m}e^{m\alpha-k\beta}) = S(r),$$

so that

$$T(r, e^{m\alpha-k\beta}) = S(r).$$

This contradicts Claim I.

Hence the functions $(-a_{k,i})e^{i\alpha-k\beta}$, $i\in J$, are linearly dependent. Thus there exist constants C_i, $i\in J$, at least one of them is not equal to 0, such that

$$\sum_{i\in J} C_i a_{k,i} e^{i\alpha-k\beta}=0,$$

so that

$$\sum_{i\in J} C_i a_{k,i} e^{i\alpha}=0.$$

This contradicts Lemma 3.

The proof of Theorem 1 is complete.

Acknowledgements. The authors are grateful to the referee for his/her many valuable suggestions.

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Received 10 September 2004