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A SELECTION THEORY FOR MULTIPLE-VALUED FUNCTIONS IN THE SENSE OF ALMGREN

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Abstract. A $\mathbf{Q}_k(\mathbf{R}^n)$-valued function is essentially a rule assigning k unordered and non necessarily distinct elements of $\mathbf{R}^n$ to each element of its domain set $A \subseteq \mathbf{R}^m$. For a $\mathbf{Q}_k(\mathbf{R})$-valued function f we construct a decomposition into k branches that naturally inherit the regularity properties of f. Next we prove that a measurable $\mathbf{Q}_k(\mathbf{R}^n)$-valued function admits a decomposition into k measurable branches. An example of a Lipschitzian $\mathbf{Q}_2(\mathbf{R}^2)$-valued function that does not admit a continuous decomposition is also provided and we state a selection result about multiple-valued functions defined on intervals. We finally give a new proof of Rademacher's theorem for multiple-valued functions. This proof is mainly based on the decomposition theory and it does not use Almgren's bi-Lipschitzian correspondence between $\mathbf{Q}_k(\mathbf{R}^n)$ and a cone $Q^* \subset \mathbf{R}^{P(n)k}$.

1. Introduction

In his big regularity paper [1], F. J. Almgren introduced the machinery of multiple-valued functions to study the partial regularity of area-minimizing integral currents. He proved that any m-dimensional mass-minimizing integral current is regular except on a set of Hausdorff dimension at most $m-2$. His regularity theory relies on a scheme of approximation of Dirichlet-minimizing multiple-valued functions. The success of Almgren's regularity theory raises the need of further studying multiple-valued functions and of making his work more accessible.

A multiple-valued function $f\colon A \subseteq \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ is essentially a rule assigning k unordered and non necessarily distinct elements of $\mathbf{R}^n$ to each element of its domain. Such maps are studied in complex analysis (see Appendix 5 in [8]). Indeed in complex function theory one often speaks of the "two-valued function $f(z) = z^{1/2}$". This is better considered as a function from $\mathbf{R}^2$ to $\mathbf{Q}_2(\mathbf{R}^2)$.

In Chapter 1 of [1], Almgren proved that the metric space $\mathbf{Q}_k(\mathbf{R}^n)$ is in explicit bi-Lipschitzian correspondence with a finite polyhedral cone Q^* included in a higher dimensional Euclidean space and his analysis is mainly based on this correspondence. In the present paper, we put this correspondence aside and we approach multiple-valued functions by selection arguments. For a $\mathbf{Q}_k(\mathbf{R}^n)$-valued function f, we construct a decomposition into k particular branches $f_1, \dots, f_k\colon \mathbf{R}^m \to \mathbf{R}^n$ and we study the properties of the branches such as continuity in accordance with those of f. For $n = 1$, we prove in Proposition 4.1 that f admits

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Lipschitzian (respectively Hölder continuous, continuous, measurable) branches if f is Lipschitzian (respectively Hölder continuous, continuous, measurable). For $n > 1$, we show in Proposition 5.1 that f can be split into measurable branches if f is measurable and we complete this result by a Lusin type theorem for multiple-valued functions. We also provide an example of a Lipschitzian $\mathbf{Q}_2(\mathbf{R}^2)$-valued function that does not admit a continuous decomposition and state a selection result already proved in [1] and [3] about multiple-valued functions defined on closed intervals.

We finally recall what is meant by "differentiability" in the context of multiple-valued functions. Essentially a multiple-valued function f is said to be affinely approximable at $a \in \mathbf{R}^m$ if it is possible to approach f near a by a multiple-valued function admitting an affine decomposition. We suggest an original proof of Rademacher's theorem based on the selection theory:

Theorem. *Let $f\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ be a Lipschitzian multiple-valued function. Then f is strongly affinely approximable at $\mathscr{L}^m$ almost all points of $\mathbf{R}^m$.*

We close our paper by using the 1-dimensional selection result to show that these affine approximations control the variation of f:

Theorem. *Let $f\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ be a Lipschitzian multiple-valued function and $[a,b] := \{a + t(b-a) \mid t \in [0,1]\}$ where $a, b \in \mathbf{R}^m$ such that $a \neq b$. If f is affinely approximable at $\mathscr{H}^1$ almost all points of $[a,b]$ then*

$$\mathscr{F}\big(f(a), f(b)\big) \le \int_{[a,b]} \|Af(x)\| \, d\mathscr{H}^1(x).$$

2. Preliminaries

The scalar product of two vectors $x, y \in \mathbf{R}^n$ is denoted $(x|y)$, and the Euclidean norm of $x \in \mathbf{R}^n$ is denoted $|x|$. For a fixed space X with a metric d, let

$$\mathbf{B}(a,r) = \{x \in X \mid d(a,x) \le r\}, \quad \mathbf{U}(a,r) = \{x \in X \mid d(a,x) < r\}$$

be the closed and open balls with center $a \in X$ and radius $r > 0$. If $L\colon \mathbf{R}^m \to \mathbf{R}^n$ is a linear mapping, we set $\|L\| = \sup\{|L(x)| \mid x \in \mathbf{R}^m \text{ with } |x| \le 1\}$. For each point $x_i \in \mathbf{R}^n$, $[\![x_i]\!]$ denotes the Dirac measure at x_i. Denote $\mathbf{Q}_k(\mathbf{R}^n) = \big\{\sum_{i=1}^k [\![x_i]\!] \mid x_i \in \mathbf{R}^n\big\}$, where x_i and x_j are not necessarily distinct for $i \neq j$ and the integer k is already fixed. We can define a topology on $\mathbf{Q}_k(\mathbf{R}^n)$ by the equivalent metrics $\mathscr{G}$ and $\mathscr{F}$:

$$\mathscr{G}\bigg(\sum_{i=1}^k [\![x_i]\!], \sum_{j=1}^k [\![y_j]\!]\bigg) = \min\bigg\{ \bigg(\sum_{i=1}^k |x_i - y_{\sigma(i)}|^2\bigg)^{1/2} \;\bigg|\; \sigma \text{ is a permutation of } \{1, \dots, k\}\bigg\},$$

$$\mathscr{F}\left(\sum_{i=1}^{k}[\![x_i]\!],\sum_{j=1}^{k}[\![y_j]\!]\right)=\min\left\{\sum_{i=1}^{k}|x_i-y_{\sigma(i)}|\mid \sigma \text{ is a permutation of } \{1,\dots,k\}\right\}.$$

We easily prove that $\big(\mathbf{Q}_k(\mathbf{R}^n),\mathscr{G}\big)$ is a complete separable metric space. A map $f\colon A\subseteq\mathbf{R}^m\to\mathbf{Q}_k(\mathbf{R}^n)$ will be called a multiple-valued function or a $\mathbf{Q}_k(\mathbf{R}^n)$-valued function. Let us note that a $\mathbf{Q}_1(\mathbf{R}^n)$-valued function is nothing else than a classical function since $\mathbf{Q}_1(\mathbf{R}^n)$ is in bijection with $\mathbf{R}^n$.

To begin with, let us pay attention to a problem that will make it possible for us to reduce significantly the size of the coming proofs. Considering $f_1,\dots,f_k\colon A\subseteq\mathbf{R}^m\to\mathbf{R}^n$, we can define the following $\mathbf{Q}_k(\mathbf{R}^n)$-valued function

$$f:=\sum_{i=1}^{k}[\![f_i]\!] \tag{1}$$

and wonder if f inherits the regularity properties of $f_1,\dots,f_k$. We can then easily check Proposition 2.1.

Proposition 2.1. *Let $f_1,\dots,f_k\colon A\subseteq\mathbf{R}^m\to\mathbf{R}^n$ and f defined by* (1)*. The following assertions are checked*:

(1) *If $f_1,\dots,f_k$ are Lipschitzian on $B\subseteq A$ then f is Lipschitzian on B and* $\mathrm{Lip}(f)\le\big(\sum_{i=1}^{k}\mathrm{Lip}^2(f_i)\big)^{1/2}$.

(2) *If $f_1,\dots,f_k$ are continuous on $B\subseteq A$ then f is continuous on B.*

To end this section, we will define what is meant by measurable $\mathbf{Q}_k(\mathbf{R}^n)$-valued function and prove a proposition which is similar to Proposition 2.1. As in [4] and [5], we call measure what is usually called outer measure.

Definition 2.1. Let μ be a measure on $\mathbf{R}^m$. A multiple-valued function $f\colon A\subseteq\mathbf{R}^m\to\mathbf{Q}_k(\mathbf{R}^n)$ is called μ-measurable if for all open $G\subseteq\mathbf{Q}_k(\mathbf{R}^n)$, $f^{-1}(G)$ is μ-measurable.

Proposition 2.2. *Let $f_1,\dots,f_k\colon A\subseteq\mathbf{R}^m\to\mathbf{R}^n$ and f defined by* (1)*. If $f_1,\dots,f_k$ are μ-measurable then f is μ-measurable.*

Proof. Since $\mathbf{Q}_k(\mathbf{R}^n)$ is separable it is enough to show that $f^{-1}(U)$ is μ-measurable for all open balls $U\subset\mathbf{Q}_k(\mathbf{R}^n)$. Let $v=\sum_{i=1}^{k}[\![x_i]\!]\in\mathbf{Q}_k(\mathbf{R}^n)$ and $r>0$. We note that

$$\begin{aligned}
f^{-1}\big(\mathbf{U}(v,r)\big)&=\left\{y\in A\;\middle|\;\mathscr{G}\left(v,\sum_{i=1}^{k}[\![f_i(y)]\!]\right)<r\right\}\\
&=\bigcup_{\sigma}\left\{y\in A\;\middle|\;\sum_{i=1}^{k}|x_i-f_{\sigma(i)}(y)|^2<r^2\right\}\\
&=\bigcup_{\sigma}\;\bigcup_{\substack{r_1,\dots,r_k\\ \text{rational}\\ r_1^2+\dots+r_k^2<r^2}}\;\bigcap_{i=1}^{k}f_{\sigma(i)}^{-1}\big(\mathbf{B}(x_i,r_i)\big)
\end{aligned}$$

so f is μ-measurable. ▫

3. Canonical decomposition

The central idea in this section consists in decomposing multiple-valued functions in the most efficient possible way. If one has a $\mathbf{Q}_k(\mathbf{R}^n)$-valued function defined on $A \subseteq \mathbf{R}^m$, it is easy to find maps $f_1, \ldots, f_k\colon A \to \mathbf{R}^n$ such that $f = \sum_{i=1}^k [\![f_i]\!]$ on A. However we will be concerned with the existence of branches which preserve some properties of f such as measurability, continuity .... We start by defining some necessary tools to this study.

For all $n, k \in \mathbf{N}_0$, we define the functions

$$\beta_{(n,k),1}\colon \mathbf{Q}_k(\mathbf{R}^n) \to \mathbf{R}^n : \; v \mapsto \min\bigl(\operatorname{spt}(v)\bigr)$$

where $\mathbf{R}^n$ is endowed with a total order. For $k \in \mathbf{N}_0$ and $v \in \mathbf{Q}_k(\mathbf{R}^n)$ fixed, we will define by induction with respect to $i \in \{1, \ldots, k\}$ a sequence of pairs $\bigl(v_i, \beta_{(n,k),i}(v)\bigr)$ where $v_i \in \mathbf{Q}_{k-i+1}(\mathbf{R}^n)$ and $\beta_{(n,k),i}(v) \in \mathbf{R}^n$. Let us assume that $v_1 = v$ and that the sequence is defined until the index i, we then set successively

$$v_{i+1} := v_i - [\![\beta_{(n,k),i}(v)]\!] \quad \text{and} \quad \beta_{(n,k),i+1}(v) := \beta_{(n,k-i),1}(v_{i+1}).$$

By this process, we have just built the functions

$$\beta_{(n,k),i}\colon \mathbf{Q}_k(\mathbf{R}^n) \to \mathbf{R}^n \tag{2}$$

for $i = 1, \ldots, k$ associated to a total order on $\mathbf{R}^n$ and such that

$$v = \sum_{i=1}^{k} [\![\beta_{(n,k),i}(v)]\!] \tag{3}$$

for all $v \in \mathbf{Q}_k(\mathbf{R}^n)$. Consequently, if we consider a multiple-valued function $f\colon A \subseteq \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$, an immediate consequence of (3) is that $f = \sum_{i=1}^k [\![f_i]\!]$ with

$$f_i := \beta_{(n,k),i} \circ f \tag{4}$$

for $i = 1, \ldots, k$. Subsequently, we will study the properties of the selection defined by (4), assuming that $\mathbf{R}^n$ is endowed with the lexicographical order.

4. Decomposition properties for $n = 1$

Lemma 4.1. *For all* $k \in \mathbf{N}_0$, $\operatorname{Lip}(\beta_{(1,k),i}) \leq 1$ *for* $i = 1, \dots, k$.

Proof. Let $k \in \mathbf{N}_0$ and $v = \sum_{j=1}^{k} [\![x_j]\!]$, $w = \sum_{j=1}^{k} [\![y_j]\!] \in \mathbf{Q}_k(\mathbf{R})$ such that $x_1 \leq \cdots \leq x_k$ and $y_1 \leq \cdots \leq y_k$. By the first part of the proof of Theorem 1.2 in [1], it is clear that

$$\mathscr{G}^2(v, w) = \sum_{j=1}^{k} |x_j - y_j|^2 \geq |x_i - y_i|^2 = |\beta_{(1,k),i}(v) - \beta_{(1,k),i}(w)|^2$$

for $i = 1, \dots, k$. ▫

We can then state Proposition 4.1 which immediately ensues from (4) and Lemma 4.1.

Proposition 4.1. *Let* $f\colon A \subseteq \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R})$ *be a multiple-valued function. Then there exist* $f_1, \dots, f_k\colon A \subseteq \mathbf{R}^m \to \mathbf{R}$ *such that* $f = \sum_{i=1}^{k} [\![f_i]\!]$ *on* A *and the following assertions hold*:

(1) *If* f *is continuous and* $\omega_{f,x}$ *denote the modulus of continuity of* f *at* x *i.e.*

$$\omega_{f,x}(\delta) = \sup\bigl\{\mathscr{G}(f(x), f(y)) \mid y \in A \text{ and } |x - y| \leq \delta\bigr\}$$

then $f_1, \dots, f_k$ *are continuous and* $\omega_{f_i,x} \leq \omega_{f,x}$ *for* $i = 1, \dots, k$ *and for all* $x \in A$ *where* $\omega_{f_i,x}$ *denotes the modulus of continuity of* f_i *at* x *i.e.*

$$\omega_{f_i,x}(\delta) = \sup\bigl\{|f_i(x) - f_i(y)| \mid y \in A \text{ and } |x - y| \leq \delta\bigr\}.$$

(2) *If* f *is* μ*-measurable then* $f_1, \dots, f_k$ *are* μ*-measurable.*

This last proposition is useful as it can help us to easily prove a few theorems that are not obvious at first sight. Nevertheless, this proposition only settles when $n = 1$. Can we hope the same for $n > 1$? The next section will attempt to answer this question. In order to illustrate the interest of Proposition 4.1, we mention two extension theorems for $\mathbf{Q}_k(\mathbf{R})$-valued functions.

Theorem 4.1. *If* $A \subset \mathbf{R}^m$ *is closed and* $f\colon A \to \mathbf{Q}_k(\mathbf{R})$ *is continuous, then* f *has a continuous extension* $\bar{f}\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R})$.

Proof. This follows from Proposition 4.1, Tietze's extension theorem and Proposition 2.1. ▫

Theorem 4.2. *If* $A \subset \mathbf{R}^m$ *and* $f\colon A \to \mathbf{Q}_k(\mathbf{R})$ *is Lipschitzian, then* f *has a Lipschitzian extension* $\bar{f}\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R})$ *with* $\operatorname{Lip}(\bar{f}) \leq k^{1/2} \operatorname{Lip}(f)$.

Proof. This follows from Proposition 4.1, Kirszbraun's extension theorem and Proposition 2.1. ▫

5. Decomposition properties for $n > 1$

As shown earlier, studying the hypothetical properties of the selection defined by (4) boils down to studying the properties of the functions (2). For $n = 1$, we showed that these maps are Lipschitzian. For $n > 1$, we will prove that they are merely Borelian. Let $e_1, \dots, e_n$ denote the standard basis vectors of $\mathbf{R}^n$. This notation will only be used in Lemma 5.1. Recall that $\mathbf{R}^n$ is endowed with the lexicographical order.

Lemma 5.1. *For all $k \in \mathbf{N}_0$ and $n \in \mathbf{N}_0 \backslash \{1\}$, $\beta_{(n,k),i}$ is Borelian for $i = 1, \dots, k$.*

Proof. Let $k \in \mathbf{N}_0$ and $n \in \mathbf{N}_0 \backslash \{1\}$. We will partition $\mathbf{Q}_k(\mathbf{R}^n)$ into a countable union of closed sets so that the restrictions of the functions $\beta_{(n,k),i}$ to these closed sets are continuous. It will then be clear that the functions $\beta_{(n,k),i}$ are Borelian.

It is obvious that

$$\mathbf{Q}_k(\mathbf{R}^n) = \bigcup_{a=1}^{\infty} K_a$$

where

$$K_a = \mathbf{Q}_k(\mathbf{R}^n) \cap \Big\{ \sum_{i=1}^{k} [\![x_i]\!] \;\Big|\text{if } (x_i|e_l) \neq (x_j|e_l) \text{ for } l \in \{1, \dots, n\} \\ \text{then } |(x_i|e_l) - (x_j|e_l)| \geq 1/a \Big\}.$$

Let us show that the sets K_a are closed and that the restrictions of the functions $\beta_{(n,k),i}$ to these sets are continuous.

Fix $a \in \mathbf{N}_0$. We take a sequence $(v_q = \sum_{i=1}^{k} [\![x_{i,q}]\!])_{q \in \mathbf{N}} \subset K_a$ such that $v_q \to v$ as $q \to \infty$ and we will show that $v \in K_a$. Let us consider any subsequence of $(v_q)_{q \in \mathbf{N}}$ still denoted by $(v_q)_{q \in \mathbf{N}}$. Since $(v_q)_{q \in \mathbf{N}}$ converges, it is bounded; hence there exists $M > 0$ such that $\mathscr{G}^2(v_q, k[\![0]\!]) = \sum_{i=1}^{k} |x_{i,q}|^2 < M$ for all $q \in \mathbf{N}$. Consequently the sequence $(x_{i,q})_{q \in \mathbf{N}}$ is bounded for $i = 1, \dots, k$. Therefore there exist $q_1 < q_2 < \cdots$ and $x_1, x_2, \dots, x_k \in \mathbf{R}^n$ such that for all $i \in \{1, \dots, k\}$, the subsequence $(x_{i,q_r})_{r \in \mathbf{N}}$ converges to x_i as $r \to \infty$. Then the subsequence $(v_{q_r})_{r \in \mathbf{N}}$ converges to $\sum_{i=1}^{k} [\![x_i]\!]$ as $r \to \infty$; hence $v = \sum_{i=1}^{k} [\![x_i]\!]$. If $(x_i|e_l) \neq (x_j|e_l)$ for $l \in \{1, \dots, n\}$ then there exists $p \in \mathbf{N}$ such that $(x_{i,q_r}|e_l) \neq (x_{j,q_r}|e_l)$ for all $r \geq p$; hence $|(x_{i,q_r}|e_l) - (x_{j,q_r}|e_l)| \geq 1/a$. By taking the limit, we deduce that $|(x_i|e_l) - (x_j|e_l)| \geq 1/a$; hence $v \in K_a$.

It only remains to prove that the functions $\beta_{(n,k),i}$ restricted to K_a are continuous. For $i, j \in \{1, \dots, k\}$ such that $i \neq j$, we state the following important remarks:

(1) If $(x_i|e_l) = (x_j|e_l)$ for $l \in \{1,\dots,n\}$ then there exists $p \in \mathbf{N}$ such that for all $r \geq p$, we have $(x_{i,q_r}|e_l) = (x_{j,q_r}|e_l)$. Otherwise, there exist $q_{r_1} < q_{r_2} < \cdots$ such that $(x_{i,q_{r_s}}|e_l) \neq (x_{j,q_{r_s}}|e_l)$ for all $s \in \mathbf{N}$; hence

$$|(x_{i,q_{r_s}}|e_l) - (x_{j,q_{r_s}}|e_l)| \geq 1/a$$

which on letting $s \to \infty$ implies the contradiction $|(x_i|e_l) - (x_j|e_l)| \geq 1/a$.

(2) If $(x_i|e_l) < (x_j|e_l)$ for $l \in \{1,\dots,n\}$ then there exists $p \in \mathbf{N}$ such that for all $r \geq p$, we have $(x_{i,q_r}|e_l) < (x_{j,q_r}|e_l)$. Otherwise, there exist $q_{r_1} < q_{r_2} < \cdots$ such that $(x_{i,q_{r_s}}|e_l) \geq (x_{j,q_{r_s}}|e_l)$ for all $s \in \mathbf{N}$ which on letting $s \to \infty$ implies the contradiction $(x_i|e_l) \geq (x_j|e_l)$.

(3) If $(x_i|e_l) > (x_j|e_l)$ for $l \in \{1,\dots,n\}$ then there exists $p \in \mathbf{N}$ such that for all $r \geq p$, we have $(x_{i,q_r}|e_l) > (x_{j,q_r}|e_l)$. Otherwise, there exist $q_{r_1} < q_{r_2} < \cdots$ such that $(x_{i,q_{r_s}}|e_l) \leq (x_{j,q_{r_s}}|e_l)$ for all $s \in \mathbf{N}$ which on letting $s \to \infty$ implies the contradiction $(x_i|e_l) \leq (x_j|e_l)$.

Consequently, if $x_1,\dots,x_k$ are ordered in a certain way then there exists $p \in \mathbf{N}$ such that $x_{1,q_r}, \dots, x_{k,q_r}$ are ordered in the same way for all $r \geq p$. Therefore $\beta_{(n,k),i}(v_{q_r}) \to \beta_{(n,k),i}(v)$ for $i = 1,\dots,k$. □

The following result is a consequence of (4) and Lemma 5.1.

Proposition 5.1. *Let μ be a measure on $\mathbf{R}^m$ and $f\colon A \subseteq \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ be a μ-measurable multiple-valued function. Then there exist μ-measurable functions $f_1,\dots,f_k\colon A \subseteq \mathbf{R}^m \to \mathbf{R}^n$ such that $f = \sum_{i=1}^k [\![f_i]\!]$ on A.*

The following theorem is nothing else than an adaptation of Lusin's theorem to multiple-valued functions.

Theorem 5.1. *Let μ be a Borel regular measure on $\mathbf{R}^m$ and $f\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ be a μ-measurable multiple-valued function. Assume $A \subset \mathbf{R}^m$ is μ-measurable and $\mu(A) < \infty$. Fix $\varepsilon > 0$. Then there exists a compact set $C \subset A$ and $f_1,\dots,f_k\colon \mathbf{R}^m \to \mathbf{R}^n$ such that*

(1) $f = \sum_{i=1}^k [\![f_i]\!]$ *on* $\mathbf{R}^m$,
(2) $\mu(A \backslash C) < \varepsilon$,
(3) $f_i|_C$ *is continuous for* $i = 1,\dots,k$, *and*
(4) $f|_C$ *is continuous.*

Proof. By Proposition 5.1, there exist μ-measurable functions $f_1,\dots,f_k\colon \mathbf{R}^m \to \mathbf{R}^n$ such that $f = \sum_{i=1}^k [\![f_i]\!]$ on $\mathbf{R}^m$. We then infer from Lusin's theorem that for $i = 1,\dots,k$, there exists a compact set $C_i \subset A$ such that $\mu(A \backslash C_i) < \varepsilon/k$ and $f_i|_{C_i}$ is continuous. We set $C := \bigcap_{i=1}^k C_i \subset A$ which is compact. Then it is clear that $\mu(A \backslash C) < \varepsilon$ and $f|_C$ is continuous by Proposition 2.1. □

To end this section, we show that a continuous $\mathbf{Q}_k(\mathbf{R}^n)$-valued function does no necessarily admit a continuous decomposition if $n > 1$. We consider the Lipschitzian $\mathbf{Q}_2(\mathbf{R}^2)$-valued function

$$f\colon \mathbf{S}^1 \subset \mathbf{R}^2 \to \mathbf{Q}_2(\mathbf{S}^1):$$
$$x = (\cos\theta, \sin\theta) \mapsto f(x) = \left[\!\!\left[\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)\right]\!\!\right] + \left[\!\!\left[\left(-\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\right)\right]\!\!\right]$$

where $\mathbf{S}^1 := \{(\cos\theta, \sin\theta) \mid \theta \in [0, 2\pi)\}$ is the unit circle centered on the origin, and we study the following selection problem: can one select for all $x \in \mathbf{S}^1$, a point $g(x) \in \operatorname{spt}(f(x))$ in such a way that g is continuous on $\mathbf{S}^1$? Assume that such a function exists. We introduce the mapping

$$h\colon \mathbf{S}^1 \to \mathbf{S}^1 : x = (\cos\theta, \sin\theta) \mapsto h(x) = (\cos 2\theta, \sin 2\theta)$$

and it is obvious that h is continuous and that its topological degree, written $\deg(h)$, is 2. On the other hand, $h \circ g = \mathrm{id}_{\mathbf{S}^1}$ so that $\deg(h \circ g) = 1$ and the topological degree of g is necessarily an integer which refutes the equality $\deg(h \circ g) = \deg(h)\deg(g)$.

Consequently, a continuous $\mathbf{Q}_k(\mathbf{R}^{n>1})$-valued function does not necessarily admit a continuous decomposition. This also proves that there is no order on $\mathbf{R}^{n>1}$ so that the functions defined by (2) are continuous.

Nevertheless, we can prove that every continuous multiple-valued function f defined on a closed interval can be split into single branches which inherit the modulus of continuity of f.

Proposition 5.2. *Let $f\colon [a,b] \subset \mathbf{R} \to \mathbf{Q}_k(\mathbf{R}^n)$ be a continuous multiple-valued function and ω_f denote the modulus of continuity of f. Then there exist continuous functions $f_1, \dots, f_k\colon [a,b] \subset \mathbf{R} \to \mathbf{R}^n$ such that $f = \sum_{i=1}^k [\![f_i]\!]$ on $[a,b]$ and*

$$\omega_{f_i} \le n^{1/2} k \omega_f$$

where ω_{f_i} denotes the modulus of continuity of f_i for $i = 1, \dots, k$.

Proof. We will only prove that if f is Lipschitzian then f admits a Lipschitzian selection $f_1, \dots, f_k$ such that $\operatorname{Lip}(f_i) \le \operatorname{Lip}(f)$ for $i = 1, \dots, k$ since it is the unique result we will need later. For a general argument, see the original Proposition 1.10 in [1] or Theorem 1.1 in [3].

We will construct multiple-valued functions $f^j\colon [a,b] \to \mathbf{Q}_k(\mathbf{R}^n)$ and functions $f_1^j, \dots, f_k^j : [a,b] \to \mathbf{R}^n$ with $f^j = \sum_{i=1}^k [\![f_i^j]\!]$ for each $j = 1, 2, \dots$ such that (passing to a subsequence of the j's) one obtains Lipschitzian functions $f_i = \lim_{j\to\infty} f_i^j$ with $f = \sum_{i=1}^k [\![f_i]\!]$.

For fixed $j \in \mathbf{N}_0$ and for each $l = 0, 1, \dots, j$ we set $t_l = a + l\Delta t$ where $\Delta t = (b-a)/j$. Then we choose $f^j\colon [a,b] \to \mathbf{Q}_k(\mathbf{R}^n)$ subject to the requirement that for each $l = 1, \dots, j$ and $t \in [t_{l-1}, t_l]$

$$f^j(t) = \sum_{i=1}^k \left[\!\!\left[\frac{t_l - t}{\Delta t} p_i + \frac{t - t_{l-1}}{\Delta t} q_i \right]\!\!\right]$$

corresponding to some choice of $p_1, \dots, p_k, q_1, \dots, q_k$ such that $f(t_{l-1}) = \sum_{i=1}^k [\![p_i]\!]$, $f(t_l) = \sum_{i=1}^k [\![q_i]\!]$ and $\mathscr{G}\big(f(t_{l-1}), f(t_l)\big) = \big(\sum_{i=1}^k |p_i - q_i|^2\big)^{1/2}$. One readily verifies the existence of piecewise affine continuous functions $f_1^j, \dots, f_k^j\colon [a,b] \to \mathbf{R}^n$ such that $f^j = \sum_{i=1}^k [\![f_i^j]\!]$. We choose and fix such functions for each j. We can then easily prove that $f(x) = \lim_{j\to\infty} f^j(x)$ for all $x \in [a,b]$. Let us fix $j \in \mathbf{N}_0$ and $u \in \{1, \dots, k\}$. We will now prove that f_u^j is in fact Lipschitzian with $\operatorname{Lip}(f_u^j) \le \operatorname{Lip}(f)$. Suppose $a \le c < d \le b$ and $r, s \in \{1, \dots, j\}$ such that $c \in [t_{r-1}, t_r]$ and $d \in [t_{s-1}, t_s]$. We consider two cases.

(1) Suppose that $r = s$. We obtain that

$$\begin{aligned} |f_u^j(c) - f_u^j(d)| &\le \left(\sum_{i=1}^k \left| \frac{t_r - d}{\Delta t} p_i + \frac{d - t_{r-1}}{\Delta t} q_i - \frac{t_r - c}{\Delta t} p_i - \frac{c - t_{r-1}}{\Delta t} q_i \right|^2 \right)^{1/2} \\ &= \left(\sum_{i=1}^k \left| \frac{c-d}{\Delta t} p_i - \frac{c-d}{\Delta t} q_i \right|^2 \right)^{1/2} = \frac{d-c}{\Delta t} \left(\sum_{i=1}^k |p_i - q_i|^2 \right)^{1/2} \\ &= \frac{d-c}{\Delta t} \mathscr{G}\big(f(t_{r-1}), f(t_r)\big) \le \operatorname{Lip}(f)(d-c). \end{aligned}$$

(2) Suppose that $r < s$. We obtain that

$$\begin{aligned} |f_u^j(c) - f_u^j(d)| &\le |f_u^j(c) - f_u^j(t_r)| + |f_u^j(t_r) - f_u^j(t_{r+1}) + \cdots \\ &\quad + |f_u^j(t_{s-2}) - f_u^j(t_{s-1})| + |f_u^j(t_{s-1}) - f_u^j(d)| \\ &\le \operatorname{Lip}(f)\big((t_r - c) + (t_{r+1} - t_r) + \cdots \\ &\quad + (t_{s-1} - t_{s-2}) + (d - t_{s-1})\big) \\ &= \operatorname{Lip}(f)(d-c). \end{aligned}$$

We conclude that the sequences $\{f_1^j\}_{j\in\mathbf{N}}, \dots, \{f_k^j\}_{j\in\mathbf{N}}$ belong to the following closed, bounded and equicontinuous family

$$\begin{aligned} \Omega = \Big\{ y \in C([a,b], \mathbf{R}^n) \;\Big|\; & |y(x) - y(\bar{x})| \le \operatorname{Lip}(f)|x - \bar{x}| \text{ for all } x,\ \bar{x} \in [a,b] \text{ and} \\ & |y(a)| \le \max\big\{|v| \;\big|\; v \in \operatorname{spt}\big(f(a)\big)\big\}\Big\}. \end{aligned}$$

By Arzela–Ascoli's theorem, there exist $j_1 < j_2 < \cdots$ and $f_1, \dots, f_k \in \Omega$ such that $\{f_i^{j_l}\}_{l\in\mathbf{N}}$ converges uniformly to f_i as $l \to \infty$ for $i = 1, \dots, k$. It is immediate that $f = \lim_{l\to\infty} f^{j_l} = \lim_{l\to\infty} \sum_{i=1}^k [\![f_i^{j_l}]\!] = \sum_{i=1}^k [\![f_i]\!]$. ▫

6. Rademacher's theorem for multiple-valued functions

Before presenting a theorem for multiple-valued functions identical to Rademacher's theorem, we will define the meaning of "differentiability" and "approximative differentiability" for multiple-valued functions in the way they were originally expressed by Almgren in [1].

Definition 6.1. A multiple-valued function $f\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ is said to be affine if and only if there exist affine maps $g_1,\dots,g_k\colon \mathbf{R}^m \to \mathbf{R}^n$ such that $f = \sum_{i=1}^k [\![g_i]\!]$.

One can check that if $f = \sum_{i=1}^k [\![g_i]\!]$ is such an affine function, then the functions $g_1,\dots,g_k$ are uniquely determined up to order.

Definition 6.2. Let $A \subseteq \mathbf{R}^m$ be an open set and $f\colon A \to \mathbf{Q}_k(\mathbf{R}^n)$. One says that f is affinely approximable at $a \in A$ if and only if there is an affine multiple-valued function $g\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathscr{G}\big(f(x), g(x)\big) \le \varepsilon|x-a| \quad \text{for all } x \in \mathbf{B}(a,\delta) \cap A.$$

Definition 6.3. Let $A \subseteq \mathbf{R}^m$ be an open set and $f\colon A \to \mathbf{Q}_k(\mathbf{R}^n)$. One says that f is approximately affinely approximable at $a \in A$ if and only if there is an affine multiple-valued function $g\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ satisfying

$$\operatorname{ap}\lim_{x\to a} \frac{\mathscr{G}\big(f(x), g(x)\big)}{|x-a|} = 0.$$

In other words, for any $\varepsilon > 0$,

$$\lim_{r\to 0} \frac{\mathscr{L}^m\big(\mathbf{B}(a,r) \cap \{x \mid \mathscr{G}\big(f(x), g(x)\big) > \varepsilon|x-a|\}\big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)} = 0.$$

In case f is affinely approximable (respectively approximately affinely approximable) at a, the affine multiple-valued function g is seen to be uniquely determined and is denoted $Af(a)$ (respectively $\operatorname{ap}Af(a)$). We set $\|Af(a)\| = \sum_{i=1}^k \|g_i - g_i(0)\|$, where $g_1,\dots,g_k$ are the unique affine functions such that $Af(a) = \sum_{i=1}^k [\![g_i]\!]$.

Definition 6.4. Let $A \subseteq \mathbf{R}^m$ be an open set and $f\colon A \to \mathbf{Q}_k(\mathbf{R}^n)$. One says that f is strongly affinely approximable (respectively strongly approximately affinely approximable) at $a \in A$ if and only if f is affinely approximable (respectively approximately affinely approximable) at a and if $g_1,\dots,g_k\colon \mathbf{R}^m \to \mathbf{R}^n$ are affine maps such that

$$Af(a) = \sum_{i=1}^k [\![g_i]\!] \quad (\text{respectively } \operatorname{ap}Af(a) = \sum_{i=1}^k [\![g_i]\!]),$$

then $g_i(a) = g_j(a)$ implies $g_i = g_j$ for all $i,j \in \{1,\dots,k\}$.

We now state two useful and analogous characterizations for the coming proofs.

Proposition 6.1. *Let $a \in A \subseteq \mathbf{R}^m$ and $f\colon A \to \mathbf{Q}_k(\mathbf{R}^n)$ such that $f = \sum_{i=1}^k [\![f_i]\!]$ where $f_i\colon A \to \mathbf{R}^n$ is differentiable at a for $i = 1, \ldots, k$. Then f is affinely approximable at a.*

Proof. For $i = 1, \ldots, k$ and for all $x \in \mathbf{R}^m$, we define $g_i(x) := f_i(a) + Df_i(a)(x-a)$ and we set $g := \sum_{i=1}^k [\![g_i]\!]$ which is obviously an affine multiple-valued function. Let $\varepsilon > 0$ and $i \in \{1, \ldots, k\}$. Since f_i is differentiable at a, there exists $\delta_i > 0$ such that $x \in A$ and $|x-a| \le \delta_i$ implies $|f_i(x) - g_i(x)| \le (\varepsilon/k^{1/2})|x-a|$. We then choose $\delta := \min\{\delta_1, \ldots, \delta_k\} > 0$ because $x \in A$ and $|x-a| \le \delta$ implies

$$\mathscr{G}\big(f(x), g(x)\big) \le \left(\sum_{i=1}^k |f_i(x) - g_i(x)|^2\right)^{1/2} \le \left(\sum_{i=1}^k \frac{\varepsilon^2}{k}|x-a|^2\right)^{1/2} = \varepsilon|x-a|. \ \square$$

Proposition 6.2. *Let $a \in A \subseteq \mathbf{R}^m$ and $f\colon A \to \mathbf{Q}_k(\mathbf{R}^n)$ such that $f = \sum_{i=1}^k [\![f_i]\!]$ where $f_i\colon A \to \mathbf{R}^n$ is approximately differentiable at a for $i = 1, \ldots, k$. Then f is approximately affinely approximable at a.*

Proof. For $i = 1, \ldots, k$ and for all $x \in \mathbf{R}^m$, we define

$$g_i(x) := f_i(a) + \mathrm{ap}Df_i(a)(x-a) \tag{5}$$

and we set

$$g := \sum_{i=1}^k [\![g_i]\!] \tag{6}$$

which is obviously an affine multiple-valued function. Let $\varepsilon > 0$ and $r > 0$. We easily check that

$$\begin{aligned}
&\frac{\mathscr{L}^m\Big(\mathbf{B}(a,r) \cap \big\{x \in \mathbf{R}^m \mid \mathscr{G}\big(f(x), g(x)\big) > \varepsilon|x-a|\big\}\Big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)} \\
&= \frac{\mathscr{L}^m\Big(\mathbf{B}(a,r) \cap \big\{x \in \mathbf{R}^m \mid \mathscr{G}\big(f(x), g(x)\big) \le \varepsilon|x-a|\big\}^c\Big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)} \\
&\le \frac{\mathscr{L}^m\Big(\mathbf{B}(a,r) \cap \big\{x \in \mathbf{R}^m \mid \sum_{i=1}^k |f_i(x) - g_i(x)|^2 \le \varepsilon^2|x-a|^2\big\}^c\Big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)} \\
&\le \frac{\mathscr{L}^m\left(\mathbf{B}(a,r) \cap \left(\bigcap_{i=1}^k \big\{x \in \mathbf{R}^m \mid |f_i(x) - g_i(x)|^2 \le \varepsilon^2 k^{-1}|x-a|^2\big\}\right)^c\right)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)}
\end{aligned}$$

$$= \frac{\mathscr{L}^m\Big(\mathbf{B}(a,r)\cap\Big(\bigcup_{i=1}^{k}\big\{x\in\mathbf{R}^m \mid |f_i(x)-g_i(x)|^2>\varepsilon^2k^{-1}|x-a|^2\big\}\Big)\Big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)}$$

$$\le \sum_{i=1}^{k}\frac{\mathscr{L}^m\big(\mathbf{B}(a,r)\cap\big\{x\in\mathbf{R}^m \mid |f_i(x)-g_i(x)|>\varepsilon k^{-1/2}|x-a|\big\}\big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)}$$

and the right-hand side converges to zero as $r\to 0$. □

Theorem 6.1. *Let $f\colon \mathbf{R}^m\to\mathbf{Q}_k(\mathbf{R})$ be a Lipschitzian multiple-valued function. Then f is strongly affinely approximable at $\mathscr{L}^m$ almost all points of $\mathbf{R}^m$.*

Proof. According to Proposition 4.1, there exist Lipschitzian functions $f_1,\dots,f_k\colon\mathbf{R}^m\to\mathbf{R}$ such that $f=\sum_{i=1}^{k}[\![f_i]\!]$. We then deduce from the classical Rademacher's theorem that there exists $B\subset\mathbf{R}^m$ such that $\mathscr{L}^m(B)=0$ and f_i is differentiable on B^c for $i=1,\dots,k$. Proposition 6.1 implies that f is affinely approximable on B^c with $Af(a)=\sum_{i=1}^{k}[\![f_i(a)+Df_i(a)(\,\cdot-a)]\!]$ for all $a\in B^c$. For all $i,j\in\{1,\dots,k\}$, we define $Z_{ij}=\{a\in B^c \mid f_i(a)=f_j(a)\}$ and we deduce from Corollary 1 in Section 3.1 of [4] that $Df_i(a)=Df_j(a)$ for $\mathscr{L}^m$ almost all $a\in Z_{ij}$. □

The proof of Rademacher's theorem is no longer easy when $n>1$ because a Lipschitzian multiple-valued function does not necessarily admit a Lipschitzian decomposition. The first proof was suggested by Almgren in [1] and is based on the existence of a bi-Lipschitzian correspondence between $\mathbf{Q}_k(\mathbf{R}^n)$ and a cone $Q^*\subset\mathbf{R}^{P(n)k}$ where $P(n)$ is an integer depending on n. The following proof does not use this correspondence.

Lemma 6.1. *Let $A\subseteq\mathbf{R}^m$ and $a\in A$. Suppose that $f_1,\dots,f_k\colon A\to\mathbf{R}^n$ are continuous at a. Then there exists $0<r(a)\le\infty$ such that*

$$\min_{\sigma}\Big\{\sum_{i=1}^{k}|f_i(x)-f_{\sigma(i)}(a)|^2\Big\}=\sum_{i=1}^{k}|f_i(x)-f_i(a)|^2 \quad \textit{for all } x\in\mathbf{U}\big(a,r(a)\big)\cap A.$$

Proof. We define $\varepsilon:=\min\{|f_i(a)-f_j(a)| \mid i,j\in\{1,\dots,k\}$ and $f_i(a)\neq f_j(a)\}$. Let $i\in\{1,\dots,k\}$. By the continuity of f_i at a, there exists $\delta_i>0$ such that $x\in A$ and $|x-a|<\delta_i$ implies $|f_i(x)-f_i(a)|<\frac{1}{2}\varepsilon$. We choose $r(a):=\min\{\delta_1,\dots,\delta_k\}>0$. Indeed for $i\in\{1,\dots,k\}$, σ a permutation of $\{1,\dots,k\}$ and $x\in\mathbf{U}\big(a,r(a)\big)\cap A$, we obtain the two following cases:

(1) If $f_{\sigma(i)}(a)=f_i(a)$ then $|f_i(x)-f_{\sigma(i)}(a)|=|f_i(x)-f_i(a)|$.
(2) If $f_{\sigma(i)}(a)\neq f_i(a)$ then it is clear that $f_{\sigma(i)}(a)\notin\mathbf{U}\big(f_i(a),\varepsilon\big)$. It ensues that $|f_i(x)-f_{\sigma(i)}(a)|\ge|f_i(x)-f_i(a)|$, otherwise we have the following contradiction

$$\varepsilon\le|f_i(a)-f_{\sigma(i)}(a)|\le|f_i(x)-f_{\sigma(i)}(a)|+|f_i(x)-f_i(a)|<2|f_i(x)-f_i(a)|<\varepsilon. \ \square$$

Theorem 6.2. *Let $f\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ be a Lipschitzian multiple-valued function and $A \subset \mathbf{R}^m$ be an $\mathscr{L}^m$-measurable set such that $\mathscr{L}^m(A) < \infty$. Fix $\varepsilon > 0$. Then there exists a compact set $C \subset A$ such that f is strongly affinely approximable at $\mathscr{L}^m$ almost all points of C and $\mathscr{L}^m(A \backslash C) < \varepsilon$.*

Proof. f is $\mathscr{L}^m$-measurable since f is Lipschitzian. By Theorem 5.1, there exist a compact set $C \subset A$ and $f_1, \dots, f_k\colon \mathbf{R}^m \to \mathbf{R}^n$ such that $f = \sum_{i=1}^k [\![f_i]\!]$, $\mathscr{L}^m(A \backslash C) < \varepsilon$ and $f_i|_C$ is continuous for $i = 1, \dots, k$.

Let $a \in C$. By Lemma 6.1, there exists $r(a) > 0$ such that $x \in \mathbf{U}\big(a, r(a)\big) \cap C$ implies

$$\sum_{i=1}^k |f_i(x) - f_i(a)|^2 = \mathscr{G}^2\big(f(x), f(a)\big) \le \operatorname{Lip}^2(f)|x-a|^2.$$

Consequently we get that

$$|f_i(x) - f_i(a)| \le \operatorname{Lip}(f)|x-a| \tag{7}$$

for $i = 1, \dots, k$ and for all $x \in \mathbf{U}\big(a, r(a)\big) \cap C$. Let

$$Z := \left\{ a \in C \;\middle|\; \lim_{r \to 0} \frac{\mathscr{L}^m\big(\mathbf{B}(a,r) \cap C\big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)} = 1 \right\},$$

so that $\mathscr{L}^m(C \backslash Z) = 0$ according to the Lebesgue density theorem. Notice also that

$$1 = \lim_{r \to 0} \frac{\mathscr{L}^m\big(\mathbf{B}(a,r) \cap C\big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)} = \lim_{r \to 0} \frac{\mathscr{L}^m\big(\mathbf{B}(a,r) \cap Z\big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)} + \underbrace{\lim_{r \to 0} \frac{\mathscr{L}^m\big(\mathbf{B}(a,r) \cap (C \backslash Z)\big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)}}_{=0}$$

for all $a \in Z$. Let $\bar{a} \in Z$ and $i \in \{1, \dots, k\}$. Let us set

$$h(x) := \frac{|f_i(x) - f_i(\bar{a})|}{|x - \bar{a}|}$$

for all $x \in \mathbf{R}^m$ and show that

$$\begin{aligned} \operatorname*{ap\,lim\,sup}_{x \to \bar{a}} h(x) &:= \inf\left\{ t \in \mathbf{R} \mid \lim_{r \to 0} \frac{\mathscr{L}^m\big(\mathbf{B}(\bar{a},r) \cap \{x \in \mathbf{R}^m \mid h(x) > t\}\big)}{\mathscr{L}^m\big(\mathbf{B}(\bar{a},r)\big)} = 0 \right\} \\ &\le \operatorname{Lip}(f). \end{aligned} \tag{8}$$

Assume that $t > \operatorname{Lip}(f)$ and $r < r(\bar{a})$. Observe that (7) implies

$$\frac{\mathscr{L}^m\left(\left\{ x \in \mathbf{R}^m \;\middle|\; \dfrac{|f_i(x) - f_i(\bar{a})|}{|x - \bar{a}|} > t \right\} \cap \mathbf{B}(\bar{a},r)\right)}{\mathscr{L}^m\big(\mathbf{B}(\bar{a},r)\big)} \le \frac{\mathscr{L}^m\big(\mathbf{B}(\bar{a},r) \backslash C\big)}{\mathscr{L}^m\big(\mathbf{B}(\bar{a},r)\big)}.$$

Since $\bar{a} \in Z$, the right-hand side converges to zero as $r \to 0$ so that the statement (8) is proved. For $i = 1, \ldots, k$, we deduce from Theorem 3.1.8[1] of [5] that f_i is approximately differentiable on $Z_i \subseteq Z$ with $\mathscr{L}^m(Z \backslash Z_i) = 0$. We fix $Z^* := \bigcap_{i=1}^k Z_i$ so that the functions f_i are approximately differentiable on Z^* and $\mathscr{L}^m(Z \backslash Z^*) = 0$. Let $a^* \in Z^*$. Proposition 6.2 implies that f is approximately affinely approximable at a^* by the affine multiple-valued function defined by (5) and (6). Let us show that f is in fact affinely approximable at a^* by the same affine multiple-valued function. Take note that the following argument is an adaptation of Lemma 3.1.5 of [5]. Let $\gamma > 0$. We choose $0 < \beta < 1$ such that

$$\beta\left(1 + \frac{\operatorname{Lip}(f) + \beta + \left(\sum_{i=1}^k \|\operatorname{ap}Df_i(a^*)\|^2\right)^{1/2}}{1-\beta}\right) < \gamma$$

and we define the set $W := \{x \in \mathbf{R}^m \mid \mathscr{G}(f(x), g(x)) \leq \beta|x - a^*|\}$. Since f is approximately affinely approximable at a^*, there exists $\delta > 0$ such as, if $0 < r < \delta$, then

$$\frac{\mathscr{L}^m\big(\mathbf{B}(a^*, r) \cap W^c\big)}{\mathscr{L}^m\big(\mathbf{B}(a^*, r)\big)} < \beta^m.$$

Let us fix $x \in \mathbf{U}\big(a^*, \delta(1-\beta)\big)$ and regard $r := (|x - a^*|)/(1-\beta)$. It is clear that $r < \delta$ and $\mathbf{B}(x, \beta r) \subseteq \mathbf{B}(a^*, r)$. On the other hand, we remark that $\mathbf{B}(x, \beta r) \cap W \neq \emptyset$ for otherwise we obtain the following contradiction

$$\begin{aligned}(\beta r)^m \alpha(m) &= \mathscr{L}^m\big(\mathbf{B}(x, \beta r)\big) = \mathscr{L}^m\big(\mathbf{B}(x, \beta r) \cap W^c\big) \\ &\leq \mathscr{L}^m\big(\mathbf{B}(a^*, r) \cap W^c\big) < \beta^m \mathscr{L}^m\big(\mathbf{B}(a^*, r)\big) = \beta^m r^m \alpha(m).\end{aligned}$$

where $\alpha(m)$ is the Lebesgue measure of the unit Euclidean ball in $\mathbf{R}^m$. Let $z \in \mathbf{B}(x, \beta r) \cap W$. We successively have

$$\begin{aligned}\mathscr{G}\big(f(x), g(x)\big) &\leq \mathscr{G}\big(f(x), f(z)\big) + \mathscr{G}\big(f(z), g(z)\big) + \mathscr{G}\big(g(z), g(x)\big) \\ &\leq \operatorname{Lip}(f)|x - z| + \beta|z - a^*| + \left(\sum_{i=1}^k |g_i(z) - g_i(x)|^2\right)^{1/2}\end{aligned}$$

[1] This theorem indicates that an approximate local growth condition on f suffices to guarantee that f is the union of a countable family of Lipschitzian functions.

$$\le \mathrm{Lip}(f)\beta r + \beta(|z-x| + |x-a^*|) + \left(\sum_{i=1}^{k} |\mathrm{ap}Df_i(a^*)(z-x)|^2\right)^{1/2}$$

$$\le \mathrm{Lip}(f)\beta r + \beta(\beta r + |x-a^*|) + |z-x|\left(\sum_{i=1}^{k} \|\mathrm{ap}Df_i(a^*)\|^2\right)^{1/2}$$

$$\le \mathrm{Lip}(f)\beta r + \beta(\beta r + |x-a^*|) + \beta r\left(\sum_{i=1}^{k} \|\mathrm{ap}Df_i(a^*)\|^2\right)^{1/2}$$

$$= \beta|x-a^*|\left(1 + \frac{\mathrm{Lip}(f) + \beta + \left(\sum_{i=1}^{k} \|\mathrm{ap}Df_i(a^*)\|^2\right)^{1/2}}{1-\beta}\right).$$

We thus obtain that $\mathscr{G}\big(f(x), g(x)\big) < \gamma|x-a^*|$ so f is affinely approximable at a^*. Thus f is affinely approximable on Z^*. For all $i, j \in \{1, \ldots, k\}$, we define $Z^*_{ij} = \{a \in Z^* \mid f_i(a) = f_j(a)\}$ and we deduce from Theorem 3 in Section 6.1 of [4] that $\mathrm{ap}Df_i(a) = \mathrm{ap}Df_j(a)$ for $\mathscr{L}^m$ almost all $a \in Z^*_{ij}$. Consequently f is strongly affinely approximable at $\mathscr{L}^m$ almost all points of C. ▫

Theorem 6.3. *Let $f\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ be a Lipschitzian multiple-valued function. Then f is strongly affinely approximable at $\mathscr{L}^m$ almost all points of $\mathbf{R}^m$.*

Proof. Let us show that there exist disjoint, compact sets $\{C_i\}_{i=1}^{\infty} \subset \mathbf{R}^m$ such that

$$\mathscr{L}^m\left(\left(\bigcup_{i=1}^{\infty} C_i\right)^c\right) = 0$$

and for each $i = 1, 2, \ldots$

$$f \text{ is strongly affinely approximable at } \mathscr{L}^m \text{ almost all points of } C_i.$$

For each positive integer n, set $\mathbf{B}_n = \mathbf{B}(0, n)$. By Theorem 6.2, there exists a compact set $C_1 \subset \mathbf{B}_1$ such that $\mathscr{L}^m(\mathbf{B}_1 \backslash C_1) \le 1$ and f is strongly affinely approximable $\mathscr{L}^m$ almost everywhere on C_1. Assume now $C_1, \ldots, C_n$ have been constructed, there exists a compact set $C_{n+1} \subset \mathbf{B}_{n+1} \backslash \bigcup_{i=1}^{n} C_i$ such that $\mathscr{L}^m(\mathbf{B}_{n+1} \backslash \bigcup_{i=1}^{n+1} C_i) \le 1/(n+1)$ and f is strongly affinely approximable at $\mathscr{L}^m$ almost all points of C_{n+1}. Consequently the starting assertion is confirmed. For all $i \in \mathbf{N}_0$, we define the set $A_i := \{a \in C_i \mid f \text{ is strongly affinely approximable at } a\}$ and note that $\mathscr{L}^m(C_i \backslash A_i) = 0$. It is clear that $\mathscr{L}^m\big(\big(\bigcup_{i=1}^{\infty} A_i\big)^c\big) = 0$ so f is strongly affinely approximable at $\mathscr{L}^m$ almost all points of $\mathbf{R}^m$. ▫

Theorem 6.4. *Let $f\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ be a Lipschitzian multiple-valued function and $[a, b] := \{a + t(b-a) \mid t \in [0, 1]\}$ where $a, b \in \mathbf{R}^m$ such that $a \ne b$.*

If f is affinely approximable at $\mathscr{H}^1$ almost all points of $[a,b]$ then

$$\mathscr{F}\big(f(a),f(b)\big) \le \int_{[a,b]} \|Af(x)\|\, d\mathscr{H}^1(x).$$

Proof. By a standard argument applied to Proposition 5.2, there exist Lipschitzian functions $f_1,\dots,f_k\colon [a,b] \to \mathbf{R}^n$ such that $f|_{[a,b]} = \sum_{i=1}^k [\![f_i]\!]$. We introduce the map $\varphi\colon [0,1] \to [a,b]\colon t \mapsto a + t(b-a)$ and the direction $u = (b-a)/(|b-a|)$. We then obtain

$$\begin{aligned}\mathscr{F}\big(f(b),f(a)\big) &\le \sum_{i=1}^k |f_i(b) - f_i(a)| = \sum_{i=1}^k \big|f_i\big(\varphi(1)\big) - f_i\big(\varphi(0)\big)\big| \\ &= \sum_{i=1}^k \left|\int_0^1 (f_i \circ \varphi)'(t)\, dt\right| \le \sum_{i=1}^k \int_0^1 \big|D_u f_i\big(\varphi(t)\big)\big|\, |\varphi'(t)|\, dt \\ &= \sum_{i=1}^k \int_{[a,b]} |D_u f_i(x)|\, d\mathscr{H}^1(x).\end{aligned}$$

So it remains to prove the following assertion:

Claim. *Let $x \in]a,b[$ such that f is affinely approximable at x and $D_u f_i(x)$ exists for $i = 1,\dots,k$. Then $\sum_{i=1}^k |D_u f_i(x)| \le \|Af(x)\|$.*

Since f is affinely approximable at x, it is clear that $Af(x)(x) = f(x) = \sum_{i=1}^k [\![f_i(x)]\!]$ so that

$$Af(x)(\,\cdot\,) = \sum_{i=1}^k [\![f_i(x) + L_i(x)(\,\cdot - x)]\!]$$

where $L_1(x),\dots,L_k(x)\colon \mathbf{R}^m \to \mathbf{R}^n$ are linear maps. Let

$$T = \{t \in \mathbf{R} \mid x + tu \in [a,b]\}.$$

For all $t \in T$, let σ_t be any permutation which attains the following minimum:

$$\min_\sigma \sum_{i=1}^k \big|f_{\sigma(i)}(x+tu) - f_i(x) - tL_i(x)(u)\big|.$$

We will now prove that there exists $\delta > 0$ such that $f_{\sigma_t(i)}(x) = f_i(x)$ for $i = 1,\dots,k$ and for all $t \in T$ satisfying $|t| < \delta$.

Fix $j \in \{1, \dots, k\}$. By contradiction, let $(t_l)_{l \in \mathbf{N}} \subset T$ be a sequence converging to 0 such that $f_{\sigma_{t_l}(j)}(x) \neq f_j(x)$ for all $l \in \mathbf{N}$. Using the continuity of every f_q satisfying $f_q(x) \neq f_j(x)$, we have that there exists $\varepsilon > 0$ such that, for l large enough,

$$|f_j(x) - f_{\sigma_{t_l}(j)}(x + t_l u)| > \varepsilon.$$

Hence, for l large enough,

$$\begin{aligned} &\frac{\min_\sigma \sum_{i=1}^k |f_{\sigma(i)}(x + t_l u) - f_i(x) - t_l L_i(x)(u)|}{|t_l|} \\ &\geq \frac{|f_{\sigma_{t_l}(j)}(x + t_l u) - f_j(x) - t_l L_j(x)(u)|}{|t_l|} \\ &\geq \frac{|f_{\sigma_{t_l}(j)}(x + t_l u) - f_j(x)|}{|t_l|} - \frac{|t_l L_j(x)(u)|}{|t_l|} \geq \frac{\varepsilon}{|t_l|} - \|L_j(x)\|. \end{aligned}$$

Consequently,

$$\lim_{l \to \infty} \frac{\min_\sigma \sum_{i=1}^k \big|f_{\sigma(i)}(x + t_l u) - f_i(x) - t_l L_i(x)(u)\big|}{|t_l|} = \infty$$

which contradicts the fact that f is affinely approximable at x by $Af(x)$. Then, there exists $\delta_j > 0$ such that $f_{\sigma_t(j)}(x) = f_j(x)$ if $t \in T$ and $|t| < \delta_j$. If we set $\delta = \min\{\delta_j \mid j = 1, \dots, k\}$ then $f_{\sigma_t(i)}(x) = f_i(x)$ for $i = 1, \dots, k$ and for all $t \in T$ such that $|t| < \delta$.

Fix $\bar{\varepsilon} > 0$. Since f is affinely approximable at x by $Af(x)$, there exists $\gamma > 0$ such that if $t \in T$ and $0 < |t| < \gamma$ then

$$\sum_{i=1}^k \left| \frac{f_{\sigma_t(i)}(x + tu) - f_i(x)}{t} - L_i(x)(u) \right| < \bar{\varepsilon}.$$

On the other hand, if $t \in T$ and $0 < |t| < \min\{\delta, \gamma\}$ then

$$\sum_{i=1}^k \left| \frac{f_{\sigma_t(i)}(x + tu) - f_{\sigma_t(i)}(x)}{t} - L_i(x)(u) \right| < \bar{\varepsilon};$$

hence

$$\min_\sigma \sum_{i=1}^k \left| \frac{f_{\sigma(i)}(x + tu) - f_{\sigma(i)}(x)}{t} - L_i(x)(u) \right| < \bar{\varepsilon}.$$

Consequently,

$$\lim_{t \to 0} \mathscr{F}\left(\sum_{i=1}^k \left[\!\!\left[\frac{f_i(x + tu) - f_i(x)}{t} \right]\!\!\right], \sum_{i=1}^k [\![L_i(x)(u)]\!] \right) = 0.$$

Finally, by the triangular inequality, we obtain for all $t \neq 0$ that

$$\begin{aligned}\mathscr{F}\Big(\sum_{i=1}^{k}[\![D_u f_i(x)]\!],\sum_{i=1}^{k}[\![L_i(x)(u)]\!]\Big)\\ \leq \mathscr{F}\Big(\sum_{i=1}^{k}[\![D_u f_i(x)]\!],\sum_{i=1}^{k}[\![\big(f_i(x+tu)-f_i(x)\big)/t]\!]\Big)\\ +\mathscr{F}\Big(\sum_{i=1}^{k}[\![\big(f_i(x+tu)-f_i(x)\big)/t]\!],\sum_{i=1}^{k}[\![L_i(x)(u)]\!]\Big)\end{aligned}$$

where the right-hand side converges to zero as $t \to 0$. Therefore, there exists a permutation σ such that $D_u f_i(x) = L_{\sigma(i)}(x)(u)$ for $i = 1, \dots, k$; hence

$$\sum_{i=1}^{k}|D_u f_i(x)| = \sum_{i=1}^{k}|L_i(x)(u)| \leq \sum_{i=1}^{k}\|L_i(x)\| = \|Af(x)\|. \square$$

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